TEST CODE: PMB

SYLLABUS

Convergence and divergence of sequence and series;
Cauchy sequence and completeness;
Bolzano-Weierstrass theorem;
continuity, uniform continuity, differentiability,
directional derivatives, Jacobians, Taylor Expansion;
integral calculus of one variable – existence of Riemann integral,
Fundamental theorem of calculus, change of variable;
elementary topological notions for metric space – open, closed and
compact sets, connectedness;
elements of ordinary differential equations.

Equivalence relations and partitions;
vector spaces, subspaces, basis, dimension, direct sum;
matrices, systems of linear equations, determinants;
diagonalization, triangular forms;
linear transformations and their representation as matrices;
groups, subgroups, quotients, homomorphisms, products,
Lagrange's theorem, Sylow's theorems;
rings, ideals, maximal ideals, prime ideals, quotients,
integral domains, unique factorization domains, polynomial rings;
fields, algebraic extensions, separable and normal extensions, finite fields.

SAMPLE QUESTIONS

1. Let k be a field and $k[x, y]$ denote the polynomial ring in the two variables x and y with coefficients from k. Prove that for any $a, b \in k$ the ideal generated by the linear polynomials $x - a$ and $y - b$ is a maximal ideal of $k[x, y]$.

2. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation. Show that there is a line L such that $T(L) = L$.

3. Let $A \subseteq \mathbb{R}^n$ and $f : A \to \mathbb{R}^m$ be a uniformly continuous function. If $\{x_n\}_{n \geq 1} \subseteq A$ is a Cauchy sequence then show that $\lim_{n\to\infty} f(x_n)$ exists.

4. Let $N > 0$ and let $f : [0, 1] \to [0, 1]$ be denoted by $f(x) = 1$ if $x = 1/i$ for some integer $i \leq N$ and $f(x) = 0$ for all other values of x. Show that f is Riemann integrable.

5. Let $F : \mathbf{R}^n \to \mathbf{R}$ be defined by
$$F(x_1, x_2, \dots, x_n) = \max\{|x_1|, |x_2|, \dots, |x_n|\}.$$
Show that F is a uniformly continuous function.

6. Show that every isometry of a compact metric space into itself is onto.

7. Let $\mathbf{T} = \{z \in \mathbf{C} : |z| = 1\}$ and $f : [0,1] \to \mathbf{C}$ be continuous with $f(0) = 0, f(1) = 2$. Show that there exists at least one t_0 in $[0,1]$ such that $f(t_0)$ is in $\mathbf{T}$.

8. Let f be a continuous function on $[0,1]$. Evaluate
$$\lim_{n\to\infty} \int_0^1 x^n f(x) dx.$$

9. Find the most general curve whose normal at each point passes though $(0,0)$. Find the particular curve through $(2,3)$.

10. Suppose f is a continuous function on $\mathbf{R}$ which is periodic with period 1, that is, $f(x+1) = f(x)$ for all x. Show that
    (i) the function f is bounded above and below,
    (ii) it achieves both its maximum and minimum and
    (iii) it is uniformly continuous.

11. Let $A = (a_{ij})$ be an $n \times n$ matrix such that $a_{ij} = 0$ whenever $i \geq j$. Prove that A^n is the zero matrix.

12. Determine the integers n for which $\mathbf{Z}_n$, the set of integers modulo n, contains elements x, y so that $x + y = 2,\ 2x - 3y = 3$.

13. Let a_1, b_1 be arbitrary positive real numbers. Define
$$a_{n+1} = \frac{a_n + b_n}{2}, b_{n+1} = \sqrt{a_n b_n}$$
for all $n \geq 1$. Show that a_n and b_n converge to a common limit.

14. Show that the only field automorphism of $\mathbf{Q}$ is the identity. Using this prove that the only field automorphism of $\mathbf{R}$ is the identity.

15. Consider a circle which is tangent to the y-axis at 0. Show that the slope at any point (x, y) satisfies $\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$.

16. Consider an $n \times n$ matrix $A = (a_{ij})$ with $a_{12} = 1, a_{ij} = 0\ \forall\ (i,j) \neq (1,2)$. Prove that there is no invertible matrix P such that PAP^{-1} is a diagonal matrix.

17. Let G be a nonabelian group of order 39. How many subgroups of order 3 does it have?

18. Let $n \in \mathbf{N}$, let p be a prime number and let $\mathbf{Z}_{p^n}$ denote the ring of integers modulo p^n under addition and multiplication modulo p^n. Let $f(x)$ and $g(x)$ be polynomials with coefficients from the ring $\mathbf{Z}_{p^n}$ such that $f(x) \cdot g(x) = 0$. Prove that $a_i b_j = 0 \ \forall \ i, j$ where a_i and b_j are the coefficients of f and g respectively.

19. Show that the fields $\mathbf{Q}(\sqrt{2})$ and $\mathbf{Q}(\sqrt{3})$ are isomorphic as $\mathbf{Q}$-vector spaces but not as fields.

20. Suppose $a_n \geq 0$ and $\sum a_n$ is convergent. Show that $\sum 1/(n^2 a_n)$ is divergent.