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Reg. No. : ....

Name : ....

M.Sc. Previous Degree Examination, August 2009
(I.D.E.)
Branch : MATHEMATICS
MM 1101 – Linear Algebra
(Prior to 2006 admission)

Time : 3 Hours Max. Marks : 85

***Instructions**: Answer either **A** or **B** of **each** question.*

I. A) a) Let $A = \begin{bmatrix} -1 & i \\ -i & 3 \\ 1 & 2 \end{bmatrix}$. Prove that the system AX = 0 has only the trivial solution. **5**

b) Suppose A, B, C are matrices such that the products BC, A (BC), AB and (AB) C are defined. Prove that A (BC) = (AB) C. **6**

c) Discover whether $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ is invertible. If invertible find A^{-1}. **6**

B) a) Prove that the set of all numbers of the form $x + y\sqrt{2}$, where x and y are rationals is a subfield of the field of complex numbers. **5**

b) Define raw equivalence of matrices. Suppose A and B are row equivalent $m \times n$ matrices over a field F. Prove that the homogeneous system of linear equations AX = 0 and BX = 0 have exactly the same solutions. **6**

c) Define raw reduced echelon matrices. Give an example. Prove that every $m \times n$ matrix is raw-equivalent to a raw reduced echelon matrix. **6**

P.T.O.

II. A) a) Prove that the only subspaces of $\mathbb{R}$ are $\mathbb{R}$ and the zero space. **4**

b) Suppose P is an $n \times n$ inverlible matrix over F. Let V be an n-dimensional vecter space over F with an ordered basis $\mathcal{B}$. Prove that there is a unique ordered $\mathcal{B}'$ of V such that $[\alpha]_{\mathcal{B}} = P[\alpha]_{\mathcal{B}'}$ and $[\alpha]_{\mathcal{B}'} = P^{-1}[\alpha]_{\mathcal{B}}$ for every vecter α in V. **7**

c) In $\mathbb{C}^3$ Let $\alpha_1 = (1, 0, -i), \alpha_2 = (1+i, 1-i, 1)$ and $\alpha_3 = (i, i, i)$. Prove that $\{\alpha_1, \alpha_2, \alpha_3\}$ is a basis for $\mathbb{C}^3$. What are the coordinates of the vector (a, b, c) in this basis ? **6**

B) a) Suppose V is a finite dimensional vecter space over a field F. Prove that every non-empty linearly independent set of vectors in V can be extended to a basis for V. **6**

b) Let V be an n-dimensional vecter space over a field F with ordered bases $\mathcal{B}$ and $\mathcal{B}'$. Prove that there is a unique, invertible by $n \times n$ matrix P with entries in F such that $[\alpha]_{\mathcal{B}} = P[\alpha]_{\mathcal{B}'}$ and $[\alpha]_{\mathcal{B}'} = P^{-1}[\alpha]_{\mathcal{B}}$ for every vector α in V. **7**

c) Let W be the subspace of $\mathbb{C}^3$ spanned by (1, 0, i) and (1 + i, 1, –1). Prove that the vector (1, i, 1 + i) lie in W. **4**

III. A) a) State and prove the rank-nullity theorem. **6**

b) Let T be a line an transformation from V into W. Prove that T is non-singular if and only if T carries each linearly independent subset of V onto a linearly independent subset of W. **6**

c) Suppose W_1 and W_2 are subspaces of a finite dimensional vector space V. Prove that $W_1 = W_2$ if and only if $W_1^0 = W_2^0$. **5**

B) a) Define the rank, nullity of a linear transformation consider the mapping $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined as $T(x_1, x_2, x_3) = (x_1 - x_2, 0, x_3)$. Prove that T is a linear transformation. Find its rank and nullity. **5**

b) Let $T : \mathbb{C}^3 \to \mathbb{C}^3$ be defined as $Te_1 = (1, 0, i)$, $Te_2 = (0, 1, 0)$, $Te_3 = (i, 1, 0)$ where (e_1, e_2, e_3) is the standard basis for $\mathbb{C}^3$. Check whether T is invertible. If invertible, compute the inverse. **6**

c) Let V be a finite dimensional vector space over a field F. Prove that V is isomorphic to V^*. **6**

IV. A) a) Define the characteristic value, the characteristic vector and the characteristic space of a linear operater. **4**

b) Let T be a linear operator on a finite dimensional vector space V with $C_1, \ldots, C_k$ as the distinct characteristic values and $W_1, \ldots, W_k$ the corresponding characteristic spaces. Prove the following are equivalent :

i) T is diagonalisable

ii) The characteristic polynomial for T is
$f = (x - C_1)^{d_1} \ldots (x - C_k)^{dk}$. With dim $W_i = d_i$, $i = 1, \ldots, k$.

iii) dim W_1 + ... + dim W_k = dim V. **7**

c) If $V = W_1 \oplus W_2 \oplus \ldots \oplus W_k$, prove that there exists k linear operators $E_1, \ldots, E_k$ on V such that :

i) each E_i is a projection

ii) $E_i E_j = 0, i \neq j$

iii) $I = E_1 + \ldots\ldots + E_k$

iv) Range of $E_i = W_i$, $i = 1, \ldots, k$. **6**

B) a) Define the characteristic value, the characteristic vector, the characteristic space of a linear operator. **4**

b) Let T be a linear operator on an n-dimensional vector space V. Prove that the characteristic and the minimal polynomial for T have the same roots except for multiplicities. **6**

c) State and prove a characterisation theorem for a linear operator on a finite dimensional space to be diagramalizable, in terms of the characteristic values and the projections. **7**

V. A) a) Define T-admissible subspaces. State the cyclic decomposition theorem. **5**

b) Describe when an $n \times n$ matrix A said to be in (i) rational form (ii) Jordan form. Give examples for each. **6**

c) If A is a complex 5×5 matric with characteristic polynomial $(x - 2)^3 (x + 7)^2$ and minimal polynomial $(x - 2)^2 (x + 7)$, write down the Jordan form of A. **6**

B) a) Define the T-annihilator of a vector α in a vecter space. If P_α is the T-annihilator of α and degree of P_α is k, prove that the vectors $\alpha, T\alpha, ..., T^{k-1}\alpha$ form a basis for $z(\alpha; T)$. **6**

b) State and prove the Generalized Caryley-Hamilton theorem. **11**
