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Reg. No. : ....

Name : ....

M.Sc. (Final) Degree Examination, July 2009
I.D.E.
Branch : MATHEMATICS
MM 1210 : Functional Analysis – I
(2006 admn. onwards)

Time : 3 Hours Max. Marks : 85

Instructions: *1) Answer either* ***Part A*** *or* ***Part B*** *of* ***each*** *question.*
2) ***All*** *questions carry* ***equal*** *marks.*

I. A) a) Show that every finite dimensional subspace of a normed space X is closed in X. **6**

b) Prove that if Y is a closed subspace of a normed space X, the quotient space $\frac{X}{Y}$ is a normed space under the quotient norm. **6**

c) Show by an example that an infinite dimensional subspace of a normed space X may not be closed in X. **5**

B) a) Let X and Y be normed spaces and $F : X \to Y$ be a linear map. Prove that F is continuous if and only if, F sends Cauchy sequences in X to Cauchy sequences in Y. **6**

b) Suppose that X and Y are normed spaces and X is finite dimensional. Prove that every linear map from X to Y is continuous. **6**

c) If X is an infinite dimensional normed space, show that there is a linear functional on X which is not continuous. **5**

II. A) a) Let X be a normed space. For every subspace Y of X and every $g \in Y'$ show that there is a unique Hahn Banach extension of g to X if and only if X' is strictly convex. **10**

b) Let Y be a subspace of a normed space X. Prove that $x \in \overline{Y}$ if and only if $x \in X$ and f(x) = 0, whenever $f \in X'$ and f/Y = 0. **7**

P.T.O.

B) a) Show that a normed space X is a Banach space if and only if every absolutely summable series of elements in X is summable in X. **6**

b) Give an example of a normed space which is not Banach. Justify your answer. **5**

c) Prove that the dual X' of a normed space X is a Banach space. **6**

III. A) a) Let X be a normed space and E be a subset of X. Show that E is bounded in X if and only if, f(E) is bounded in K for every $f \in X'$. **6**

b) Let X and Y be normed spaces and $F : X \to Y$ be linear. Prove that F is continuous if and only if $g_o F$ is continuous for every $g \in Y'$. **6**

c) If X is a normed space and $P : X \to X$ is a projection, prove that P is a closed map if and only if the subspaces R(P) and Z(P) are closed in X. **5**

B) a) Let X and Y be Banach spaces and $F : X \to Y$ a closed linear map. Show that F is continuous. **8**

b) Let X and Y be Banach spaces and $F \in BL(X, Y)$ be bijective. Show that $F^{-1} \in BL(Y, X)$. **9**

IV. A) a) Let X be a normed space and $A \in BL(X)$. Define the spectrum, eigen spectrum and approximate eigen spectrum of A. **5**

b) Let X = l^p with norm $\| \ \|_p$ $1 \le p < \infty$. Let $A : X \to X$ be defined by

$$A\,(x(1), x(2)......) = \left(x(1), \frac{x(2)}{2}, \frac{x(3)}{3} .... \right)$$. Show that $A \in BL(X)$.

Compute $\sigma_e(A), \sigma_a(A)$ and $\sigma(A)$. **6**

c) If $A \in BL(X)$ is invertible, show that $\sigma(A^{-1}) = \left\{ k^{-1} : k \in \sigma(A) \right\}$. **6**

B) a) Let X be a normed space. If X' is separable, prove that X is separable. Is the converse true ? Justify. **6**

b) Let X and Y be normed spaces. If F' denote the transpose of $F \in BL(X, Y)$, show that $\|F'\| = \|F\| = \|F''\|$. **6**

c) If $A \in BL(X)$, show that $\sigma(A') \subset \sigma(A)$. **5**

V. A) a) Define reflexive normed space. Show that every closed subspace of a reflexive normed space is reflexive. **5**

b) Prove that a finite dimensional normed space is reflexive. **6**

c) If X is a normed space and Y is a Banach space, show that CL(X, Y) is closed in BL(X, Y). **6**

B) a) If X is a normed space and $A \in CL(X)$, prove that the eigen spectrum of A is countable. **6**

b) If X is an infinite dimensional normed space and $A \in CL(X)$, show that $0 \in \sigma_a(A)$. **6**

c) If X is an infinite dimensional normed space and $F \in CL(X)$, prove that F is not invertible. **5**

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