B.E.(M.D.U.)
First Semester Examination, 2009-10
Mathematics-1 (Math-1)

Note : Attempt five questions in all, selecting two questions from each part.

Part—(A)

Q. 1. (a) Test the convergence or divergence of the series :

$$\sum\left[\sqrt{n^4+1}-\sqrt{n^4-1}\right]$$

Ans. Let
$$u_n=\sqrt{n^4+1}-\sqrt{n^4-1}$$
$$=n^2\sqrt{1+\frac{1}{n^4}}-n^2\sqrt{1-\frac{1}{n^4}}$$
$$u_n=n^2\left[\sqrt{1+\frac{1}{n^4}}-\sqrt{1-\frac{1}{n^4}}\right]$$
$$=n^2\left[\left(1+\frac{1}{n^4}\right)^{1/2}-\left(1-\frac{1}{n^4}\right)^{1/2}\right]$$
$$=n^2\left[\left\{1+\frac{1}{2}\cdot\frac{1}{n^4}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\frac{1}{n^8}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}\frac{1}{n^{12}}+\ldots\ldots\right\},\right.$$
$$\left.-\left\{1-\frac{1}{2}\cdot\frac{1}{n^4}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\frac{1}{n^8}-\ldots\ldots\right\}\right]$$
$$=n^2\left[2\left\{\frac{1}{2n^4}+\frac{1}{16n^{12}}+\ldots\ldots\right\}\right]$$
$$=\frac{1}{n^2}+\frac{1}{8n^{10}}+\ldots\ldots$$

Let us take the auxiliary series $\Sigma v_n=\Sigma\frac{1}{n^2}$

Now
$$\lim\frac{u_n}{v_n}=\lim\frac{\frac{1}{n^2}\left[1+\frac{1}{8n^8}+\ldots\ldots\right]}{\frac{1}{n^2}}$$

$$= 1 \text{ (which is finite and non-zero)}$$

Since the auxiliary series $\Sigma v_n = \Sigma \frac{1}{n^2}$ is convergent (as $p = 2 > 1$). Hence by comparison test the given series Σu_n is also convergent.

Q. 1. (b) Discuss the convergence of the series

$$x + \frac{2^2 x^2}{2!} + \frac{3^3 x^3}{3!} + \frac{4^4 x^4}{4!} + \ldots\ldots \infty (x > 0)$$

Ans. Let
$$u_n = \frac{n^n x^n}{n!}$$

$$u_{n+1} = \frac{(n+1)^{n+1} x^{n+1}}{(n+1)!}$$

$\therefore$
$$\frac{u_n}{u_{n+1}} = \frac{1}{x} \cdot \frac{n^n}{n!} \cdot \frac{(n+1)!}{(n+1)^{n+1}}$$

$$= \frac{1}{x} \frac{n^n (n+1)}{(n+1)^{n+1}}$$

$$= \frac{1}{x} \frac{n^n}{(n+1)^n}$$

$$= \frac{1}{x} \cdot \frac{1}{\left(1 + \frac{1}{n}\right)^n}$$

$$\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \frac{1}{x} \lim_{n \to \infty} \frac{1}{\left(1 + \frac{1}{n}\right)^n}$$

$$= \frac{1}{x} \cdot \frac{1}{e}$$

$$= \frac{1}{ex}$$

$\therefore$ By ratio test, Σu_n is convergent if

$$\frac{1}{ex} > 1 \text{ i.e., } x < \frac{1}{e}$$

& Σu_n is divergent if
$$\frac{1}{ex} < 1 \text{ i.e., } x > \frac{1}{e}$$

For $\frac{1}{ex} = 1$, i.e.. for $x = \frac{1}{e}$, the ratio test fails.

Applying log test.

When $$x = \frac{1}{e}$$

$$\frac{u_n}{u_{n+1}} = e\frac{1}{\left(1+\frac{1}{n}\right)^n}$$

$$\therefore \quad n\log\frac{u_n}{u_{n+1}} = n\log\left[e\frac{1}{\left(1+\frac{1}{n}\right)^n}\right]$$

$$= n\left[\log e - \log\left(1+\frac{1}{n}\right)^n\right]$$

$$= n\left[1 - n\log\left(1+\frac{1}{n}\right)\right]$$

$$= n\left[1 - n\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3}\ldots\ldots\right)\right]$$

$$= n\left(\frac{1}{2n} - \frac{1}{3n^2} + \ldots\ldots\right)$$

$$= \frac{1}{2} - \frac{1}{3n} + \ldots\ldots$$

$$\therefore \quad \lim_{n\to\infty} n\log\left(\frac{u_n}{u_{n+1}}\right) = \frac{1}{2} < 1$$

By log test, Σu_n is divergent for $x = \frac{1}{e}$. Hence the given series is convergent if $x < \frac{1}{e}$ and is divergent if $x \geq \frac{1}{e}$.

Q. 1. (c) State, with reasons, the values of x for which the series

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots\infty \textbf{ converges}$$

Ans. Let $$\Sigma u_n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots$$

The series Σu_n is absolutely convergent if the series $\Sigma|u_n|$ is convergent. Applying ratio test.

$$\left|\frac{u_n}{u_{n+1}}\right| = \left|\frac{x^n}{n}\cdot\frac{n+1}{x^{n+1}}\right|$$

$$= \frac{n+1}{n}\,\frac{1}{|x|}$$

$$=\left(1+\frac{1}{n}\right)\frac{1}{|x|}$$

$$\lim\left|\frac{u_n}{u_{n+1}}\right| = \lim\left[\left(1+\frac{1}{n}\right)\frac{1}{|x|}\right] = \frac{1}{|x|}$$

So by ratio test, the series $\Sigma|u_n|$ is convergent if $\frac{1}{|x|} > 1$ i.e., $|x| < 1$ i.e., $-1 < x < 1$

$\therefore$ The given series is absolutely convergent and hence also convergent if $-1 < x < 1$ if $|x| < 1$

When $x = 1$, the given series is

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\ldots.$$

Which converges by Leibnitz's test but converges conditionally.

Q.2. (a) Compute to four decimal places, the value of cos 32°, by use of Taylor's series.

Ans. To find cos 32° $\quad \cos 32^\circ = \cos\frac{32\pi}{180}$

$$=\cos\frac{8\pi}{45}$$

Let us take cos 32° in the neighbours house of cos 30°

$$\cos\left(\frac{8\pi}{45}+\frac{\pi}{6}-\frac{\pi}{6}\right) = \cos(x+h-h)$$

$$= f(x+h-h)$$

$$= f(a+h)$$

$$h = \left(\frac{8\pi}{45}-\frac{\pi}{6}\right), \quad a = \frac{\pi}{6}$$

$$h = \frac{48\pi-45\pi}{270} = \frac{\pi}{90}$$

$$a = \frac{\pi}{6}$$

$$f(x) = \cos x \qquad f\left(\frac{\pi}{6}\right) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

$$f'(x) = -\sin x, \qquad f'\left(\frac{\pi}{6}\right) = -\sin\frac{\pi}{6} = -\frac{1}{2}$$

$$f''(x) = -\cos x, \qquad f''\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$

$$f'''(x) = \sin x, \qquad f'''\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

$$f^{iv}(x) = \cos x, \qquad f^{iv}\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \text{ and so on}$$

Now using Taylor's series $\quad f(x) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots\ldots$

$$\cos\left(\frac{8\pi}{45}\right) = \frac{\sqrt{3}}{2} + \frac{\Pi}{90}\left(-\frac{1}{2}\right) + \frac{\Pi^2}{90^2} \times \frac{1}{2!}\left(\frac{-\sqrt{3}}{2}\right) + \left(\frac{\pi}{90}\right)^3 \frac{1}{3!}\left(\frac{1}{2}\right) + \ldots\ldots$$

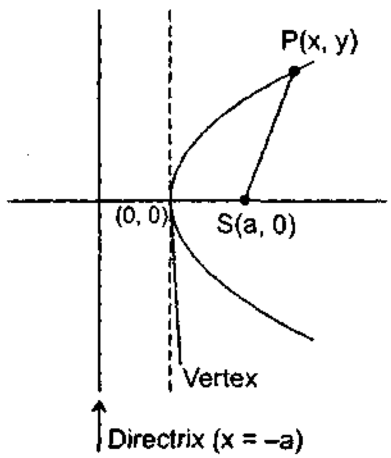
$$\cos 32^\circ = \frac{\sqrt{3}}{2} - \frac{1}{2}(0.03492) - \frac{\sqrt{3}}{4}(0.00121945) + \frac{1}{12}(0.0000425832) + \ldots$$

$$= 0.8660254 - 0.01746 - 0.00052802 + 0.0000035486$$

$$\cos 32^\circ = 0.84804 \quad \textbf{Ans.}$$

Q. 2. (b) If ρ be the radius of curvature at any point P on the parabola $y^2 = 4ax$ and S be its focus, then show that ρ^2 varies as $(SP)^3$.

Ans.



$$SP = \sqrt{(x-a)^2 + (y-0)^2}$$

$$(SP)^2 = (x-a)^2 + y^2$$

$$= x^2 + a^2 - 2ax + 4ax \qquad [\because y^2 = 4ax]$$

$$(SP)^2 = (x+a)^2$$

$$SP = x + a$$

$$(SP)^3 = (x+a)^3 \qquad \ldots\text{(i)}$$

Now, given $y^2 = 4ax$

Differentiating both sides w.r. to x,

$$2y\frac{dy}{dx} = 4a$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2a}{y} \qquad \text{..... (ii)}$$

$$1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{4a^2}{y^2} = 1+\frac{4a^2}{4ax}$$

$$1+\left(\frac{dy}{dx}\right)^2 = \left(\frac{x+a}{x}\right)$$

Again differentiating equation (ii) w.r. to x,

$$\frac{d^2y}{dx^2} = -\frac{2a}{y^2}\frac{dy}{dx} = -\frac{2a}{4ax}.\frac{2a}{y}$$

$$\frac{d^2y}{dx^2} = -\frac{a}{xy}$$

Radius of curvature (ρ) is given by

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$\rho = \frac{\left(\frac{x+a}{x}\right)^{3/2}}{\left(-\frac{a}{xy}\right)}$$

$$\rho^2 = \left(\frac{x+a}{x}\right)^3 \times \frac{x^2y^2}{a^2}$$

Or

$$\rho^2 = \left(\frac{x+a}{x}\right)^3 \times \frac{x^2.4ax}{a^2}$$

$$\rho^2 = \frac{4}{a}(x+a)^3$$

$$\rho^2 = \frac{4}{a}(SP)^3 \quad \text{[from equation (i)]}$$

$\rho^2 \propto (SP)^3$ **Hence proved**

Q. 2. (c) Show that the asymptotes of the curve $x^2y^2 = a^2(x^2+y^2)$ from a square of side $2a$.

Ans. Given curve is $\quad x^2y^2 = a^2(x^2+y^2)$

Since all powers of x and y are even asymptotes arr parallel to x and y-axis.

Asymptotes Parallel to x-axis : Equating coefficient of highest power of x to zero i.e.,

$$y^2 - a^2 = 0$$

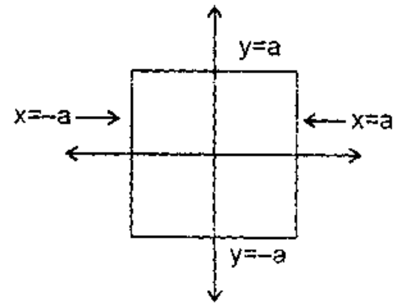
$y = \pm\ a$ are the asymptotes parallel to x-axis

Asymptotes Parallel to y-axis : By equating the coefficient of highest power of y to zero

$$x^2 - a^2 = 0$$

$x = \pm a$, are the asymptotes parallel to y-axis.

Since equation of the curve is of degree 4, it cannot have more than four asymptotes. Thus, four asymptotes are $x = \pm a$, $y = \pm a$, which form a square



Q. 3. (a) If $u = x^2 \tan^{-1}(y/x) - y^2 \tan^{-1}(x/y)$, evaluate

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2}$$

Ans. Given : $$u = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$$

Let $$u = v - w$$

Where, $v = x^2 \tan^{-1}\left(\frac{y}{x}\right)$, $\quad w = y^2 \tan^{-1}\left(\frac{x}{y}\right)$

v is a homogeneous function of degree $n = 2$ in x, y.

Thus, $$x^2 \frac{\partial^2 v}{\partial x^2} + 2xy \frac{\partial^2 v}{\partial x \partial y} + y^2 \frac{\partial^2 v}{\partial y^2} = n(n-1)v$$

$$= 2(2-1)v$$

$$= 2v \qquad \text{..... (i)}$$

Also, w is also a homogeneous function of degree 2 in x, y.

$$x^2 \frac{\partial^2 w}{\partial x^2} + 2xy \frac{\partial^2 w}{\partial x \partial y} + y^2 \frac{\partial^2 w}{\partial y^2} = 2w \qquad \text{..... (ii)}$$

Subtracting equation (ii) from equation (i)

$$x^2 \frac{\partial^2}{\partial x^2}(v-w) + 2xy \frac{\partial^2}{\partial x \partial y}(v-w) + y^2 \frac{\partial^2}{\partial y^2}(v-w)$$

$$= 2(v-w)$$

$$\Rightarrow \quad x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 2u$$

Thus, we have

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 2\left\{x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)\right\}$$ **Ans.**

Q. 3. (b) If $f(x, y) = \tan^{-1}(xy)$, compute $f(0.9, -1.2)$ approximately.

Ans. $$f(x, y) = \tan^{-1}(xy)$$

Let us expand $f(x, y)$ near the point $(1, -1)$

$$f(0.9, -1.2) = f(1-0.1, -1, -0.2)$$

$$= f(1, -1) + \left[(-0.1)\frac{\partial f}{\partial x} + (-0.2)\frac{\partial f}{\partial y}\right]$$

$$+ \frac{1}{2!}\left[(-0.1)^2 \frac{\partial^2 f}{\partial x^2} + 2(-0.1)(-0.2)\frac{\partial^2 f}{\partial x \partial y} + (-0.2)^2 \frac{\partial^2 f}{\partial y^2}\right] + \ldots \quad \ldots\text{(i)}$$

Now,

$f(x, y) = \tan^{-1}(x, y)$	$f(1, -1) = -\frac{\pi}{4}$
$\frac{\partial f}{\partial x} = \frac{y}{1 + x^2 y^2}$	$\frac{\partial f}{\partial x}(1, -1) = \frac{1}{2}$
$\frac{\partial f}{\partial y} = \frac{x}{1 + x^2 y^2}$	$\frac{\partial f}{\partial y}(1, -1) = \frac{1}{2}$
$\frac{\partial^2 f}{\partial x^2} = \frac{-2xy}{(1 + x^2 y^2)^2}$	$\frac{\partial^2 f}{\partial x^2}(1, -1) = \frac{1}{2}$
$\frac{\partial^2 f}{\partial x \partial y} = \frac{1 + x^2 y^2 - 2x^2 y^2}{(1 + x^2 y^2)^2}$	$\frac{\partial^2 f}{\partial x \partial y}(1, -1) = 0$
$\frac{\partial^2 f}{\partial y^2} = \frac{-x(2x^2 y)}{(1 + x^2 y^2)^2}$	$\frac{\partial^2 f}{\partial x^2}(1, -1) = \frac{1}{2}$

Putting all these values in equation (i),

$$f(0.9, -1.2) = -\frac{\pi}{4} + (-(0.1)\left(-\frac{1}{2}\right) + (-0.2)\left(\frac{1}{2}\right)$$

$$+ \frac{1}{2}\left[(-0.1)^2\left(\frac{1}{2}\right) + 2(-0.1)(-0.2)(0) + (-0.2)^2\left(\frac{1}{2}\right)\right] + \ldots\ldots$$

$$= -\frac{\pi}{4} + 0.05 - 0.1 + \frac{1}{2}(0.005 + 0.02)$$

$$= -\frac{\pi}{4} + 0.05 - 0.1 + 0.0125$$

$$f(0.9, -1.2) = -0.823$$ **Ans.**

Q. 4. (a) Find the maximum and minimum distances from the origin to the curve $5x^2 + 6xy + 5y^2 - 8 = 0$

Ans. Let $p(x, y)$ be any point on the curve. Distance of the point $A(0, 0)$ from $P(x, y)$ is

$$\sqrt{(x-0)^2 + (y-0)^2}$$

If the distance is maximum or minimum, so will be the square of the distance.

Let $\qquad f(x, y) = x^2 + y^2 \qquad$ ..... **(i)**

Subject to the condition $\qquad \phi(x, y) \equiv 5x^2 + 6xy + 5y^2 - 8 = 0 \qquad$ ..... **(ii)**

Consider Lagrange's function

$$F(x, y) = f(x, y) + \lambda\phi(x, y)$$

$$F(x, y) = x^2 + y^2 + \lambda(5x^2 + 6xy + 5y^2 - 8)$$

For stationary values $\qquad dF = 0$

$$\{2x + \lambda(10x + 6y)\}dx + \{2y + \lambda(6x + 10y)\}dy = 0$$

$$2x + \lambda(10x + 6y) = 0$$

$$2y + \lambda(6x + 10y) = 0$$

Multiplying equation (iii) by x and equation (iv) by y and on adding

$$2(x^2 + y^2) + \lambda(10x^2 + 6xy + 6xy + 10y^2) = 0$$

$$2(x^2 + y^2) + 2\lambda(5x^2 + 6xy + 5y^2) = 0$$

$\Rightarrow$
$$f(x, y) + \lambda(8) = 0$$

$$\lambda = -\frac{f}{8} \quad \text{(using equations (i) and (ii))}$$

From equations (iii) and (iv)

$$2x - \frac{f}{8}(10x + 6y) = 0$$

$$2y - \frac{f}{8}(6x + 10y) = 0$$

$\Rightarrow$
$$4x - f(5x + 3y) = 0$$

$$4y - f(3x + 5y) = 0$$

Or
$$(4 - 5f)x - 3fy = 0 \qquad \text{.... (v)}$$

$$-3fx + (4 - 5l)y = 0 \qquad \text{...... (vi)}$$

Solving equations (v) and (vi)

$$(3f)^2 = (4 - 5f)^2$$

$$9f^2 = 16 + 25f^2 - 40f$$

$$16f^2 - 40f + 16 = 0$$

$$2f^2 - 5f + 2 = 0$$

$$2f^2 - 4f - f + 2 = 0$$

$$2f(f-2) - 1(f-2) = 0$$

$$(2f-1)(f-2) = 0$$

$$f = \frac{1}{2}, 2$$

Thus, the maximum distance $= \sqrt{2}$

$= 1.414$

& minimum distance $= \sqrt{\frac{1}{2}}$

$= 0.7072$ **Ans.**

Q. 4. (b) Evaluate $\int_0^{\alpha} \frac{\log(1+\alpha x)}{1+x^2} dx$

Ans. $\int_0^{\alpha} \frac{\log(1+\alpha x)}{1+x^2} dx$

Let us take $\alpha = 1$

$$I = \int_0^1 \frac{\log(1+x)}{1+x^2} dx$$

Now putting $x = \tan\theta$

$$dx = \sec^2\theta d\theta$$

When $x = 0,\ \theta = 0$

When $x = 1,\ \theta = \frac{\pi}{4}$

$$I = \int_0^{\pi/4} \frac{\log(1+\tan\theta)}{(1+\tan^2\theta)} \sec^2\theta d\theta$$

$$I = \int_0^{\pi/4} \log(1+\tan\theta) d\theta$$

Applying property equation (iv) of definite integral

$$I = \int_0^{\pi/4} \log\left\{1+\tan\left(\frac{\pi}{4}-\theta\right)\right\} d\theta$$

$$= \int_0^{\pi/4} \log\left\{1+\left(\frac{\tan\frac{\pi}{4}-\tan\theta}{1+\tan\frac{\pi}{4}\tan\theta}\right)\right\} d\theta$$

$$= \int_0^{\pi/4} \log\left(1+\frac{1-\tan\theta}{1+\tan\theta}\right)d\theta$$

$$= \int_0^{\pi/4} \log\left(\frac{2}{1+\tan\theta}\right)d\theta$$

$$I = \int_0^{\pi/4} \log 2\, d\theta - \int_0^{\pi/4} \log(1+\tan\theta)d\theta$$

$$I = \int_0^{\pi/4} \log 2 d\theta - I$$

$$2I = \int_0^{\pi/4} \log 2\, d\theta$$

$$= \log 2 \int_0^{\pi/4} 1\, d\theta$$

$$2I = \log 2[\theta]_0^{\pi/4}$$

$$2I = \frac{\pi}{4}\log 2$$

$$I = \frac{\pi}{8}\log 2 \quad \textbf{Ans.}$$

Part—(B)

Q. 5. (a) Find the volume of solid formed by revolving a loop of the lemniscate $r^2 = a^2 \cos 2\theta$ about the initial line.

Ans. For the upper half of the loop θ varies from 0 to $\pi/4$. The curve is revolving about the initial line i.e., x-axis

Required volume $= \frac{2}{3}\pi \int_0^{\pi/4} r^3 \sin\theta d\theta$

$$= \frac{2}{3}\pi \int_0^{\pi/4} \{a\sqrt{\cos 2\theta}\}^3 \sin\theta d\theta \qquad [\because r^2 = a^2 \cos 2\theta]$$

$$= \frac{2\pi a^3}{3} \int_0^{\pi/4} (2\cos^2\theta - 1)^{3/2} \sin\theta d\theta$$

Put $\sqrt{2}\cos\theta = \sec\phi$

$$-\sqrt{2}\sin\theta d\theta = \sec\phi\tan\phi d\phi$$

& when $\theta = 0$, $\phi = \pi/4$ and when $\theta = \frac{\pi}{4}$, $\phi = 0$

$$= \frac{2\pi a^3}{3}\int_{\pi/4}^{0}(\sec^2\theta - 1)^{3/2}\frac{(-\sec\phi\tan\phi)}{\sqrt{2}}d\phi$$

$$= \frac{\sqrt{2}\pi a^3}{3}\int_0^{\pi/4}\tan^4\phi\sec\phi d\phi$$

$$= \frac{\sqrt{2}\pi a^3}{3}\int_0^{\pi/4}(\sec^2\phi - 1)^2\sec\phi d\phi$$

$$= \frac{\sqrt{2}\pi a^3}{3}\int_0^{\pi/4}(\sec^5\phi - 2\sec^3\phi + \sec\phi)d\phi \qquad \text{..... (i)}$$

Now using the reduction formula

$$\int\sec^n\phi d\phi = \frac{\sec^{n-2}\phi\tan\phi}{n-1} + \frac{(n-2)}{(n-1)}\int\sec^{n-2}\phi d\phi$$

Thus, $$\int_0^{\pi/4}\sec^5\phi d\phi = \left[\frac{\sec^3\phi\tan\phi}{4}\right]_0^{\pi/4} + \frac{3}{4}\int_0^{\pi/4}\sec^3\phi d\phi$$

$$= \frac{\sqrt{2}}{2} + \frac{3}{4}\left[\left\{\frac{\sec\phi\tan\phi}{2}\right\}_0^{\pi/4} + \frac{1}{2}\int_0^{\pi/4}\sec\phi d\phi\right]$$

$$= \frac{\sqrt{2}}{2} + \frac{3}{4}\left[\frac{\sqrt{2}}{2} + \frac{1}{2}\{\log(\sec\phi + \tan\phi)\}_0^{\pi/4}\right]$$

$$= \frac{\sqrt{2}}{2} + \frac{3\sqrt{2}}{8} + \frac{3}{8}\log(\sqrt{2}+1) = \frac{7\sqrt{2}}{8} + \frac{3}{8}\log(\sqrt{2}+1)$$

& $$\int_0^{\pi/4}\sec^3\phi d\phi = \left[\frac{\sec\phi\tan\phi}{2}\right]_0^{\pi/4} + \frac{1}{2}\int_0^{\pi/4}\sec\phi d\phi$$

$$= \frac{\sqrt{2}}{2} + \frac{1}{2}\log(\sqrt{2}+1)$$

& $$\int_0^{\pi/4}\sec\phi d\phi = \log(\sqrt{2}+1)$$

From equation (i), Required Volume

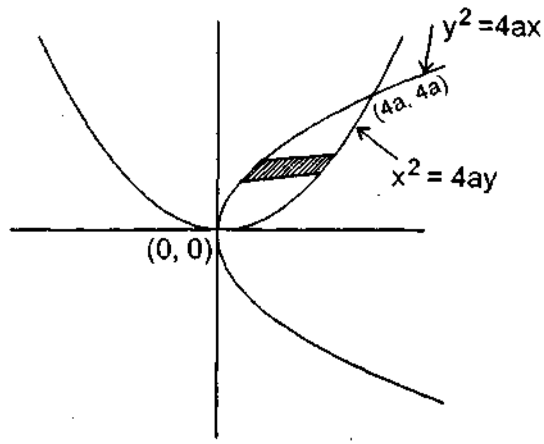
$$= \frac{\sqrt{2}\pi a^3}{3}\left[\frac{7\sqrt{2}}{8} + \frac{3}{8}\log(\sqrt{2}+1) - 2\left\{\frac{\sqrt{2}}{2} + \frac{1}{2}\log(\sqrt{2}+1)\right\} + \log(\sqrt{2}+1)\right]$$

$$= \frac{\sqrt{2}\pi a^3}{3}\left[\frac{3}{8}\log(\sqrt{2}+1) - \frac{\sqrt{2}}{8}\right]$$

$$V = \frac{\pi a^3\sqrt{2}}{24}[3\log(\sqrt{2}+1) - \sqrt{2}] \quad \textbf{Ans.}$$

Q. 5. (b) Evaluate $\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dydx$ by changing the order of integration.

Ans. Let
$$I = \int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dydx$$



By changing the order of integration

$$I = \int_0^{4a}\int_{y^2/4a}^{2\sqrt{ay}} dxdy$$

$$I = \int_0^{4a}[x]_{y^2/4a}^{2\sqrt{ay}}\, dy$$

$$= \int_0^{4a}\left[2\sqrt{ay} - \frac{y^2}{4a}\right]dy$$

$$= 2\sqrt{a}\left[\frac{2}{3}y^{3/2}\right]_0^{4a} - \frac{1}{4a}\left[\frac{y^3}{3}\right]_0^{4a}$$

$$= \frac{4\sqrt{a}}{3}(4a)^{3/2} - \frac{1}{12a}(4a)^3$$

$$= \frac{32a^2}{3} - \frac{16a^2}{3}$$

$$= \frac{16a^2}{3} \quad \textbf{Ans.}$$

Q. 6. (a) Evaluate $\iiint (x+y+z)dxdydz$ over the tetrahedron bounded by the planes $x=0$, $y=0$, $z=0$ and $x+y+z=1$.

Ans. To evaluate $\iiint (x+y+z)dxdydz$ over the tetrahedron bounded by the planes $x=0$, $y=0$, $z=0$ and $x+y+z=1$

$$\iiint (x+y+z)dxdydz$$

$$= \iiint x^{1-1}y^{1-1}z^{1-1}(x+y+z)dxdydz$$

Where $0 \le x+y+z \le 1$

Using Liouville's extension $= \frac{1!\,1!\,1!}{1+1+1!}\int_0^1 u.u^{1+1+1-1}.du$

$$= \frac{1}{3!}\int_0^1 u.u^2 du$$

$$= \frac{1}{2}\int_0^1 u^3 du$$

$$= \frac{1}{2}\left[\frac{u^4}{4}\right]_0^1$$

$$= \frac{1}{8}[u^4]_0^1$$

$$= \frac{1}{8} \quad \textbf{Ans.}$$

Q. 6. (b) Show that

$$\int_0^{\pi/2}\sqrt{\sin\theta}\,d\theta \times \int_0^{\pi/2}\frac{d\theta}{\sqrt{\sin\theta}} = \pi$$

Ans. Taking L.H.S. $\int_0^{\pi/2}\sqrt{\sin\theta}\,d\theta \times \int_0^{\pi/2}\frac{d\theta}{\sqrt{\sin\theta}}$

$$\Rightarrow \quad \int_0^{\pi/2} \sin^{1/2}\theta\cos^0\theta\,d\theta \times \int_0^{\pi/2} \sin^{-1/2}\theta\cos^0\theta\,d\theta$$

$$\left[\because \int_0^{\pi/2} \sin^p\theta\cos^q\theta\,d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{p+q+2}{2}\right)}\right]$$

$$\Rightarrow \quad \frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\,\Gamma\left(\frac{\frac{1}{2}+0+2}{2}\right)} \times \frac{\Gamma\left(\frac{\frac{-1}{2}+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\,\Gamma\left(\frac{-\frac{1}{2}+0+2}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{2\,\Gamma\left(\frac{5}{4}\right)} \times \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{2\,\Gamma\left(\frac{3}{4}\right)}$$

$$= \frac{\left(\Gamma\left(\frac{1}{2}\right)\right)^2\Gamma\left(\frac{1}{4}\right)}{4\,\Gamma\left(\frac{5}{4}\right)}$$

$$= \frac{(\sqrt{\pi})^2\,\Gamma\left(\frac{1}{4}\right)}{4\times\frac{1}{4}\,\Gamma\left(\frac{1}{4}\right)} \qquad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \quad \& \quad n! = (n-1)\,\overline{n-1}!\right]$$

$$= (\sqrt{\pi})^2$$

$$= \pi = \text{R.H.S.}$$

Q. 7. (a) Find the constants a and b so that the surface $ax^2 - byz = (a+2)x$ is orthogonal to the surface $4x^2y + z^3 = 4$ at the point (1, –1, 2).

Ans. Given the two surfaces are

$$ax^2 - byz = (a+2)x$$

& $$4x^2y + z^3 = 4$$

Let $$\phi_1 \equiv ax^2 - byz - (a+2)x \qquad \text{.... (i)}$$

$$\phi_2 = 4x^2y + z^3 - 4 \qquad \text{.... (ii)}$$

$$\nabla\phi_1 \equiv \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(ax^2 - byz - ax - 2x)$$

$$= \{2ax-(a+2)\}\hat{i}+(-bz)\hat{j}+(-by)\hat{k}$$

At $(1, -1, 2)$

$$\nabla\phi_1 = (a-2)\hat{i} - 2b\hat{j} + b\hat{k} \quad \text{.... (iii)}$$

Also

$$\nabla\phi_2 = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(4x^2y+z^3-4)$$

$$\nabla\phi_2 = 8xy\hat{i} + 4x^2\hat{j} + 3z^2\hat{k}$$

At $(1, -1, 2)$

$$\nabla\phi_2 = -8\hat{i} + 4\hat{j} + 12\hat{k} \quad \text{.... (iv)}$$

Since both surfaces are orthogonal,

$$\nabla\phi_1 . \nabla\phi_2 = 0$$

$$\{(a-2)\hat{i} - 2b\hat{j} + b\hat{k}\}.\{-8\hat{i} + 4\hat{j} + 12\hat{k}\} = 0$$

$$\Rightarrow \quad -8(a-2) - 8b + 12b = 0$$

$$-8a + 4b + 16 = 0 \quad \text{.... (v)}$$

Also since both surfaces are orthogonal at $(1, -1, 2)$ so this point will satisfy the surfaces. Thus, from first surface

$$a(1)^2 - b(-1)(2) = (a+2)(1)$$

$$a + 2b - a - 2 = 0$$

$$2b = 2$$

$$b = 1$$

Putting $b = 1$ in equation (v)

$$-8a + 4(1) + 16 = 0$$

$$-8a = -20$$

$$a = \frac{20}{8}$$

$$a = \frac{5}{2}$$

The values of a and b are $\frac{5}{2}$ and 1.

Q. 7. (b) If v_1 and v_2 be the vectors joining the fixed points (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively to a variable point (x, y, z), prove that

$$\textbf{curl}\ (v_1 \times v_2) = 2(v_1 - v_2)$$

Ans.

$$\vec{V_1} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$$

$$\vec{V_2} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$$

$$\vec{V_1} \times \vec{V_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

$$= \hat{i}(y_1 z_2 - y_2 z_1) - \hat{j}(x_1 z_2 - x_2 z_1) + \hat{k}(x_1 y_2 - x_2 y_1)$$

$$\vec{V_1} \times \vec{V_2} = \hat{i}(y_1 z_2 - y_2 z_1) + \hat{j}(x_2 z_1 - x_1 z_2) + \hat{k}(x_1 y_2 - x_2 y_1)$$

Now curl $(\vec{V_1} \times \vec{V_2}) = \nabla \times (\vec{V_1} \times \vec{V_2})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (y_1 z_2 - y_2 z_1) & (x_2 z_1 - x_1 z_2) & (x_1 y_2 - x_2 y_1) \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(x_1 y_2 - x_2 y_1) - \frac{\partial}{\partial z}(x_2 z_1 - x_1 z_2)\right] - \hat{j}\left[\frac{\partial}{\partial x}(x_1 y_2 - x_2 y_1) - \frac{\partial}{\partial z}(y_1 z_2 - y_2 z_1)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(x_2 z_1 - x_1 z_2) - \frac{\partial}{\partial y}(y_1 z_2 - y_2 z_1)\right]$$

$$= \hat{i}[(x_1 - x_2) - (x_2 - x_1)] - \hat{j}[(y_2 - y_1) - (y_1 - y_2)] + \hat{k}[(z_1 - z_2) - (z_2 - z_1)]$$

$$= \hat{i}(2x_1 - 2x_2) - \hat{j}(2y_2 - 2y_1) + \hat{k}(2z_1 - 2z_2)$$

$$= 2[\hat{i}(x_1 - x_2) + \hat{j}(y_1 - y_2) + \hat{k}(z_1 - z_2)]$$

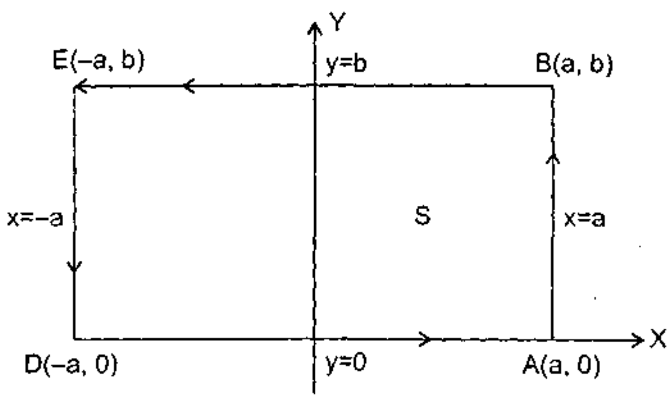
$$= 2[(x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) - (x_2\hat{i} + y_2\hat{j} + z_2\hat{k})]$$

$$= 2(V_1 - V_2)$$

= R.H.S.

Q. 8. (a) Verify Stoke's theorem for $F = (x^2 + y^2)\hat{i} - 2xy\,\hat{j}$ taken around the rectangle bounded by the lines $x = \pm a$, $y = 0$, $y = b$.

Ans. Let C denote the boundary of the rectangle ABED, then



$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C [(x^2+y^2)\hat{i} - 2xy\hat{j}]\cdot(\hat{i}dx + \hat{j}dy)$$

$$= \oint_C (x^2+y^2)dx - 2xy\,dy$$

The curve C consists of four lines AB, BE, ED and DA.

Along AB, $x = a$, $dx = 0$, y varies from 0 to b

$$\int_{AB} (x^2+y^2)dx - 2xydy$$

$$= \int_0^b -2ay\,dy = -2a\left[\frac{y^2}{2}\right]_0^b = -ab^2 \qquad \text{.... (i)}$$

Along BE, $y = b$, $dy = 0$, x varies from a to $-a$

$$\int_{BE} (x^2+y^2)dx - 2xydy = \int_{+a}^{-a} (x^2+b^2)dx$$

$$= \left[\frac{x^3}{3} + b^2x\right]_{+a}^{-a}$$

$$= \frac{-a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2$$

$$= \frac{-2a^3}{3} - 2ab^2 \qquad \text{.... (ii)}$$

Along ED, $x = -a$, $dx = 0$, y varies from b to 0

$$\int_{ED} (x^2+y^2)dx - 2xydy = \int_b^0 2aydy = -ab^2 \qquad \text{.... (iii)}$$

Along DA, $y = 0$, $dy = 0$, x varies from $-a$ to a

$$\int_{DA} (x^2+y^2)dx - 2xydy = \int_{-a}^{a} x^2dx = \frac{2a^3}{3} \qquad \text{..... (iv)}$$

On adding equations (i), (ii), (iii) and (iv)

$$\oint_C \vec{F}\cdot d\vec{r} = -ab^2 - \frac{2a^3}{3} - 2ab^2 - ab^2 + \frac{2a^3}{3}$$

$$= -4ab^2 \qquad \text{.... (v)}$$

Now $\qquad$ curl $\vec{F} = \nabla \times \vec{F}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2 + y^2) & -2xy & 0 \end{vmatrix}$$

$$= (-2y - 2y)\hat{k} = -4y\hat{k}$$

For the surface S, $\hat{n} = \hat{k}$

$\Rightarrow$ $\text{curl } \vec{F}.\hat{n} = -4y\hat{k}.\hat{k} = -4y$

Now $\iint_S \text{curl } \vec{F}.\hat{n}dS = \int_0^b \int_{-a}^{a} -4y\,dxdy$

$$= \int_0^b -4y[x]_{-a}^{a}\,dy$$

$$= -8a\int_0^b y\,dy$$

$$= -4a[y^2]_0^b$$

$$= -4ab^2 \quad \text{.... (vi)}$$

From equations (v) and (vi)

$$\oint_C \vec{F}.\vec{dr} = \iint_S \text{curl } \vec{F}.\hat{n}dS$$

Hence verifies Stoke's theorem.

Q. 8. (b) Using divergence theorem, evaluate $\int_S R.N ds$ where S is the surface of the sphere $x^2 + y^2 + z^2 = 9$.

Ans. To evaluate $\iint_S \vec{r}.\hat{n}dS$

Using divergence theorem,

$$\iint_S \vec{r}.\hat{n}dS = \iiint_V div\,\vec{r}\,dV \quad \text{... (i)}$$

We know that $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$div\,\vec{r} = \nabla.\vec{r}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$$

$$= 1 + 1 + 1$$

$$= 3$$

From equation (i) $\iint_S r.\hat{n}dS = \iiint_V 3dV$

$$= 3\iiint_V dV$$

= 3 Volume of the given sphere with radius 3

$$= 3.\frac{4}{3}\pi r^3$$

$$= 4\pi r^3$$

$$= 4\pi(3)^3$$

$\iint_S r.\hat{n}dS = 108\pi$ **Ans.**