Midterm Exam Key Math 428: Operations Research

Name: ______________________ **Score:** __________

Instructions: Write out your solutions on the paper provided. Show your work and give reasons for your answers. NO notes, calculators, laptops, cell phones or other electronic equipment allowed. Each problem is worth 20 points *for a total of* 100 *points.* Work problem 1 and any four of the remaining five problems. *Clearly indicate which four you want graded.* Give exact answers (like $1/3$ or $\sqrt{2}$) where calculations are expected.

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1. Answer True or False for each of the following statements about LP problems and justify your answer.

(a) Although any CPF (corner point feasible) solution can be chosen to be the initial CPF solution, the simplex method always chooses the origin.

(b) An LP problem cannot handle variables that could be negative.

(c) If there is no leaving variable in a column selected for an entering basic variable, then the objective function is unbounded.

(d) If a maximization problem in standard form and its dual have feasible solutions, then both problems have optimal solutions.

(e) If the final tableau of the simplex method applied to LP has a nonbasic variable with a coefficient of 0 in row 0, then the problem has multiple solutions.

Solution.

(a) **False.** The origin may not be a CPF solution, in which case Big M or two-phase methods will select the origin as an initial BF solution, but this does not correspond to any CPF solution.

(b) **False.** One writes the possibly negative variable as $x_j = x_j^+ - x_j^-$, where x_j^+, x_j^- are nonnegative variables. (Or: if the variable is bounded below, do change of variables so resulting variable is nonnegative.)

(c) **True.** This means that improvements to this variable will increase the objective function at a nonzero constant rate and no upper bound is imposed on the variable.

(d) **True.** We know from weak duality that in this case if $\mathbf{x}, \mathbf{y}$ are BF solutions to the primal and dual, respectively, then $\mathbf{b}^T\mathbf{x} \le \mathbf{c}^T\mathbf{y}$. It follows that there is an upper bound on $\mathbf{b}^T\mathbf{x}$ and lower bound on $\mathbf{c}^T\mathbf{y}$, over all BF $\mathbf{x}, \mathbf{y}$, hence each problem has a max, min, respectively.

(e) **False.** This is true, *provided* the simplex method finds an optimal solution, but in the case that it terminates with no solution or an unbounded Z, we may have a nonbasic variable with coefficient of 0.

2. Consider the following LP problem:

$$\text{Minimize } 2x_1 + 3x_2$$

subject to the constraints $x_i \ge 0$, $i = 1, 2$, and

$$\begin{aligned} x_1 + 2x_2 &\ge 4 \\ x_1 + x_2 &\ge 3. \end{aligned}$$

(a) Convert this problem to a maximization problem.
(b) Solve the problem of (a) by the graphical method.

Solution.

(a) We could do this by taking the negative of the objective function:

$$\text{Maximize } -2x_1 - 3x_2$$

subject to the constraints $x_i \geq 0$, $i = 1,2$, and

$$\begin{aligned} x_1 + 2x_2 &\geq 4 \\ x_1 + x_2 &\geq 3. \end{aligned}$$

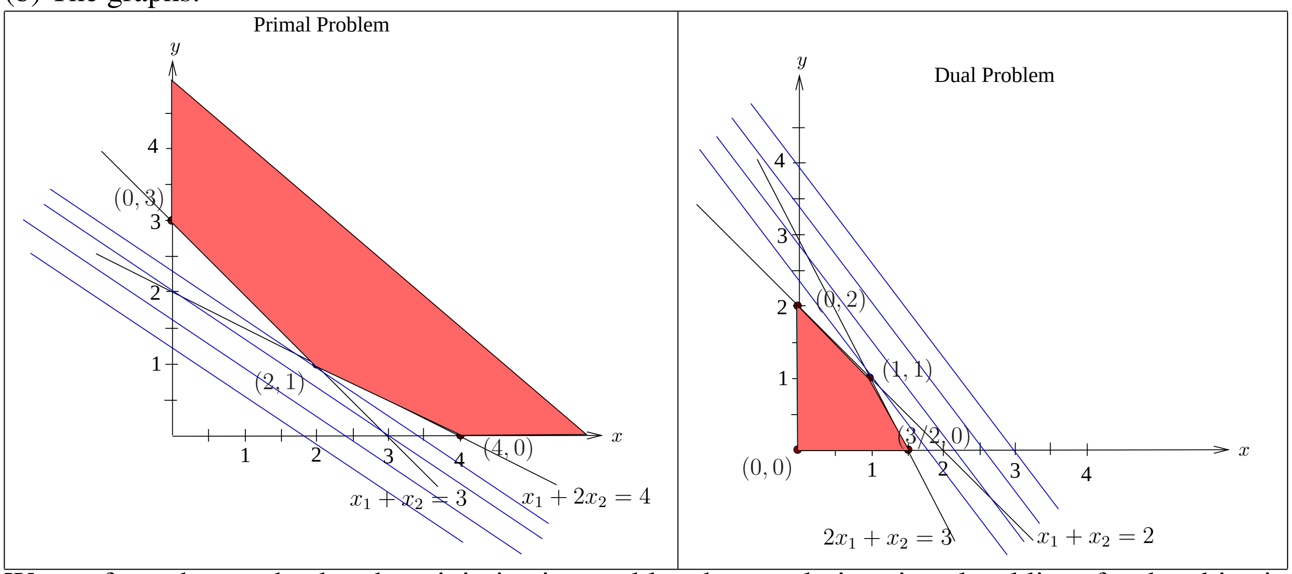
or by looking at the dual:

$$\text{Maximize } 4x_1 + 3x_2$$

subject to the constraints $x_i \geq 0$, $i = 1,2$, and

$$\begin{aligned} x_1 + x_2 &\leq 2 \\ 2x_1 + x_2 &\leq 3. \end{aligned}$$

(b) The graphs:



We see from the graphs that the minimization problem has a solution since level lines for the objective function move the the left as Z decreases and we need only test the CPF solutions:

$$\begin{aligned} Z(0,3) &= 9 \\ Z(2,1) &= 7 \\ Z(4,0) &= 8. \end{aligned}$$

Thus the minimum value is $Z = 7$ which occurs at CPF $(2,1)$.

We could also see the same with the dual, namely that the maximum value of the dual objective is $W = 7$ at the dual point $(1,1)$, but this does not identify the primal optimal point.

3. Use the simplex method to solve the problem

$$\text{Max } Z = x_1 + 2x_3$$

Sbj.

$$\begin{aligned} x_1 + 2x_2 + x_3 &\leq 2 \\ x_3 &\leq 1 \\ x_1 \geq 0, x_2 \geq 0, x_3 &\geq 0. \end{aligned}$$

At each stage say which variables are entering and exiting basic variables.

Solution. This is a straightforward optimization problem with an obvious initial BF solution at the origin. Introduce slack variables x_4, x_5 for the two inequalities and we have initial tableau with columns for x_1, x_2, x_3, x_4, x_5, rhs:

$$\begin{bmatrix} -1 & 0 & -2 & 0 & 0 & 0 \\ 1 & 2 & 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

The column of x_3 has largest negative, so x_3 will enter and $x_4$will exit by the minimum ratio test with suitable row ops using third row for pivot:

$$\begin{array}{c} \longrightarrow \\ E_{23}(-1) \\ E_{13}(2) \end{array} \begin{bmatrix} -1 & 0 & 0 & 0 & 2 & 2 \\ 1 & 2 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

At the next step $x_1$has the largest negative, so x_1 will enter and x_4 will exit:

$$\begin{array}{c} \longrightarrow \\ E_{12}(1) \end{array} \begin{bmatrix} 0 & 2 & 0 & 1 & 1 & 3 \\ 1 & 2 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

With this step we are done, since no negative entries reside in the zeroth row, hence the optimal solution is $Z = 3$, which occurs at the BF solution $(1,0,1,0,0)$, i.e., CPF solution $(1,0,1)$.

4. Consider the problem

$$\begin{aligned} &\text{Max } Z = x_1 + 3x_2 \\ &\text{Sbj.} \\ &3x_1 + x_2 \leq 3 \\ &x_1 - x_2 \geq 2 \\ &x_1 \geq 0, x_2 \geq 0. \end{aligned}$$

(a) Use the Big M method or phase 1 of the two-phase method to show that there is no feasible solution to this problem.

(b) Exhibit iteration 1 of simplex tableau of the method not used in (a) and identify the initial BF solution.

Solution. (a) If we use Big M, the variables used are decision variables x_1, x_2, slack variable x_3, surplus variable x_4 and artificial variable $\bar{x}_5$ with rhs to yield the tableau

$$\begin{bmatrix} -1 & -3 & 0 & 0 & M & 0 \\ 3 & 1 & 1 & 0 & 0 & 3 \\ 1 & -1 & 0 & -1 & 1 & 2 \end{bmatrix}.$$

To reach the initial simplex tableau, we add $-M$ times the third row to the first, yielding

$$\begin{bmatrix} -1-M & -3+M & 0 & M & 0 & -2M \\ 3 & 1 & 1 & 0 & 0 & 3 \\ 1 & -1 & 0 & -1 & 1 & 2 \end{bmatrix}.$$

Thus the initial BF solution is $(0,0,3,0,2)$.

To finish the problem, we would use x_1 as entering, and x_3 as exiting by minimum ratio test to obtain

$$\begin{matrix} \longrightarrow \\ E_2\left(\frac{1}{3}\right) \\ E_{32}(-1) \\ E_{12}(1+M) \end{matrix} \begin{bmatrix} 0 & -\frac{8}{3}+\frac{4}{3}M & \frac{1}{3}(1+M) & M & 0 & M \\ 1 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 1 \\ 0 & -\frac{4}{3} & -\frac{1}{3} & 0-1 & 1 & 1 \end{bmatrix}.$$

This is the final tableau, which shows $Z = M$ at a BF solution $(1,0,0,0,1)$, which has the artificial variable $\bar{x}_5 = 1$, and hence does not correspond to a BF solution to the original problem. Hence, it has no BF solutions.

(b) If we use the two-phase method, the variables used for phase 1 are decision variables x_1, x_2, slack variable x_3, surplus variable x_4 and artificial variable $\bar{x}_5$ with rhs, and we attempt to minimize $Z = \bar{x}_5$, i.e., maximize $-\bar{x}_5$ to yield the tableau

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 3 \\ 1 & -1 & 0 & -1 & 1 & 2 \end{bmatrix}.$$

To reach the initial simplex tableau, we add -1 times the third row to the first, yielding

$$\begin{bmatrix} -1 & 1 & 0 & 1 & 0 & -2 \\ 3 & 1 & 1 & 0 & 0 & 3 \\ 1 & -1 & 0 & -1 & 1 & 2 \end{bmatrix}.$$

Thus the initial BF solution is $(0,0,3,0,2)$.

To finish the problem, we would use x_1 as entering, and x_3 as exiting by minimum ratio test to obtain

$$\begin{matrix} \longrightarrow \\ E_2\left(\frac{1}{3}\right) \\ E_{32}(-1) \\ E_{12}(1) \end{matrix} \begin{bmatrix} 1 & \frac{4}{3} & \frac{1}{3} & 1 & 0 & -1 \\ 1 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 1 \\ 0 & -\frac{4}{3} & -\frac{1}{3} & 0-1 & 1 & 1 \end{bmatrix}.$$

This is the final tableau, which shows $Z = 1$ at a BF solution $(1,0,0,0,1)$, which has the artificial variable $\bar{x}_5 = 1$, and hence does not correspond to a BF solution to the original problem. Hence, it has no BF solutions.

5. Consider the problem

$$\begin{aligned} &\text{Max } Z = 5x_1 + 7x_2 \\ &\text{Sbj.} \\ &x_1 + 2x_2 \leq 4 \\ &x_1 + x_2 \leq 3 \\ &x_1 \geq 0, x_2 \geq 0. \end{aligned}$$

(a) Find the dual to this LP.

(b) Exhibit the BF solutions of the simplex method applied to the primal along with the corresponding complementary basic solutions to the dual. You may use the simplex method or graphical methods. (You may assume that the simplex method moves along CPF solutions $(0,0) \to (0,2) \to (2,1)$.)

Solution.

(a) The dual problem is

$$\begin{aligned} &\text{Min } W = 4y_1 + 3y_2 \\ &\text{Sbj.} \\ &y_1 + y_2 \geq 5 \\ &2y_1 + y_2 \geq 7 \\ &y_1 \geq 0, y_2 \geq 0. \end{aligned}$$

(b) Using simplex methods on the primal problem, we have the sequence

$$\begin{bmatrix} -5 & -7 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 4 \\ 1 & 1 & 0 & 1 & 3 \end{bmatrix} \begin{array}{c} \longrightarrow \\ E_2\left(\frac{1}{3}\right) \\ E_{32}(-1) \\ E_{12}(1) \end{array} \begin{bmatrix} -\frac{3}{2} & 0 & \frac{7}{2} & 0 & 14 \\ \frac{1}{2} & 1 & \frac{1}{2} & 0 & 2 \\ \frac{1}{2} & 0 & -\frac{1}{2} & 1 & 1 \end{bmatrix}$$

$$\begin{array}{c} \longrightarrow \\ E_3(2) \\ E_{23}\left(-\frac{1}{2}\right) \\ E_{13}\left(\frac{3}{2}\right) \end{array} \begin{bmatrix} 0 & 0 & 2 & 3 & 17 \\ 0 & 1 & 1 & -1 & 1 \\ 1 & 0 & -1 & 2 & 2 \end{bmatrix}.$$

Thus we see that the sequence BF solutions and complementary dual solutions (from reversing first two and last two entries of zeroth row) are given as

(x_1, x_2, x_3, x_4)	(y_1, y_2, y_3, y_4)
$(0,0,4,3)$	$(0,0,-5,-7)$
$(0,2,0,1)$	$\left(\frac{7}{2},0,-\frac{3}{2},0\right)$
$(2,1,0,0)$	$(2,3,0,0)$

If we use graphical methods, we will graph the two feasible sets and observe that the primal CPF solutions are $(0,0),(0,2),(1,2)$ and the dual CP solutions are $(0,0)\,(5,0)\,(0,5),(2,3),(0,7),(7/2,0)$. Now compute objective functions at each point and obtain that

$$\begin{aligned} Z(0,0) &= 0 = W(0,0) \\ Z(0,2) &= 14 = W(7/2,0) \\ Z(2,1) &= 17 = W(2,3). \end{aligned}$$

This gives the CP correspondence. Fill in the slack variables for each problem and we have the table above.

6. A zero-sum game has the following payoff table for player 1:

	Strategy	Player 2: 1	2	3	4
Player 1	1	0	-2	2	1
	2	5	4	-3	5
	3	2	3	-4	3

(a) Use the dominated strategies technique to eliminate rows and/or columns, if possible.
(b) Use maximin/minimax to find a saddle point of the game resulting from (a), if there is one.
(c) Express the solution to the game of (a) as a linear programming problem (do not solve.)

Solution. (a) Inspection shows that row 3 is dominated by row 2, so player 1 should discard row 3. Rows 1 and 2 are incomparable.

Having done so, we see that column 4 is dominated by column 1, so player 2 should discard column 4. Next, we see that column 1 is dominated by column 2, so player 2 should discard move 1. Doing so leaves us with this table of incomparable moves and no saddle point:

Strategy		Player 2: 2	3
Player 1	1	-2	2
	2	4	-3

(b) We calculate the maximin payoff for player 1 to be $\max\{-2,-3\} = -2$, so player 1 would choose move 1 as an optimal pure strategy with a payoff of -2.

On the other hand, the minimax strategy for player 2 is $\min\{4,2\} = 2$, so player 2 would choose move 3 as an optimal pure strategy with a payoff of 2. Thus we see that payoffs do not match, so there is no saddle point.

(c) Consider player 1 strategy. As a linear programming problem, we would convert the payoff v to the difference $v = x_5 - x_6$ (I'm making room for the slack variables, but this isn't necessary) of two nonnegative variables. So if we focus on the primal (player 1), the problem is

$$\begin{aligned} \text{Max } Z &= x_5 - x_6 \\ &\text{Sbj.} \\ 2x_1 - 4x_2 + x_5 - x_6 &\le 0 \\ -2x_1 + 3x_2 + x_5 - x_6 &\le 0 \\ x_j &\ge 0, j = 1,2,5,6. \end{aligned}$$

(Or any number of variations on this.)