15. With the usual notation show that if $T \in A(V)$ has all its characteristic roots in F then there is a basis of V in which the matrix of T is triangular.

16. Prove with the usual notation :

(a) if $T \in A(V)$ is such that $\langle vt, v \rangle = 0$ for all $v \in V$, then $T = 0$.

(b) a linear transformation T on V is unitary if and only if it takes an orthonormal basis of V into an orthonormal basis of V.

M.Sc Maths (R)

2535/KM1 (11C, after) **OCTOBER 2011**

Paper I — ALGEBRA – I

(For those who joined in July 2009 and after)

Time : Three hours Maximum : 100 marks

PART A — $(4 \times 10 = 40$ marks$)$

Answer any FOUR questions.

1. When do you say that two elements of a group G are conjugates? Show that conjugacy is an equivalence relation. If $c(a)$ is the equivalence class containing a with C_α elements what can you say about ΣC_α ?

2. For $s \in N$, G an abelian group, define $G(s) = \{x \in G : x^s = e\}$. Is $G(s)$ a subgroup of G? If G, G' are isomorphic finite abelian groups then show that for every integer s, $G(s)$, $G'(s)$ are isomorphic?

3. Let I be a two-sided ideal of a ring R. Show that R/I is a ring and is a homomorphic image of R.

4. Define the term Euclidean Ring. Show that if a, b are two elements of a Euclidean ring R then they have a greatest common divisor d and that $d = \alpha a + \beta b$ for some $\alpha, \beta \in R$.

5. Define the term primitive polynomial. Show that if $f(x)$, $g(x)$ are two primitive polynomial then their product is also a primitive polynomial.

6. Define an Inner Product Space. Establish the Schwarz inequality.

7. Define invertibility of $T \in A(V)$. Show that if V is a finite dimensional vector space over a field F, then $T \in A(V)$ is invertible if and only if the constant term of the minimal polynomial is non-zero.

8. Let V be a vector space over F. With the usual notation for $T \in A(V)$ define the Hermitian adjoint T^* of T. Show that

(a) $(T^*)^* = T$.

(b) $(S+T)^* = S^* + T^*$.

(c) $(\lambda S)^* = \bar{\lambda} S^*$ and

(d) $(ST)^* = T^* S^*$ for all $S, T \in A(V)$ and all $\lambda \in F$.

PART B — ($4 \times 15 = 60$ marks)

Answer any FOUR questions.

9. Write the class equation. Let G be a finite group of order n and let p be a prime dividing n. Show that G has an element of order p. What is the number of conjugate classes in S_n ?

10. State the Sylow's theorem and give the second proof.

11. Show that two abelian groups of order p^n are isomorphic if and only if they have the same invariants.

12. State and prove

(a) the division algorithm for polynomials.

(b) the Eisenstein criterion.

13. Let V, W be finite dimensional vector spaces over a field F of dimension m, n respectively. Define $Hom(V, W)$. Show that $Hom(V, W)$ is a vector space of dimension mn over F.

14. Suppose A is an algebra with identity over a field F, show that with the usual notation, A is isomorphic to a subalgebra of $A(V)$ for some vector space V over F.

2536/KM2 **OCTOBER 2011**

ANALYSIS – I
(REAL AND COMPLEX ANALYSIS)

(For those who joined in July 2009 and after)

Time : Three hours Maximum : 100 marks

PART A — ($4 \times 10 = 40$ marks)

Answer any FOUR questions.

1. If $E_1, E_2, E_3, \cdots$ is a sequence of countable sets show that their union is again countable.

2. Suppose $K \subset X \subset Y$. Show that K is compact relative to X if and only if K is compact relative to Y.

3. State and prove the Root test or the Ratio test on the convergence of a series.

4. Let f be a continuous mapping of a compact metric space into another metric space. Prove that f is uniformly continuous.

5. State and prove the chain rule for differentiation.

6. If a, b are complex numbers show that $\left| \frac{a-b}{1-\bar{a}b} \right| = 1$ if and only if either $|a| = 1$ or $|b| = 1$.

7. Define the term 'radius of convergence' of a power series about a point a. Prove that a series converges absolutely in the open disc center a and radius given by the radius of convergence.

8. Obtain the Cauchy Integral Formula.

PART B — $(4 \times 15 = 60$ marks)

Answer any FOUR questions.

9. State and prove the Heine-Borel theorem for the Euclidean space $\boldsymbol{R}^k$.

10. Show that $\lim_{n \to \infty} (1 + 1/n)^n = e$. Prove also that e is irrational.

11. Let f be a monotonically increasing function in (a, b). Show that for any $x \in (a, b)$ both $f(x+)$ and $f(x-)$ exists. Prove also that for such a function the discontinuities is atmost countable.

12. State and prove Taylor's theorem for a real valued function defined on a closed interval.

13. Define an analytic function. Show that every analytic function satisfies the Cauchy-Riemann differential equations. Prove that if $f = u + iv$, where u and v have continuous first order partial derivatives satisfying the $C - R$ equations then f is analytic.

14. Define a linear transformation on the complex plane. Prove :

(a) if z_1, z_2, z_3, z_4 are distinct points in the extended complex plane then for any linear transformation T,

$$\left(T_{z_1}, T_{z_2}, T_{z_3}, T_{z_4}\right) = \left(z_1, z_2, z_3, z_4\right);$$

(b) a linear transformation carries circles to circles.

15. State and prove the Cauchy's theorem for a rectangle.

16. Define removable singularity. State and prove Weierstrass theorem regarding essential singularity.

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16. Let X be a non-zero Banach space over C, the complex field and $A \in BL(X)$. Prove that with the usual notation

(a) $\sigma(A)$ is non-empty.

(b) $r_\sigma(A) = \inf_{n=1,2,\cdots} \| A^n \|^{1/n} = \lim_{n \to \infty} \| A^n \|^{1/n}$.

2537/KM3

OCTOBER 2011

ANALYSIS – II
(TOPOLOGY AND FUNCTIONAL ANALYSIS)

(For those who joined in July 2009 and after)

Time : Three hours Maximum : 100 marks

PART A — (4 × 10 = 40 marks)

Answer any FOUR questions.

1. Define the term 'Topological space'. Explain complement finite topology. Show that it is indeed a topological space.

2. Define the term 'Basis for the topology'. Describe a basis for the usual topology of the real line. If B is a basis for a topology on X and if Y is a subspace show that the collection $B \cap Y$ yields a basis for the subspace topology on Y.

3. Let (X, d) be a metric space. Show that $d'(x, y) = d(x, y)/(1 + d(x, y))$ is also a metric on X. Prove that d and d' induce the same topology on X.

4. Show that a closed subset of a compact space is compact and that a compact subset of Hausdorff space is closed.

5. Define the terms : Second countable space, separable space. Show that a separable metric space is second countable.

6. Under what operations and norm the Euclidean space $\boldsymbol{R}^n$ is a normed linear space over $\boldsymbol{R}$? Establish the Minkoswki and Holder inequalities in $\boldsymbol{R}^n$.

7. What is a Banach space? If X is a Banach space and M a closed subspace of X? Define the norm on the quotient space X/M and show that it is indeed a complete norm.

8. When is a map $T : X \to Y$, where X, Y are normed linear spaces, said to be a bounded linear transformation? For a bounded linear transformation define its norm. Show that the norm is given by the following two :

(a) $\sup\{\| f(x) \| / \| x \| : x \neq 0\}$.

(b) $\sup\{\| f(x) \| : \| x \| = 1\}$.

PART B — (4 × 15 = 60 marks)

Answer any FOUR questions.

9. If β is a basis for a topology on X, then show that the topology generated by β equals the intersection of all topologies on X that contains β. Prove the same if β is a sub-basis.

10. Explain how the product topology is defined on an arbitrary product of topological spaces. Prove that the topologies on $\boldsymbol{R}^n$ induced by the Euclidean metric d and the square metric ρ are the same as the product topology on $\boldsymbol{R}^n$.

11. Show that compactness implies limit point compactness. State and prove the Lebesgue number lemma.

12. Show that a well ordered set X is normal in the order topology.

13. Let (X, τ) be a compact Hausdorff space and let $C(X, \boldsymbol{R})$ be the space of all continuous real functions on X. Show that $\| f \| = \sup\{| f(x) | : x \in X\}$ defines a norm on $C(X, \boldsymbol{R})$ making it into a Banach space.

14. Let X be a normed linear space over K, E a non-empty open convex subset of X. Let Y be a subspace of X such that $E \cap Y = \phi$. Show that there is $f \in X'$ such that $f(x) = 0$ for every $x \in Y$ but $\operatorname{Re} f(x) \neq 0$ for every $x \in E$.

15. State and prove the principle of uniform boundedness. Deduce that a subset E of a normed linear space X is bounded if and only if $f(E)$ is bounded in K for every $f \in X'$.

15. Consider the Laguerre equation $xy'' + (1 - x)y' + \alpha y = 0$, where α is a constant. Show that this equation has a regular singular point at $x = 0$. Compute the indicial polynomial and its roots. Show that if $\alpha = n$, a non-negative integer, then there is a polynomial solution of degree n.

16. Find:

(a) the general solution of

$$x^2 \frac{\partial z}{\partial x} + y^2 \frac{\partial z}{\partial y} = (x + y) z .$$

(b) the integral surface of the linear partial differential equation

$$x\left(y^2 + z\right) p - y\left(x^2 + z\right) q = \left(x^2 - y^2\right) z$$

which contains the straight line $x + y = 0$, $z = 1$.

2538/KM4

OCTOBER 2011

NUMERICAL METHODS AND DIFFERENTIAL EQUATIONS

(For those who joined in July 2009 and after)

Time : Three hours Maximum : 100 marks

PART A — (4 × 10 = 40 marks)

Answer any FOUR questions.

1. A real root of the equation $f(x) = x^3 - 5x + 1 = 0$ lies in the interval $(0, 1)$. Perform four iterations of the Secan method and the Regula-Falsi method to contain this root.

2. Find the inverse of the coefficient matrix of the system $\begin{pmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ 4 \end{pmatrix}$ by Gauss-Jordan method with partial pivoting and hence solve the system.

3. Given the following values of the function $f(x) = \sin x + \cos x$.

x	10°	20°	30°
$f(x)$:	1.1585	1.2817	1.3660

construct the quadratic interpolating polynomial that fits the data and find $f(\pi/12)$.

4. Write the formula to find the remainder of the Simpson three-eight rule for equally spaced points. Use this to find the approximate value of the integral $\int_0^1 \frac{dx}{1+x}$.

5. Find the solution of the initial value problem $\frac{du}{dt} = Au$, $u(0) = [1, 0]^T$ where A is the 2×2 matrix given by $A = \begin{pmatrix} -2 & 1 \\ 1 & -20 \end{pmatrix}$ and $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$. Is the system asymptotically stable?

6. Show that any set of n linearly independent solutions of $L(y) = 0$ on I is a basis for the solutions of $L(y) = 0$ on I.

7. Show that the differential equation $y' = \frac{3x^2 - 2xy}{x^2 - 2y}$ is exact and obtain its solution.

8. Find the solution of the equation $z = (1/2)(p^2 + q^2) + (p - x)(q - y)$ which passes through the x-axis.

PART B — ($4 \times 15 = 60$ marks)

Answer any FOUR questions.

9. Explain the Birge-Vieta method and find the smallest positive root of the polynomial $P_3 = 2x^3 - 5x + 1 = 0$, using the initial approximation $p_0 = 0.5$. Also obtain the deflated polynomial.

10. Solve $4x_1 + x_2 + x_3 = 4$; $x_1 + 4x_2 - 2x_3 = 4$; $3x_1 + 2x_2 - 4x_3 = 6$ by

(a) Gauss elimination method.

(b) by the LU-decomposition method.

11. Explain quadratic Spline Interpolation.

12. Evaluate the integral $I = \int_0^1 dx/(1+x)$ using

(a) composite trapezoidal rule.

(b) composite Simpson's rule with 4 equal subintervals and

(c) Romberg integration.

13. Use the Euler method to solve numerically the initial value problem $n' = -2ta^2$, $u(0) = 1$, with $h = 0.2, 0.1$ and 0.05 on the interval $[0, 1]$. Neglecting the round off errors, determine the bound for the error. Apply Richardson's extrapolation to improve the computed value $u(1, 0)$.

14. For the Legendra equation

$$L(y) = (1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0,$$

show that convergent power series solutions exist and find a basis for the solutions.

2539/KMA **OCTOBER 2011**

MECHANICS AND FUZZY SETS

(For those who joined in July 2009 and after)

Time : Three hours Maximum : 100 marks

PART A — (4 × 10 = 40 marks)

Answer any FOUR questions.

1. State and prove the conversion theorem for the angular momentum of a particle.

2. Obtain Lagrange's equation for conservative and non-conservative holonomic and scaleronomic system.

3. Discuss the motion of a loop on an inclined plane using Langrange undetermined multipliers.

4. A particle describes the curves $r^n = a^n \cos n\theta$ under a force p to the pole find the laws of force.

5. Show that in the case of an elliptical orbit, the major axis depends upon E the total energy.

6. Define the terms : Scalar cardinality and Fuzzy cardinality. Obtain the cardinalities for the fuzzy set $B = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$.

7. Let

$$M_P = \begin{pmatrix} 0.3 & 0.5 & 0.8 \\ 0 & 0.7 & 1 \\ 0.4 & 0.6 & 0.5 \end{pmatrix} \text{ and } M_Q = \begin{pmatrix} 0.9 & 0.5 & 0.7 & 0.7 \\ 0.3 & 0.2 & 0 & 0.9 \\ 1 & 0 & 0.5 & 0.5 \end{pmatrix}.$$

Draw the sagittal diagram and also find $P \odot Q$.

8. With the usual notation find B' and conclude "y is B", when $A = \frac{0.6}{x_1} + \frac{1}{x_2} + \frac{0.9}{x_3}$, $B = \frac{0.6}{y_1} + \frac{1}{y_2}$ and $A' = \frac{0.5}{x_1} + \frac{0.9}{x_2} + \frac{1}{x_3}$.

PART B — (4 × 15 = 60 marks)

Answer any FOUR questions.

9. Show that the total angular momentum about a point is the angular momentum of a system concentrated at the centre of mass and the angular momentum of motion about the centre of mass.

10. Obtain the Lagrange equations of motion for a spherical pendulum i.e. a mass point suspended by a rigid weightless rod.

11. State and prove the Virial theorem.

12. With the usual notation derive the equation of the central orbits under inverse square law of force. Show also that the force of attraction is balanced by a centrifugal force in the case of circular motion.

13. Show that all α-cuts of any fuzzy set A defined on R^n $(n \geq 1)$ are convex if and only if $\mu_A(\lambda r + (1-\lambda)_S) \geq \min[\mu_A(r), \mu_A(s)]$ for all $r, s \in R^n$ and all $\lambda \in [0, 1]$.

14. Prove that if $A \subseteq E$ and $B \subseteq F$, then

(a) $A + B \subseteq E + F$

(b) $A - B \subseteq E - F$.

(c) $A \cdot B \subseteq E \cdot F$ and

(d) $A/B \subseteq E/F$ $(0 \notin F$, in this case), where A, B, E and F are closed intervals.

15. Find R_T if $M_R = \begin{bmatrix} 0.7 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0.8 & 0 \end{bmatrix}$.

16. Discuss in detail on any two areas of applications of fuzzy set theory.

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