

GRAPH THEORY AND COMBINATORICS

(Common to CSE and ISE)

Sub code : 06CS42

UNIT 1

Introduction to Graph Theory : Definition and Examples Subgraphs Complements, and Graph Isomorphism Vertex Degree, Euler Trails and Circuits.

Reference Books :

- 1. Discrete and combinatorial Mathematics, Ralph P Gridaldi, 5th Edition, PHI/Perarson Education , 2004.**
- 2. Introduction to Graph Tehroy, charted Zhang, TMH, 2006**
- 3. Graph Theory to combinatorics, Dr. C S chandrasekharaiah, Prism, 2005.**
- 4. Introductory combinatorics, Richard A, Brualdi, 4th Edition, PHI, 2004.**
- 5. Graph Theory Modling, Applications, and algorithms, Geir Agnasson and Raymond Geenlaw, PHI, 2007**

INTRODUCTION :

This topic is about a branch of discrete mathematics called graph theory. Discrete mathematics – the study of discrete structure (usually finite collections) and their properties include combinatorics (the study of combination and enumeration of objects) algorithms for computing properties of collections of objects, and graph theory (the study of objects and their relations).

Many problem in discrete mathematics can be stated and solved using graph theory therefore graph theory is considered by many to be one of the most important and vibrant fields within discrete mathematics.

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DISCOVERY

It is no coincidence that graph theory has been independently discovered many times, since it may quite properly be regarded as an area of applied mathematics. Indeed the earliest recorded mention of the subject occurs in the works of Euler, and although the original problem he was considering might be regarded as a some what frivolous puzzle, it did arise from the physical world.

Kirchhoff's investigations of electric network led to his development of the basic concepts and theorems concerning trees in graphs. While Cayley considered trees arising from the enumeration of organic chemical isomer's. Another puzzle approach to graphs was proposed by Hamilton. After this, the celebrated four color conjecture came into prominence and has been notorious ever since. In the present century, there have already been a great many rediscoveries of graph theory which we can only mention most briefly in this chronological account.

WHY STUDY GRAPH?

The best way to illustrate the utility of graphs is via a “cook's tour” of several simple problem that can be stated and solved via graph theory. Graph theory has many practical applications in various disciplines including, to name a few, biology, computer

science, economics, engineering, informatics, linguistics, mathematics, medicine, and social science, (As will become evident after reading this chapter) graphs are excellent modeling tools, we now look at several classic problems.

We begin with the bridges of Königsberg. This problem has a historical significance, as it was the first problem to be stated and then solved using what is now known as graph theory. Leonard Euler fathered graph theory in 1736 when his general solution to such problems was published Euler not only solved this particular problem but more importantly introduced the terminology for graph theory

1.THE KONIGSBERG BRIDGE PROBLEM

Euler (1707-- 1782) became the father of graph theory as well as topology when in 1736 he settled a famous unsolved problem of his day called the Königsberg bridge problem. The city of Königsberg was located on the Pregel river in Prussia, the city occupied two island plus areas on both banks. These region were linked by seven bridges as shown in fig(1.1).

The problem was to begin at any of the four land areas, walk across each bridge exactly once and return to the starting point one can easily try to solve this problem empirically but all attempts must be unsuccessful, for the tremendous contribution of Euler in this case was negative.

In proving that the problem is unsolvable, Euler replaced each land area by a point and each bridge by a line joining the corresponding points these by producing a “graph” this graph is shown in fig(1.2) where the points are labeled to correspond to the four land areas of fig(1.1) showing that the problem is unsolvable is equivalent to showing that the graph of fig(1.2) cannot be traversed in a certain way.

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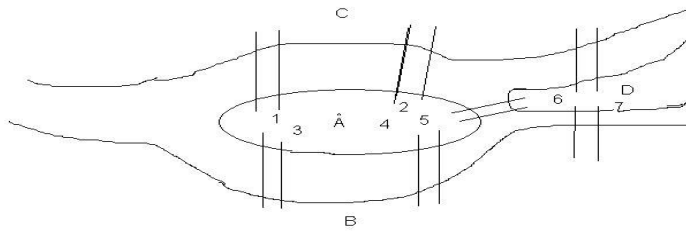


Figure1.1: A park in Königsberg 1736

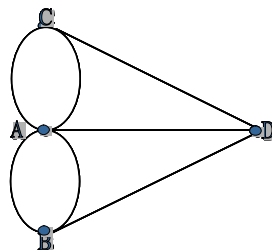


Figure1.2: The Graph of the Königsberg bridge problem

Rather than treating this specific situation, Euler generalized the problem and developed a criterion for a given graph to be so traversable; namely that it is connected and every point is incident with an even number of lines. While the graph in fig(1.2) is connected, not every point incident with an even number of lines.

2.ELECTRIC NETWORKS

Kirchhoffs developed the theory of trees in 1847 in order to solve the system of simultaneous linear equations linear equations which gives the current in each branch and around each circuit of an electric network..

Although a physicist he thought like a mathematician when he abstracted an electric network with its resistances, condensers, inductances, etc, and replaced it by its corresponding combinatorial structure consisting only of points and lines without any indication of the type of electrical element represented by individual lines. Thus, in effect, Kirchhoff replaced each electrical network by its underlying graph and showed that it is not necessary to consider every cycle in the graph of an electric network separating in order to solve the system of equation.

Instead, he pointed out by a simple but powerful construction, which has since became std procedure, that the independent cycles of a graph determined by any of its

“spanning trees” will suffice. A contrived electrical network N , its underlying graph G , and a spanning tree T are shown in fig(1.3)

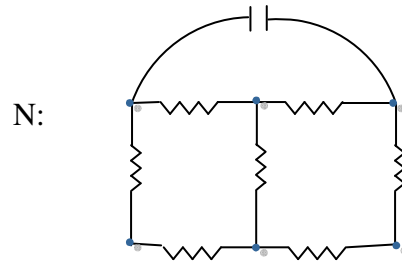
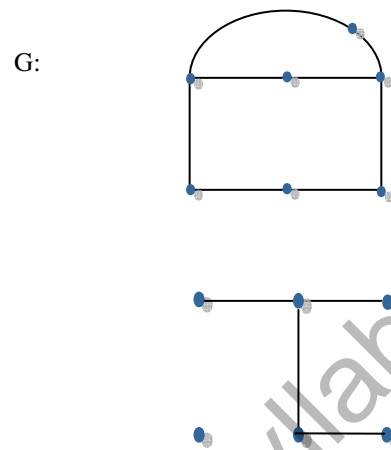
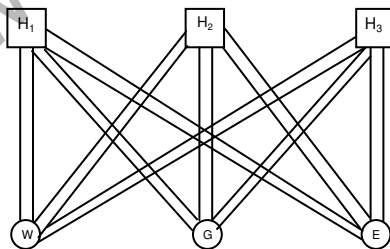


Fig (1.3)- A network N , its underlying graph G , and a spanning tree T



3.UTILITIES PROBLEM

These are three houses fig(1.4) H_1 , H_2 , and H_3 , each to be connected to each of the three utilities water(w), gas(G), and electricity(E)- by means of conduits, is it possible to make such connection without any crossovers of the conduits?

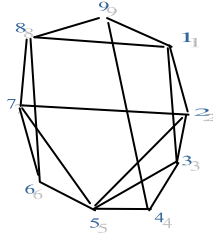


Fig(1.4)- three – utilities problem

Fig(1.4) shows how this problem can be represented by a graph – the conduits are shown as edges while the houses and utility supply centers are vertices

4. SEATING PROBLEM

Nine members of a new club meet each day for lunch at a round table they decide to sit such that every members has different neighbors at each lunch



Fig(1.5) – Arrangements at a dinner table

How many days can this arrangement last?

This situation can be represented by a graph with nine vertices such that each vertex represent a member, and an edge joining two vertices represents the relationship of sitting next to each other. Fig(1.5) shows two possible seating arrangement – these are 1 2 3 4 5 6 7 8 9 1 (solid lines), and 1 3 5 2 7 4 9 6 8 1 (dashed lines) it can be shown by graph – theoretic considerations that there are only two more arrangement possible. They are 1 5 7 3 9 2 8 4 6 1 and 1 7 9 5 8 3 6 2 4 1. In general it can be shown that for n people the number of such possible arrangements is $(n-1)/2$, if n is odd. $(n-2)/2$, if n is even

WHAT IS A GRAPH?

A linear graph (or simply a graph) $G = (V, E)$ consists of a set of objects $V = \{v_1, v_2, \dots\}$ called vertices, and another set $E = \{e_1, e_2, \dots\}$ whose elements are called edges, such that each edge e_k is identified with an unordered pair (v_i, v_j) of vertices. The vertices v_i, v_j associated with edge e_k are called the end vertices of e_k . The most common representation of a graph is by means of a diagram, in which the vertices are represented as points and each edge as a line segment joining its end vertices

The object shown in fig (a)

The Object Shown in Fig.(a)

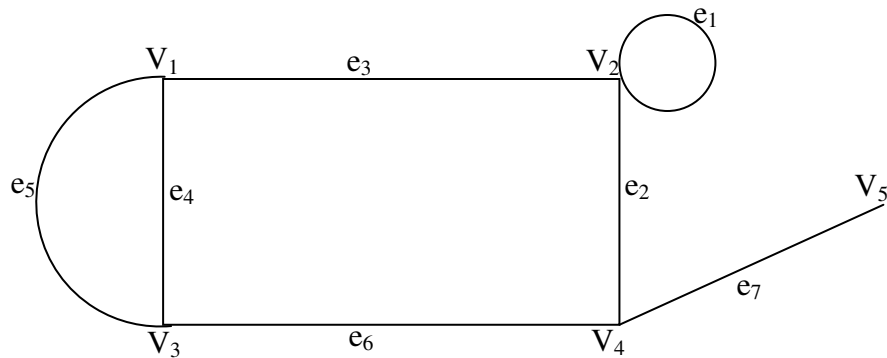
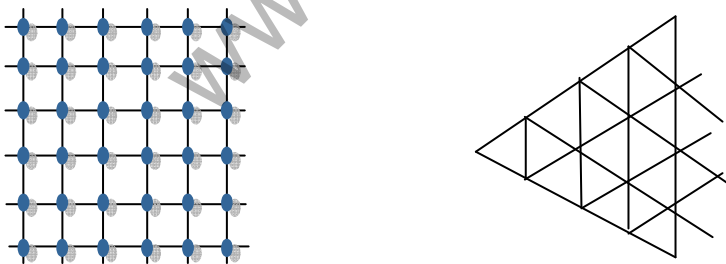


Fig (a) – Graph with five vertices and seven edges

Observe that this definition permits an edge to be associated with a vertex pair (v_i, v_j) such an edge having the same vertex as both its end vertices is called a self-loop. Edge e_1 in fig (a) is a self-loop. Also note that the definition allows more one edge associated with a given pair of vertices, for example, edges e_4 and e_5 in fig (a), such edges are referred to as ‘parallel edges’. A graph that has neither self-loops nor **parallel edges** is called a ‘**simple graph**’.

FINITE AND INFINITE GRAPHS

Although in our definition of a graph neither the vertex set V nor the edge set E need be finite, in most of the theory and almost all application these sets are finite. A graph with a finite number of vertices as well as a finite number of edge is called a ‘**finite graph**’: otherwise it is an **infinite graph**. The graphs in fig (a), (1.2), are all examples of finite graphs. Portions of two infinite graphs are shown below



Fig(1.6) – Portion of two infinite graphs

INCIDENCE AND DEGREE

When a vertex v_i is an end vertex of same edge e_j , v_i and e_j are said to be incident with (on or to) each other. In fig (a), for examples, edges e_2 , e_6 and e_7 are incident with vertex v_4 . Two nonparallel edges are said to be adjacent if there are incident on a common vertex. For example, e_2 and e_7 in fig (a) are adjacent. Similarly, two

vertices are said to be adjacent if they are the end vertices of the same edge in fig (a), v_4 and v_5 are adjacent, but v_1 and v_4 are not.

The number of edges incident on a vertex v_i , with self-loops counted twice, is called the degree, $d(v_i)$, of vertex v_i , in fig (a) for example $d(v_1) = d(v_2) = d(v_3) = 3$, $d(v_4) = 4$ and $d(v_5) = 1$. The degree of a vertex is same times also referred to as its valency.

Let us now considered a graph G with e edges and n vertices v_1, v_2, \dots, v_n since each edge contributes two degrees

The sum of the degrees of all vertices in G is twice the number of edges in G that is

$$\sum_{i=1}^n d(v_i) = 2e \quad \text{---(1.1)}$$

Taking fig (a) as an example, once more $d(v_1) + d(v_2) + d(v_3) + d(v_4) + d(v_5) = 3 + 4 + 3 + 3 + 1 = 14 =$ twice the number of edges.

From equation (1.1) we shall derive the following interesting result.

THEOREM 1.1

“The number of vertices of odd degree in a graph is always even”.

Proof : If we consider the vertices with odd and even degree separately, the quantity in the left side of equation (1.1) can be expressed as the sum of two sum, each taken over vertices of even and odd degree respectively, as follows.

$$\sum_{i=1}^n d(v_i) = \sum_{\text{even}} d(v_j) + \sum_{\text{odd}} d(v_k) \quad \text{---(1.2)}$$

Since the left hand side in equation (1.2) is even, and the first expression on the right hand side is even (being a sum of even numbers), the second expression must also be even

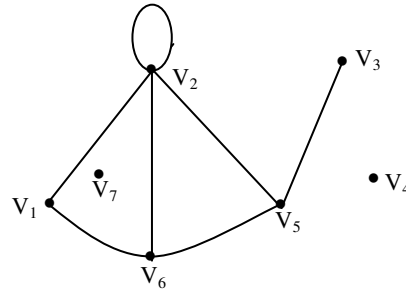
$$\sum_{\text{odd}} d(v_k) = \text{an even number} \quad \text{---(1.3)}$$

Because in equation (1.3) each $d(v_k)$ is odd, the total number of terms in the sum must be even to make the sum an even number. Hence the theorem.

A graph in which all vertices are of equal degree is called a **‘regular graph’** (or simply a regular).

DEFINITION:

ISOLATED VERTEX , PENDANT VERTEX AND NULL GRAPH



Fig(1.7) – Graph containing isolated vertices, series edges, and a pendent vertex.

A vertex having no incident edge is called an **‘isolated vertex’**. In other words, isolated vertices are vertices with zero degree. Vertices v_4 and v_7 in fig(1.7), for example, are isolated vertices a vertex of degree one is called a pendent vertex or an end vertex v_3 in fig(1.7) is a pendent vertex. Two adjacent edges are said to be in series if their common vertex is of degree two in fig(1.7), the two edges incident on v_1 are in series.

In the definition of a graph $G = (V,E)$, it is possible for the edge set E to be empty. Such a graph , without any edges is called a **‘null graph’**. In other words, every vertex in a null graph is an isolated vertex. A null graph of six vertices is shown in fig (1.8). Although the edge set E may empty the vertex set V must not be empty; otherwise there is no graph. In other words, by definition, a graph must have atleast one vertex

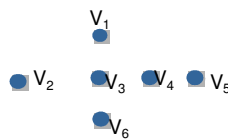


Fig 1.8: Null graph of Six Vertices

A BRIEF HISTORY OF GRAPH THEORY

As mentioned before, graph theory was born in 1736 with Euler's paper in which he solved Konigsberg bridge problem. For the next 100 years nothing more was done in the field.

In 1847, G.R. Kirchhoff (1824-1887) developed the theory of trees for their applications in Electrical network. Ten years later, A. Cayley (1821-1895) discovered trees while he was trying to enumerate the isomers of saturated hydrocarbons C_nH_{2n+2} .

About the time of Kirchhoff and Cayley, two other milestones in graph theory were laid. One was the four-color conjecture, which states that four colors are sufficient for coloring any atlas (a map on a plane) such that the countries with common boundaries have different colors.

It is believed that A.F. Mobius (1790-1868) first presented four-color problem in one of his lectures in 1840.

About 10 years later A. De Morgan (1806-1871) discussed this problem with his fellow mathematicians in London. De Morgan's letter is the first authenticated reference to the four-color problem. The problem became well known after Cayley published it in 1879 in the first volume of the **Proceedings of the Royal Geographic Society**. To this day, the four-color conjecture is by far the most famous unsolved problem in Graph theory. It has stimulated an enormous amount of research in the field.

The other milestone is due to Sir W.R. Hamilton (1805-1865). In the year 1859, he invented a puzzle and sold it for 25 guineas to a game manufacturer in Dublin. The puzzle consisted of a wooden, regular Dodecahedron (A polyhedron with 12 faces and 20 corners, each face being a regular pentagon and three edges meeting at each corner). The corners were marked with the names of 20 important cities; London, New York, Delhi, Paris and so on. The object in the puzzle was to find a route along the edges of the Dodecahedron, passing through each of the 20 cities exactly once.

Although the solution of this specific problem is easy to obtain, to date no one has found a necessary and sufficient condition for the existence of such a route (called Hamiltonian circuit) in an arbitrary graph.

This fertile period was followed by half a century of relative inactivity. Then a resurgence of interest in graphs started during the 1920's. One of the pioneers in this

period was D. Konig. He organized the work of other mathematicians and his own and wrote the first book on the subject which was published in 1936.

The past 30 years has been a period of intense activity in graph theory both pure and applied. A great deal of research has been done and is being done in this area. Thousands of papers have been published and more than hundred of books written during the past decade. **Among the current leaders in the field are Claude Berg, Oystein Ore, Paul Erdos, William Tutte and Frank Harary.**

DIRECTED GRAPHS AND GRAPHS:

DIRECTED GRAPHS :

Look at the diagram shown below. This diagram consists of four vertices A,B,C,D and three edges AB,CD,CA with directions attached to them. The directions being indicated by arrows.

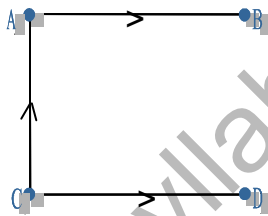


Fig. 1.1

Because of attaching directions to the edges, the edge AB has to be interpreted as an edge from the vertex A to the vertex B and it cannot be written as BA. Similarly the edge CD is from C to D and cannot be written as DC and the edge CA is from C to A and cannot be written as AC. Thus here the edges AB, CD, CA are directed edges.

The directed edge AB is determined by the vertices A and B in that order and may therefore be represented by the ordered pair (A,B). Similarly, the directed edge CD and CA may be represented by the ordered pair (C,D) and (C,A) respectively. Thus the diagram in fig(1.1) consists of a nonempty set of vertices, namely {A,B,C,D} and a set of directed edges represented by ordered pairs

{(A,B),(C,D),(C,A)}. Such a diagram is called a diagram of a directed graph.

DEFINITION OF A DIRECTED GRAPH :

A directed graph (or digraph) is a pair (V,E) , where V is a non empty set and E is a set of ordered pairs of elements taken from the set V .

For a directed graph (V, E) , the elements of V are called **Vertices** (points or nodes) and the elements of E are called "**Directed Edges**". The set V is called the **vertex set** and the set E is called the **directed edge set**

The directed graph (V,E) is also denoted by $D=(V,E)$ or $D =D(V,E)$.

The geometrical figure that depicts a directed graph for which the vertex set is $V=\{A,B,C,D\}$ and the edge set is

$$E=\{AB,CD,CA\}=\{(A,B),(C,D),(C,A)\}$$

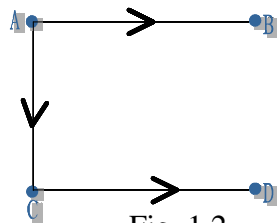


Fig. 1.2

Fig(1.2) depicts the directed graph for which the vertex set is $V=\{A,B,C,D\}$ and the edge set is $E=\{AB,CD,AC\}=\{(A,B),(C,D),(A,C)\}$.

It has to be mentioned that in a diagram of a directed graph the directed edges need not be straight line segments, they can be curve lines (arcs)Also.

For example, a directed edge AB of a directed graph can be represented by an arbitrary arc drawn from the vertex A to the vertex B as shown in fig(1.3).

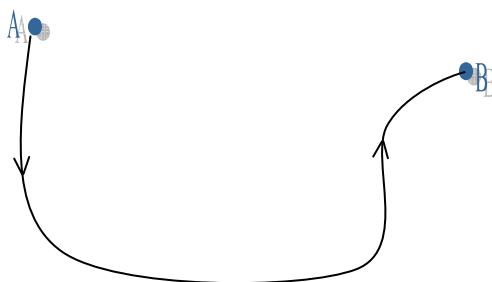


Fig.1.3

In fig (1.1) every directed edge of a digraph (directed graph) is determined by two vertices of the digraph- a vertex from which it begins and a vertex at which it ends. Thus

,if AB is a directed edge of a digraph D . Then it is understood that this directed edge begins at the vertex A of D and terminates at the vertex B of D . Here we say that A is the **initial vertex** and B is the **terminal vertex** of AB .

It should be mentioned that for a directed edge (in a digraph) the initial vertex and the terminal vertex need not be different. A directed edge beginning and ending at the same vertex A is denoted by AA or (A,A) and is called **directed loop**. The directed edge shown in Fig.(1.4) is a directed loop which begins and ends at the vertex A .

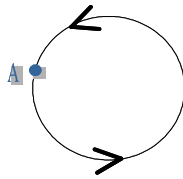


Fig. 1.4

A digraph can have more than one directed edge having the same initial vertex and the same terminal vertex. Two directed edges having the same initial vertex and the same terminal vertex are called **parallel directed edges**.

Two parallel directed edges are shown in fig(1.5)(a).



Fig. 1.5(a)

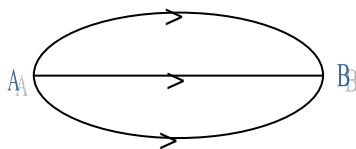


Fig. 1.5(b)

Two or more directed edges having the same initial vertex and the same terminal vertex are called “**multiple directed edges**”. Three multiple edges are shown in fig(1.5)(b).

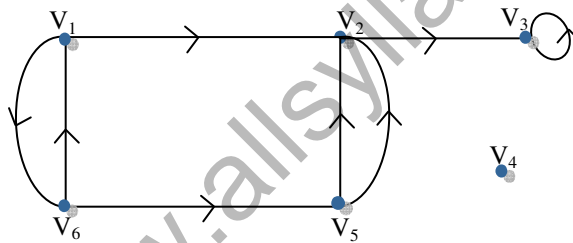
IN- DEGREE AND OUT –DEGREE

If V is the vertex of a digraph D , the number of edges for which V is the initial vertex is called the **outgoing degree** or the **out degree** of V and the number of edges for which V is the terminal vertex is called the **incoming degree** or the **in degree** of V . The out degree of V is denoted by $d^+(v)$ or $o d(v)$ and the in degree of V is denoted by $d^-(v)$ or $i d(v)$.

It follows that

- i. $d^+(v) = 0$, if V is a sink
- ii. $d^-(v) = 0$, if V is a source
- iii. $d^+(v) = d^-(v) = 0$, if V is an isolated vertex.

For the digraph shown in fig(1.6) the out degrees and the in degrees of the vertices are as given below



$d^+(v_1) = 2$	$d^-(v_1) = 1$
$d^+(v_2) = 1$	$d^-(v_2) = 3$
$d^+(v_3) = 1$	$d^-(v_3) = 2$
$d^+(v_4) = 0$	$d^-(v_4) = 0$
$d^+(v_5) = 2$	$d^-(v_5) = 1$
$d^+(v_6) = 2$	$d^-(v_6) = 1$

We note that, in the above digraph, there is a directed loop at the vertex v_3 and this loop contributes a count 1 to each of $d^+(v_3)$ and $d^-(v_3)$.

We further observe that the above digraph has 6 vertices and 8 edges and the sums of the out-degrees and in-degrees of its vertices are

$$\sum_{i=1}^6 d^+(v_j) = 8, \quad \sum_{i=1}^6 d^-(v_i) = 8$$

Example 1: Find the in- degrees and the out-degrees of the vertices of the digraph shown in fig (1.8)

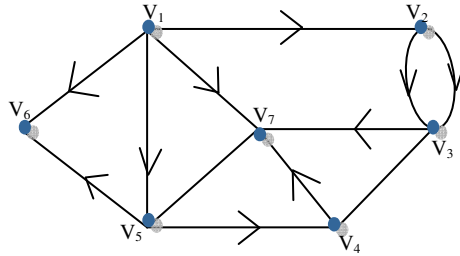


Fig. (1.8)

SOLUTION:

The given digraph has 7 vertices and 12 directed edges. The out-degree of a vertex is got by counting the number of edges that go out of the vertex and the in-degree of a vertex is got by counting the number of edges that end at the vertex. Thus we obtain the following data

Vertex	V ₁	V ₂	V ₃	V ₄	V ₅	V ₆	V ₇
Out-degree	4	2	2	1	3	0	0
In-degree	0	1	2	2	1	2	4

This table gives the out-degrees and in-degrees of all the vertices. We note that v_1 is a source and v_6 and v_7 are sinks.

We also check that sum of out-degrees = sum of in – degrees = 12 = No of edges.

Example 2: Write down the vertex set and the directed edge set of each of the following digraphs.

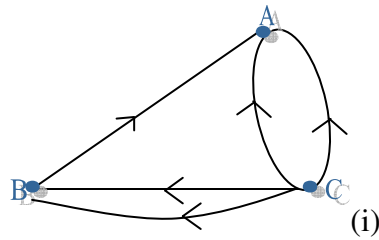


Fig.1.9

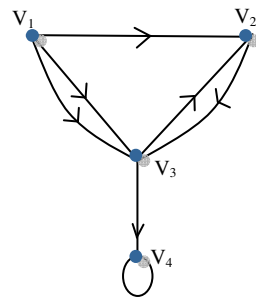
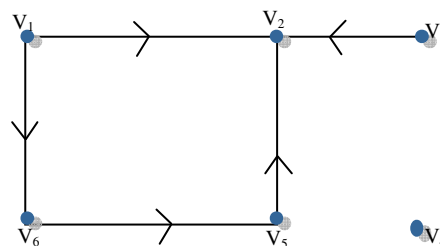


Fig. (ii)

Solution of graph (i) & (ii):

- i) This is a digraph whose vertex set is $V=\{A,B,C\}$ and the directed edge set $E=\{(B,A),(C,A),(C,A),(C,B),(C,B)\}$.
- ii) This is a digraph whose vertex set is $V=\{V_1,V_2,V_3, V_4\}$ and the directed edge set $E=\{(V_1,V_2),(V_1,V_3),(V_2,V_3), (V_2,V_3),(V_3,V_2),(V_3,V_4),(V_4,V_4)\}$.

Example 3: For the digraph shown in fig, determine the out-degrees and in-degrees of all the vertices



Solution: $d^+(V_1) = 0, d^+(V_2) = 3, d^+(V_3) = 0, d^+(V_4) = 0, d^+(V_5) = 1, d^+(V_6) = 1$

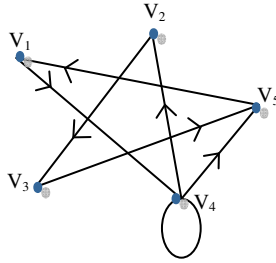
$$d^+(v_1) = 2, d^+(v_2) = 0, d^+(v_3) = 1, d^+(v_4) = 0, d^+(v_5) = 1, d^+(v_6) = 1$$

Example 4: Let D be the digraph whose vertex set

$V = \{V_1, V_2, V_3, V_4, V_5\}$ and the directed edge set is

$E = \{(V_1, V_4), (V_2, V_3), (V_3, V_5), (V_4, V_2), (V_4, V_4), (V_4, V_5), (V_5, V_1)\}$.

Write down a diagram of D and indicate the out-degrees and in-degrees of all the vertices



vertices	V_1	V_2	V_3	V_4	V_5
D^+	1	1	1	3	1
d^-	1	1	1	2	2

DEFINITION :

SIMPLE GRAPH :

A graph which does not contain loops and multiple edges is called simple graph.

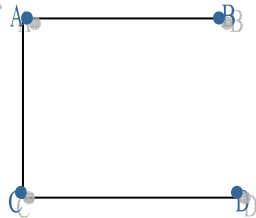


Fig. Simple Graph

LOOP FREE GRAPH.

A graph which does not contain loop is called loop free graph.

MULTIGRAPH

A graph which contains multiple edges but no loops is called multigraph.

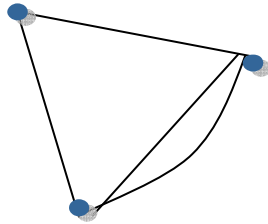


Fig. Multigraph

GENERAL GRAPH

A graph which contains multiple edges or loops (or both) is called general graph.

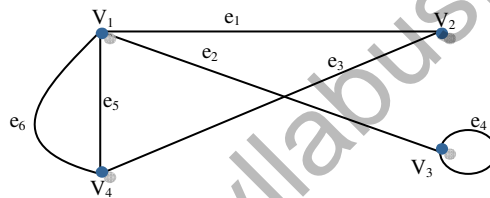


Fig. General Graph

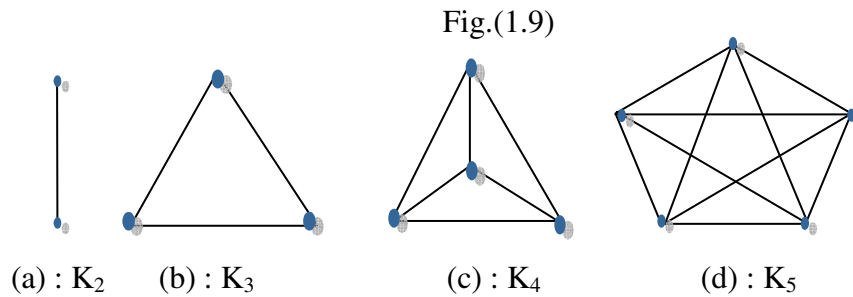
COMPLETE GRAPH :

A simple graph of order ≥ 2 in which there is an edge between every pair of vertices is called a complete graph (or a full graph).

In other words a complete graph is a simple graph in which every pair of distinct vertices are adjacent.

A complete graph with $n \geq 2$ vertices is denoted by K_n .

A complete graph with 2,3,4,5 vertices are shown in fig (1.9)(a) to (1.9)(d) respectively. Of these complete graphs, the complete graph with 5 vertices namely K_5 (shown in fig.1.9 (d), is of great importance. This graph is called the Kuratowski's first graph



BIPARTITE GRAPH

Suppose a simple graph G is such that its vertex set V is the union of two of its mutually disjoint non-empty subsets V_1 and V_2 which are such that each edge in G joins a vertex in V_1 and a vertex in V_2 . Then G is called a **bipartite graph**. If E is the edge set of this graph, the graph is denoted by $G = (V_1, V_2; E)$, or $G = G(V_1, V_2; E)$. The sets V_1 and V_2 are called **bipartites** (or partitions) of the vertex set V .

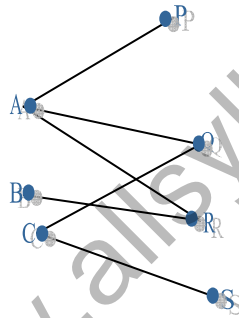


Fig. (1.10)

For example, consider the graph G in fig(1.10) for which the vertex set is

$V = \{A, B, C, P, Q, R, S\}$ and the edge set is

$E = \{AP, AQ, AR, BR, CQ, CS\}$. Note that the set V is the union of two of its subsets $V_1 = \{A, B, C\}$ and $V_2 = \{P, Q, R, S\}$ which are such that

- i) V_1 and V_2 are disjoint.
- ii) Every edge in G joins a vertex in V_1 and a vertex in V_2 .
- iii) G contains no edge that joins two vertices both of which are in V_1 or V_2 . This graph is a bipartite graph with $V_1 = \{A, B, C\}$ and $V_2 = \{P, Q, R, S\}$ as bipartites.

COMPLETE BIPARTITE GRAPH

A bipartite graph $G = \{V_1, V_2; E\}$ is called a complete bipartite graph, if there is an edge between every vertex in V_1 and every vertex in V_2 .

The bipartite graph shown in fig (1.10) is not a complete bipartite graph. Observe for example that the graph does not contain an edge joining A and S.

A complete bipartite graph $G = \{ V_1, V_2 ; E \}$ in which the bipartites V_1 and V_2 contain r and s vertices respectively, with $r \leq s$ is denoted by $K_{r,s}$. In this graph each of r vertices in V_1 is joined to each of s vertices in V_2 . Thus $K_{r,s}$ has $r+s$ vertices and rs edges. That is $K_{r,s}$ is of order $r+s$ and size rs . It is therefore a $(r+s, rs)$ graph

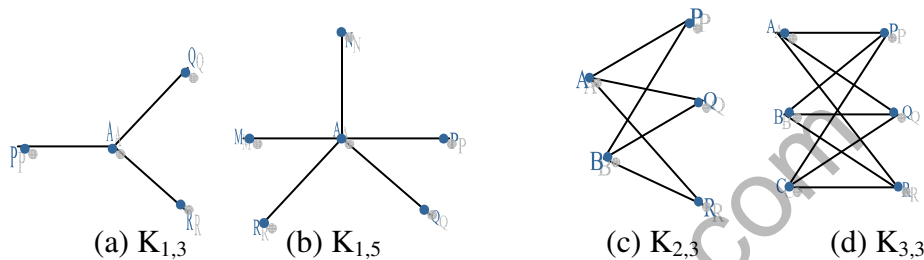


Fig. 1.11

Fig 1.11 (a) to (d) depict some bipartite graphs. Observe that in fig 1.11(a), the bipartites are $V_1 = \{ A \}$ and $V_2 = \{ P, Q, R \}$; the vertex A is joined to each of the vertices P, Q, R by an edge. In fig 1.11(b), the bipartites are $V_1 = \{ A \}$ and

$V_2 = \{ M, N, P, Q, R \}$; the vertex A is joined to each of the vertices M, N, P, Q, R by an edge. In fig 1.11(c), the bipartites are $V_1 = \{ A, B \}$ and $V_2 = \{ P, Q, R \}$; each of the vertices A and B is joined to each of the vertices P, Q, R by an edge. In fig 1.11(d), the bipartites are $V_1 = \{ A, B, C \}$ and $V_2 = \{ P, Q, R \}$; each of the vertices A, B, C is joined to each of the vertices P, Q, R. Of these complete bipartite graphs the graph $K_{3,3}$ shown in fig 1.11(d), is of great importance. This is known as **Kuratowski's second graph**.

Example 1. Draw a diagram of the graph $G = (V, E)$ in each of the following cases.

a) $V = \{ A, B, C, D \}$, $E = \{ AB, AC, AD, CD \}$

b) $V = \{ V_1, V_2, V_3, V_4, V_5 \}$,

$E = \{ V_1V_2, V_1V_3, V_2V_3, V_4V_5 \}$.

c) $V = \{ P, Q, R, S, T \}$, $E = \{ PS, QR, QS \}$

d) $V = \{ V_1, V_2, V_3, V_4, V_5, V_6 \}$,

$E = \{ V_1V_4, V_1V_6, V_4V_6, V_3V_2, V_3V_5, V_2V_5 \}$

Solution : The required diagram are shown below

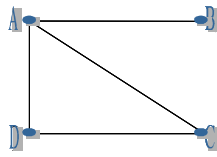


Fig: (a)

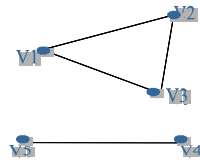


Fig: (b)

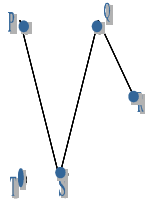


Fig: (c)

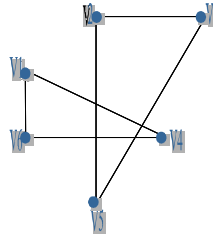
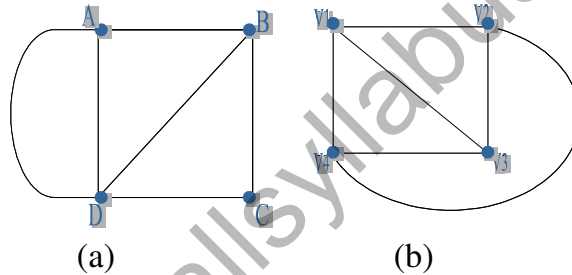


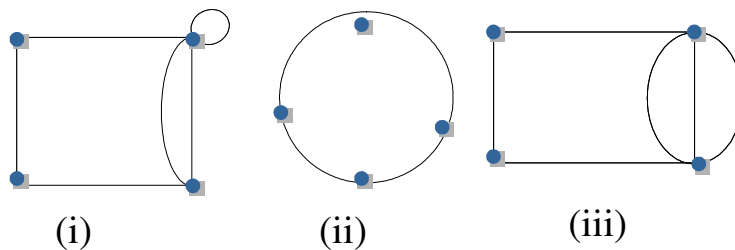
Fig: (d)

Example 2: Which of the following is a complete graph?



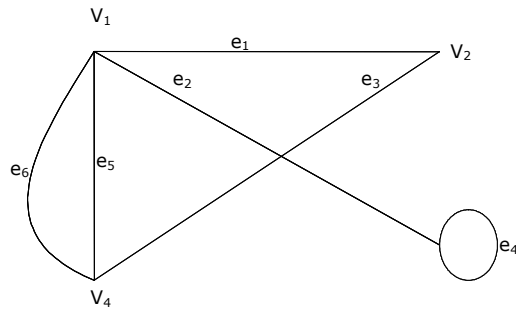
Solution: The first of the graph is not complete. It is not simple on the one hand and there is no edge between A and C on the other hand. The second of the graphs is complete. It is a simple graph and there is an edge between every pair of vertices.

Example 3: Which of the following graphs is a simple graph? a multigraph ? a general graph ?



- Solution:** (i) General Graph,
(ii) Simple Graph,
(iii) Multigraph

Example 4: Identify the adjacent vertices and adjacent edges in the graph shown in Figure.



Solution :

Adjacent Vertices : V_1 & V_2 , V_1 & V_3 , V_1 & V_4 , V_2 & V_4 .

Adjacent edges : e_1 & e_2 , e_1 & e_3 , e_1 & e_5 , e_1 & e_6 , e_2 & e_4 , e_2 & e_5 , e_2 & e_6 , e_3 & e_5 , e_3 and e_6 .

VERTEX DEGREE AND HANDSHAKING PROPERTY :

Let $G = (V, E)$ be a graph and V be a vertex of G . Then the number of edges of G that are incident on V (that is, the number of edges that join V to other vertices of G) with the loops counted twice is called the **degree** of the vertex V and is denoted by $\deg(v)$ or $d(V)$.

The degree of the vertices of a graph arranged in non-decreasing order is called the **degree sequence** of the graph. Also, the minimum of the degree of a graph is called the **degree of the graph**

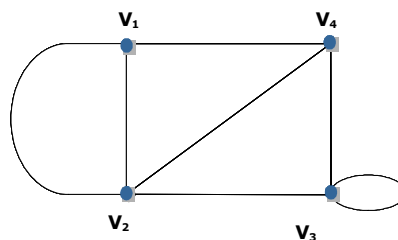


Figure (1.12)

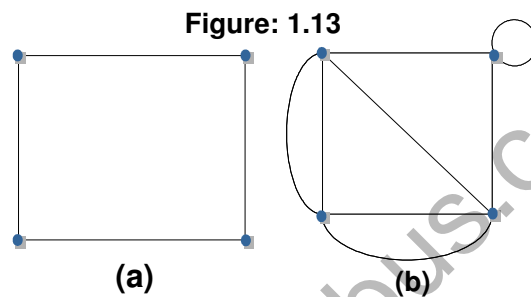
For example, the degrees of vertices of the graph shown in fig are as given below

$$d(V_1) = 3, d(V_2) = 4, d(V_3) = 4, d(V_4) = 3$$

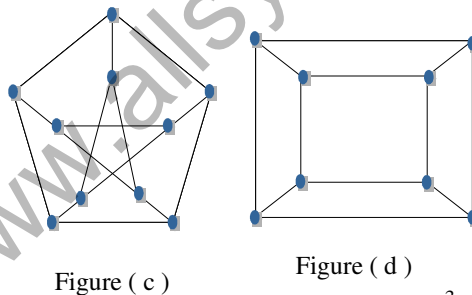
Therefore, the degree sequence of the graph is 3,3,4,4 and the degree of the graph is 3.

Regular Graph : A graph in which all the vertices are of the same degree K is called a regular graph of degree K , or a K - regular graph. In particular, a 3-regular graph is called a **cubic graph**.

The graph shown in figures 1.13 (a) and (b) are 2- regular and 4 - regular graph respectively.



The graph shown in fig1.13 (c) is a 3-regular graph (cubic graph). This particular cubic graph, which contains 10 vertices and 15 edges, is called the **Peterson Graph**.



The graph shown in fig (d) is a cubic graph with $8 = 2^3$ vertices. This particular graph is called the three dimensional hyper cube and is denoted by Q_3 .

Handshaking property :

Let us refer back to degree of the graph shown in fig 1.14. we have, in this graph,

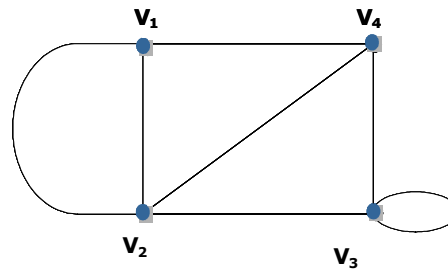


fig (1.14)

$$d(V_1) = 3, d(V_2) = 4, d(V_3) = 4, d(V_4) = 3$$

Also, the graph has 7 edges, we observe that $\deg(V_1) + \deg(V_2) + \deg(V_3) + \deg(V_4) = 14 = 2 \times 7$

Property: The sum of the degrees of all the vertices in a graph is an even number, and this number is equal to twice the number of edges in the graph.

In an alternative form, this property reads as follows:

For a graph $G = (V, E)$ $\sum_{v \in V} \deg(v) = 2|E|$

This property is obvious from the fact that while counting the degree of vertices, each edge is counted twice (once at each end).

The aforesaid property is popularly called the '**handshaking property**'

Because, it essentially states that if several people shake hands, then the total number of hands shaken must be even, because just two hands are involved in each hand shake.

Theorem : In every graph the number of vertices of odd degrees is even

Proof : Consider a graph with n vertices. Suppose k of these vertices are of odd degree so that the remaining $n-k$ vertices are of even degree. Denote the vertices with odd degree by $V_1, V_2, V_3, \dots, V_k$ and the vertices with even degree by $V_{k+1}, V_{k+2}, \dots, V_n$ then the sum of the degrees of vertices is

$$\sum_{i=1}^n \deg(v_i) = \sum_{i=1}^k \deg(v_i) + \sum_{i=k+1}^n \deg(v_i) \quad \text{---(1)}$$

In view of the hand shaking property, the sum on the left hand side of the above expression is equal to twice the number of edges in the graph. As such, this sum is even. Further, the second sum in the right hand side is the sum of the degrees of the vertices with even degrees. As such this sum is also even. Therefore, the first sum in the right hand side must be even; that is,

$$\deg(V_1) + \deg(V_2) + \dots + \deg(V_k) = \text{Even} \text{---(ii)}$$

But, each of $\deg(V_1), \deg(V_2), \dots, \deg(V_k)$ is odd. Therefore, the number of terms in the left hand side of (ii) must be even; that is, K is even

Example : For the graph shown in fig 1.15 indicating the degree of each vertex and verify the handshaking property

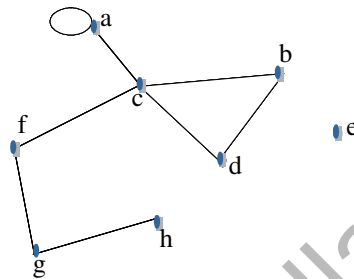


Fig (1.15)

Solution : By examining the graph, we find that the degrees of its vertices are as given below:

$$\deg(a) = 3, \deg(b) = 2, \deg(c) = 4, \deg(d) = 2, \deg(e) = 0, \deg(f) = 2, \deg(g) = 2, \deg(h) = 1.$$

We note that e is an isolated vertex and h is a pendant vertex.

Further, we observe that the sum of the degrees of vertices is equal to 16. Also, the graph has 8 edges. Thus, the sum of the degrees of vertices is equal to twice the number of edges.

This verifies the handshaking property for the given graph.

Example : For a graph with n -vertices and m edges, if δ is the minimum and Δ is the maximum of the degrees of vertices, show that

$$\delta \leq \frac{2m}{n} \leq \Delta$$

Solution : Let d_1, d_2, \dots, d_n , be the degrees of the vertices. Then, by handshaking property, we have $d_1 + d_2 + d_3 + \dots + d_n = 2m$ -----(i)

Since $\delta = \min(d_1, d_2, \dots, d_n)$, we have $d_1 \geq \delta$,

$$d_2 \geq \delta, \dots, d_n \geq \delta.$$

Adding these n inequalities, we get

$$d_1 + d_2 + \dots + d_n \geq n \delta \text{ -----(ii)}$$

Similarly, since $\Delta = \max (d_1, d_2, \dots, d_n)$, we get

$$d_1 + d_2 + \dots + d_n \leq n\Delta \text{ -----(iii)}$$

From (i), (ii) and (iii), we get $2m \geq n \delta$ and $2m \leq n\Delta$, so that $n \delta \leq 2m \leq n\Delta$,

$$\text{or } \delta \leq \frac{2m}{n} \leq \Delta$$

SUBGRAPHS

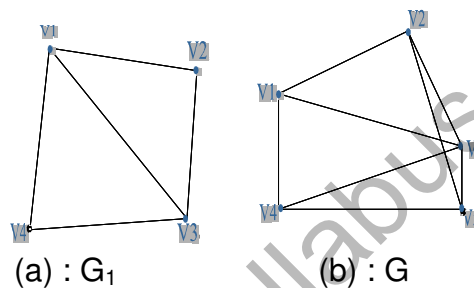


Fig. (1.6)

Given two graphs G and G_1 , we say that G_1 is a **subgraph** of G if the following conditions hold:

- (1). All the vertices and all the edges of G_1 are in G .
- (2). Each edges of G_1 has the same end vertices in G as in G_1 .

Essentially, a subgraph is a graph which is a part of another graph. Any graph isomorphic to a subgraph of a graph G is also referred to as a subgraph of G .

Consider the two graphs G_1 and G shown in figures 1.16(a) and 1.16(b) respectively, we observe that all vertices and all edges of the graph G_1 are in the graphs G and that every edge in G_1 has same end vertices in G as in G_1 . Therefore G_1 is a subgraph of G . In the diagram of G , the part G_1 is shown in thick lines.

The following observation can be made immediately.

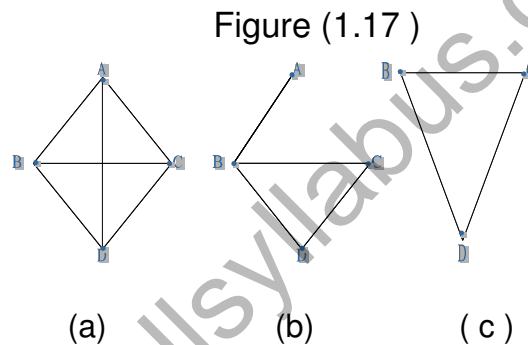
- i) Every graph is a sub-graph of itself.
- ii) Every simple graph of n vertices is a subgraph of the complete graph K_n .

- iii) If G_1 is a subgraph of a graph G_2 and G_2 is a subgraph of a graph G , then G_1 is a subgraph of a graph G .
- iv) A single vertex in a graph G is a subgraph of a graph G .
- v) A single edge in a graph G together with its end vertices, is a subgraph of G

SPANNING SUBGRAPH :

Given a graph $G=(V, E)$, if there is a subgraph $G_1=(V_1, E_1)$ of G such that $V_1=V$ then G_1 is called a spanning subgraph of G .

In other words, a subgraph G_1 of a graph G is a spanning subgraph of G whenever the vertex set of G_1 contains all vertices of G . Thus a graph and all its spanning subgraphs have the same vertex set. Obviously every graph is its own spanning subgraph.



For example, for the graph shown in fig1.17(a), the graph shown in fig 1.17(b) is a spanning subgraph whereas the graph shown in fig1.17(c) is a subgraph but not a spanning subgraph

INDUCED SUBGRAPH

Given a graph $G=(V,E)$, suppose there is a subgraph $G_1=(V_1, E_1)$ of G such that every edge $\{A,B\}$ of G , where $A, B \in V_1$ is an edge of G_1 also. then G_1 is called an induced subgraph of G (induced by V_1) and is denoted by $\langle V_1 \rangle$.

It follows that a subgraph $G_1=(V_1, E_1)$ of a graph $G=(V,E)$ is not an induced subgraph of G , if for some $A, B \in V_1$, there is an edge $\{A,B\}$ which is in G but not in G_1 .

For example, for the graph shown in the figure 1.18 (a), the graph shown in the figure 1.18 (b), is an induced subgraph, induced by the set of vertices $V_1= \{v_1, v_2, v_3, v_5\}$ whereas the graph shown in the figure 1.18 (c) is not an induced subgraph

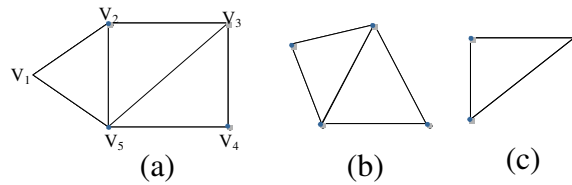


Figure 1.18 (a, b & c)

EDGE-DISJOINT AND VERTEX-DISJOINT SUBGRAPHS

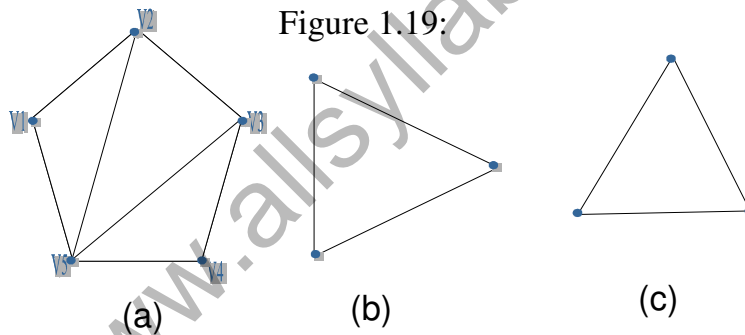
Let G be a graph and G_1 and G_2 be two subgraphs of G . then

G_1 and G_2 are said to be edge disjoint if they do not have any common edge.

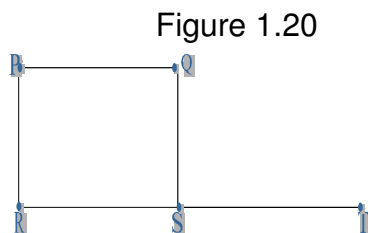
G_1 and G_2 are said to be vertex disjoint if they do not have any common edge and any common vertex.

It is to be noted that edge disjoint subgraphs may have common vertices. Subgraphs that have no vertices in common cannot possibly have edges in common.

For example, for the graph shown in the figure 1.19 (a), the graph shown in the figure 1.19 (b) and 1.19 (c) are edge disjoint but not vertex disjoint subgraphs.

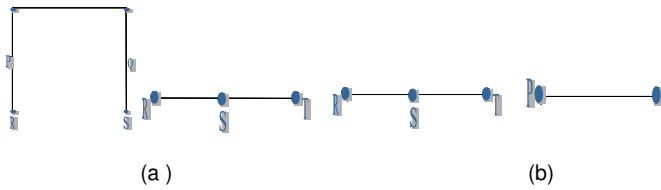


Example : For the graph shown in fig 1.20, find two edge-disjoint subgraphs and two vertex-disjoint subgraphs.



Solution:for the given graph, two edge-disjoint subgraphs are shown in fig 1.21(a) and two vertex-disjoint subgraphs are shown in fig 1.21(b).

fig 1.21



OPERATIONS ON GRAPHS

Consider two graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ then the graph whose vertex set is $V_1 \cup V_2$ and edge set is $E_1 \cup E_2$ is called the union of G_1 and G_2 and is denoted by $G_1 \cup G_2$.

Thus $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$.

Similarly, if $V_1 \cap V_2 \neq \phi$, the graph whose vertex set is $V_1 \cap V_2$ and the edge set $E_1 \cap E_2$ is called intersection of G_1 and G_2 . It is denoted by $G_1 \cap G_2$. Thus $G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$, if $V_1 \cap V_2 \neq \phi$.

Next suppose we consider the graph whose vertex set is $V_1 \cup V_2$ and edge set is $E_1 \Delta E_2$ where $E_1 \Delta E_2$ is the symmetric difference of E_1 and E_2 . This graph is called the ring sum of G_1 and G_2 . It is denoted by $G_1 \Delta G_2$. Thus $G_1 \Delta G_2 = (V_1 \cup V_2, E_1 \Delta E_2)$.

For the two graphs G_1 and G_2 shown in figures 1.22 (a) and (b), their union, intersection and ring sum are shown in figures 1.23 (a), (b) and (c) respectively.

Fig 1.22

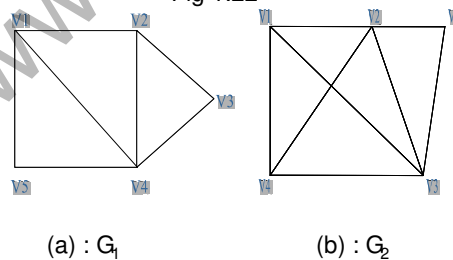
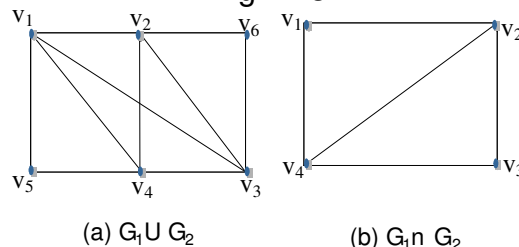
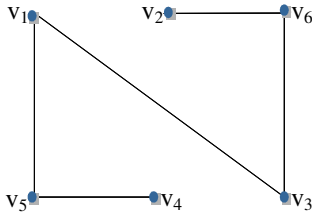
(a) : G_1 (b) : G_2

Fig 1.23

(a) $G_1 \cup G_2$ (b) $G_1 \cap G_2$

Fig 1.23: (c) G_2 **DECOMPOSITION**

We say that a graph G is decomposed (or partitioned) into two subgraphs G_1 & G_2 if $G_1 \cup G_2 = G$ & $G_1 \cap G_2 = \text{null graph}$

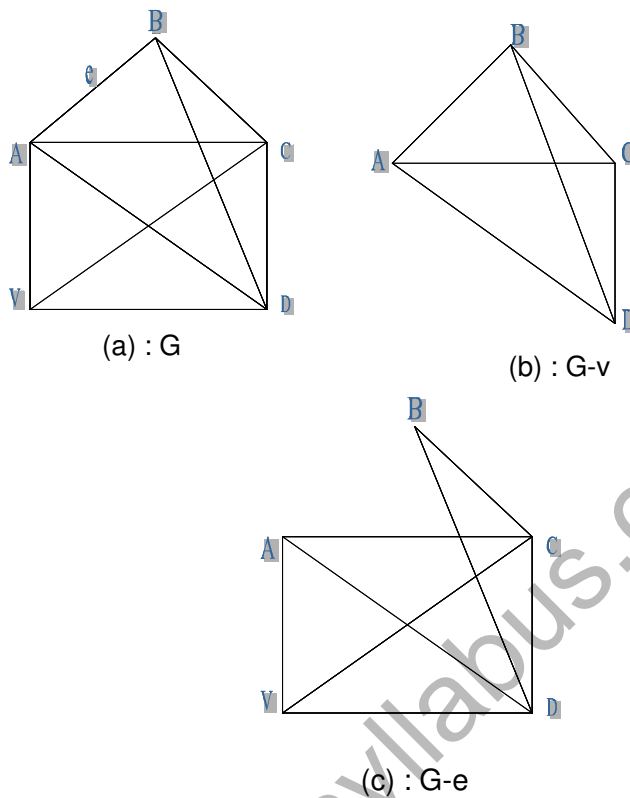
DELETION:

If V is a vertex in a graph G , then $G - V$ denotes the subgraph of G obtained by deleting V and all edges incident in V , from G this subgraph $G - u$, is referred to as **vertex deleted subgraph** of G .

It should be noted that, the deletion of a vertex always results in the deletion of all edges incident on that vertex.

If e is an edge in a graph G , then $G - e$ denotes the subgraph of G obtained by deleting e (but not its end vertices) from G . This subgraph, $G - e$, is referred to as **edge - deleted subgraph** of G . For the graph G shown in figure 1.24 (a), the subgraphs $G - V$ and $G - e$ are shown in figure 1.24 (b) and 1.24 (c) respectively.

Figure 1.24 (a, b, c)



COMPLEMENT OF A SUBGRAPH

Given a graph G and a subgraph G_1 of G , the subgraph of G obtained by deleting from all the edges that belongs to G_1 is called the complement of G_1 in G ; it is denoted by $G-G_1$ or $\overline{G_1}$

In other words, if E_1 is the set of all edges of G_1 then the complement of G_1 in G is given by $\overline{G_1} = G - E_1$. We can check that $\overline{G_1} = G \Delta G_1$.

For example :

Consider the graph G shown in fig 1.25(a). Let G_1 be the subgraph of G shown by thick lines in this figure. The complement of G_1 in G , namely $\overline{G_1}$, is as shown in fig 1.25(b)

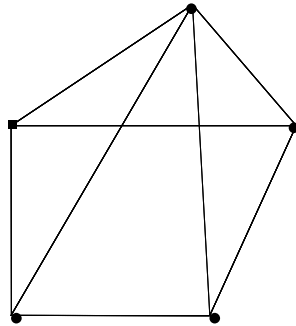


Fig. 1.25(a)

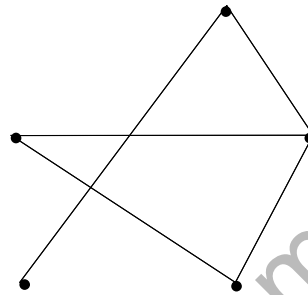


Fig.1.25 (b)

COMPLEMENT OF A SIMPLE GRAPH

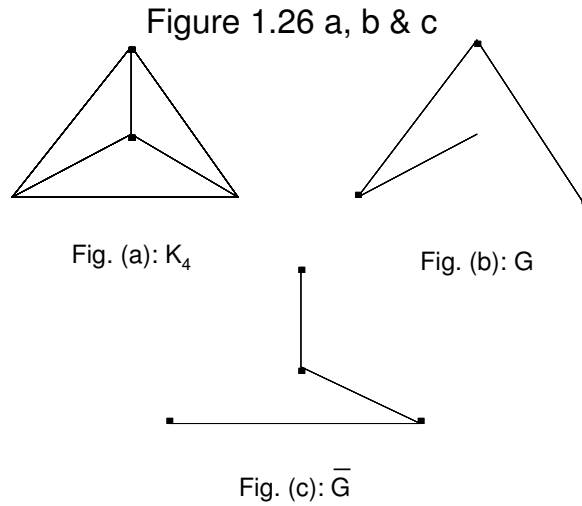
Earlier we have noted that every simple graph of order n is a subgraph of the complete graph K_n . If G is a simple graph of order n , then the complement of G in K_n is called the **complement of G** , it is denoted by \overline{G} .

Thus, the complement \overline{G} of a simple graph G with n vertices is that graph which is obtained by deleting those edges of K_n which belongs to G . Thus $\overline{G} = K_n - G = K_n \Delta G$.

Evidently K_n , G and \overline{G} have the same vertex set and two vertices are adjacent in G if and only if they are not adjacent in \overline{G} . Obviously, \overline{G} is also a simple graph and the complement of \overline{G} is G that is $\overline{\overline{G}} = G$

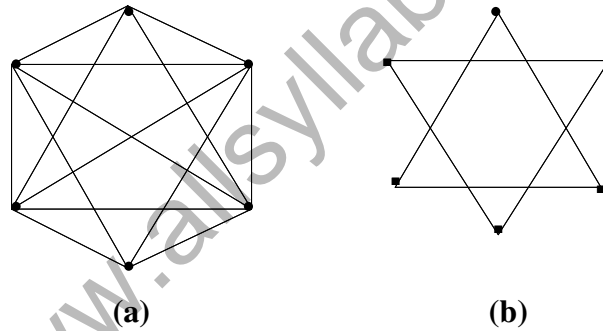
In fig 1.26(a), the complete graph K_4 is shown. A simple graph G of order 4 is shown in fig 1.26(b). The complement \overline{G} , of G is shown in fig 1.26(c).

Observe that G , \overline{G} & K_4 have the same vertices and that the edges of \overline{G} are got by deleting those edges from K_4 which belong to G .



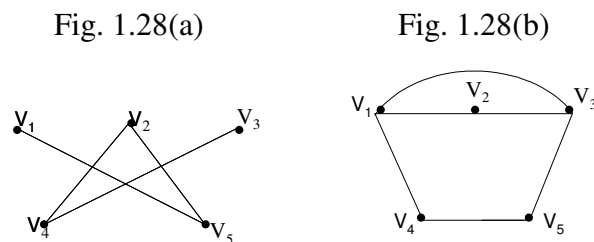
In fig1.27(a) ,a graph of order 6 is shown as a subgraph of K_6 ,the edges of G being shown in thick lines .Its complement \bar{G} , is shown in fig1.27(b).The graph shown in fig1.27(b) is known as **David Graph**.

Fig. 1.27



Example 1.Show that the complement of a bipartite graph need not be a bipartite graph.

Solution: Fig 1.28(a) shows a bipartite graph which is of order 5.The complement of this graph is shown in fig1.28(b),this is not a bipartite graph.



Example 2. Let G be a simple graph of order n . If the size of graph G is 56 and size of \bar{G} is 80. What is n ?

Solution: We know that $\bar{G} = K_n - G$ therefore

$$\text{Size of } \bar{G} = (\text{Size of } K_n) - (\text{Size of } G)$$

Since size of K_n (ie the number of edges in K_n) is $\frac{1}{2}n(n-1)$, this yields

$$80 = \frac{1}{2}n(n-1) - 56$$

$$\text{or } n(n-1) = 160 + 112 = 272 = 17 \times 16$$

thus, $n = 17$, (that is, G is of order 17)

Example 3: Find the union, intersection and the ring sum of the graph G_1 and G_2 shown below.

Fig. 1.29(G_1)

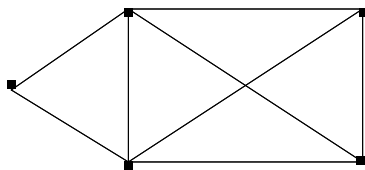
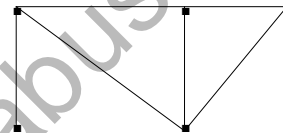


Fig. 1.29(\bar{G}_1)



Solution :

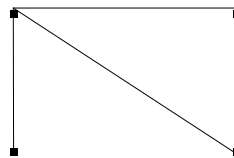
$G_1 \cup G_2$

Union :-



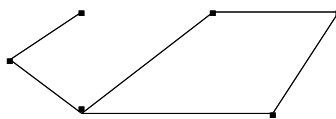
Intersection :-

$G_1 \cap G_2$



Ring Sum :

$G_1 \Delta G_2$



Example 4: For the graph G shown below, find $G-v$ and $G-e$.

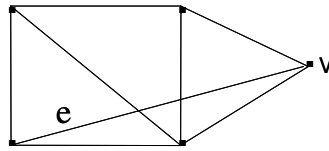
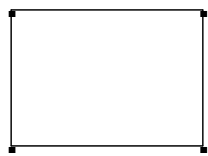


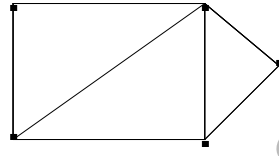
Fig 1.30

Solution :

Fig. 1.31



$G-v$



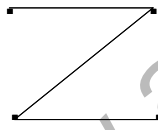
$G-e$

Example 5: Find the complement of each of the following simple graphs

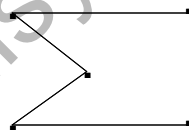
Fig. 1.32



(a)



(b)



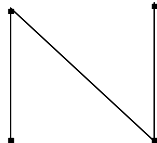
(c)

Solution :

Fig. 1.33



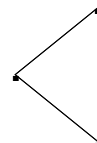
(a)



(b)



(c)



(d)

Example 6: Find the complement of the complete bipartite graph $K_{3,3}$

Solution :

Fig. 1.34



WALKS AND THEIR CLASSIFICATION

WALK:

Let G be a graph having atleast one edge. In G , consider a finite, alternating sequence of vertices and edges of the form $v_i e_j v_{i+1} e_{j+1} v_{i+2}, \dots, e_k v_m$ which begin and ends with vertices and which is such that each edge in the sequence is incident on the vertices preceding and following it in the sequence. Such a sequence is called a **walk** in G . In a walk, a vertex or an edge (or both) can appear more than once.

The number of edges present in a walk is called its '**length**'.

For example : Consider the graph shown below;

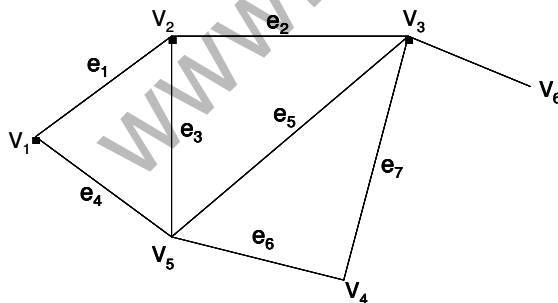


Fig.1.35

In this graph,

i) The sequence $v_1 e_1 v_2 e_2 v_3, e_8 v_6$ is a walk of length 3 (because this walk contains 3 edges; e_1, e_2, e_8). In this walk, no vertex and no edge is repeated.

ii) The sequence $V_1, e_4, V_5, e_3, V_2, e_2, V_3, e_5, V_5, e_6, V_4$ is a walk of length 5. In this walk, the vertex v_5 is repeated; but no edge is repeated.

iii) The sequence $V_1, e_1, V_2, e_3, V_5, e_3, V_2, e_2, V_3$ is a walk of length 4. In this walk, the edge e_3 is repeated and the vertex V_2 is repeated

A walk that begins and ends at the same vertex is called a **closed walk**. In other words, a closed walk is a walk in which the terminal vertices are coincident.

A walk which is not closed is called an **open walk**. In other words, an open walk is a walk that begins and ends at two different vertices.

For Example, in the graph shown in figure (1.35) $V_1, e_1, V_2, e_3, V_5, e_4, V_1$ is a **closed walk** and $V_1, e_1, V_2, e_2, V_3, e_5, V_5$ is our **open walk**.

TRAIL AND CIRCUIT:

In a walk, vertices and /or edges may appear more than once, if in an open walk no edge appears more than once, then the walk is called a **trail**. A closed walk in which no edge appears more than once is called a **circuit**.

For example: In fig (1.35), the open walk $V_1, e_1, V_2, e_3, V_5, e_3, V_2, e_2, V_3$ (shown separately in figure 1.36(a) is not a trail (because, in this walk, the edge e_3 is repeated) where as

Fig. 1.36 (a) :Not a trail

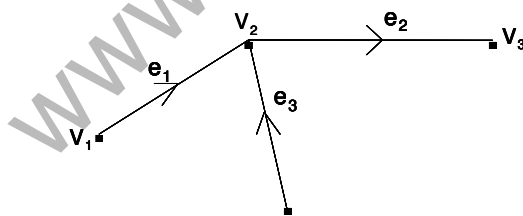
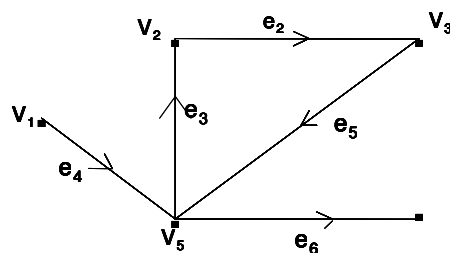


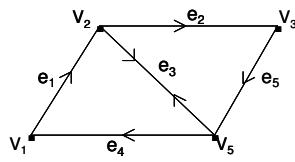
Fig. 1.36 (b): trail



The open walk $V_1e_4V_5e_3V_2V_3e_5V_5e_6V_4$ (shown separately in fig 1.36(b)) is trail.

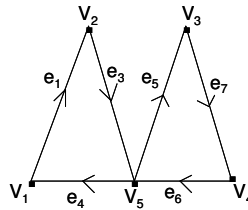
Also, in the same fig (ie., in fig1.35), the closed walk $V_1 e_1V_2 e_3 V_5 e_3 V_2 e_2 V_3 e_5 V_5 e_4 V_1$ (shown separately in fig 1.37(a)) is not a circuit (because e_3 is repeated) where as the closed walk $V_1e_1V_2e_3V_5e_5V_3e_7V_4e_6V_5e_4V_1$ (shown separately in fig1.37(b)) is a circuit.

Fig. 1.37(a)



(a): Not a circuit

Fig. 1.37(b)



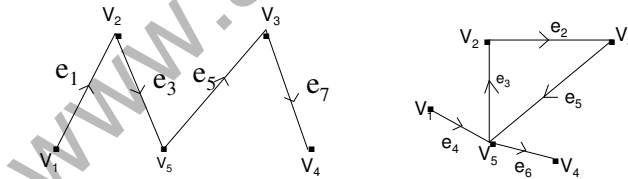
(b): Circuit

PATH AND CYCLE:

A trail in which no vertex appears more than once is called a **path**.

A Circuit in which the terminal vertex does not appear as an internal vertex (also) and no internal vertex is repeated is called a '**cycle**'.

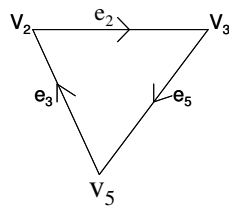
Fig. 1.38



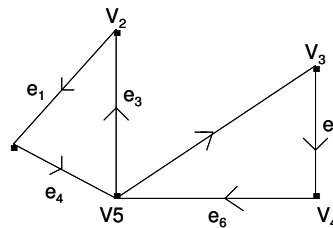
(a): Path

(a): Not a path

Fig. 1.39



(a): Cycle



(b): Not a Cycle

For example, in figure (1.35), the trail $V_1e_1e_3V_5e_5V_3e_7V_4$ (shown separately in fig 1.38(a)) is a path while as the trail $V_1e_4V_5e_3V_2e_2e_5V_5e_6V_4$ (shown separately in fig 1.38(b)) is not a path (because in this trail, v_5 appears twice).

Also, in the same fig, the circuit $V_2e_2V_3e_3V_5e_3V_2$ (shown separately in fig 1.39(a)) is a cycle while as the circuit $V_2e_1V_1e_4V_5e_5V_3e_7V_4e_6V_5e_3V_2$ (shown separately in fig 1.39(b)) is not a cycle (because, in this circuit, v_5 appears twice)

The following facts are to be emphasized.

1. A walk can be open or closed. In a walk (closed or open), a vertex and / or an edge can appear more than once.
2. A trail is an open walk in which a vertex can appear more than once but an edge cannot appear more than once.
3. A circuit is a closed walk in which a vertex can appear more than once but an edge cannot appear more than once.
4. A path is an open walk in which neither a vertex nor an edge can appear more than once. Every path is a trail; but a trail need not be a path.
5. A cycle is a closed walk in which neither a vertex nor an edge can appear more than once.

Every cycle is a circuit; but, a circuit need not be a cycle.

Example:

For the graph shown in figure 1.40 indicate the nature of the following walks.

$V_1e_1V_2e_2V_3e_2V_2$

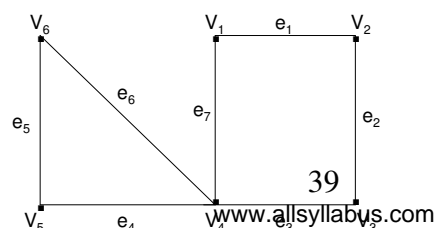
$V_4e_7V_1e_1V_2e_2V_3e_3V_4e_4V_5$

$V_1e_1V_2e_2V_3e_3V_4e_4V_5$

$V_1e_1V_2e_2V_3e_3V_4e_7V_1$

$V_6e_5V_5e_4V_4e_3V_3e_2V_2e_1V_1e_7V_4e_6V_6$

Fig. 1.40



Solution:

1. Open walk which is not a trail the edge e_2 is repeated.
2. Trail which is not a path (the vertex v_4 is repeated)
3. Trail which is a path
4. Closed walk which is a cycle.
5. Closed walk which is a circuit but not a cycle (the vertex v_4 is repeated)

EULER CIRCUITS AND EULER TRAILS.

Consider a connected graph G . If there is a circuit in G that contains all the edges of G . Then that circuit is called an **Euler circuit** (or Eulerian line, or Euler tour) in G . If there is a trail in G that contains all the edges of G , than that trail is called an **Euler trail**.

Recall that in a trail and a circuit no edge can appear more than once but a vertex can appear more than once. This property is carried to Euler trails and Euler Circuits also.

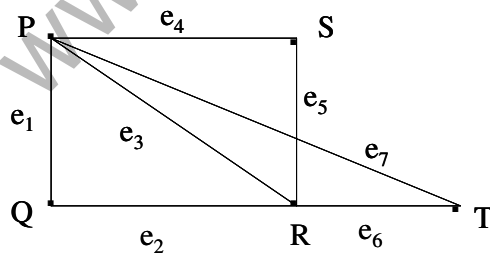
Since Euler circuits and Euler trails include all edge, then automatically should include all vertices as well.

A connected graph that contains an Euler circuit is called a **Semi Euler graph** (or a Semi Eulerian graph).

For Example, in the graph shown in figure 1.41 closed walk.

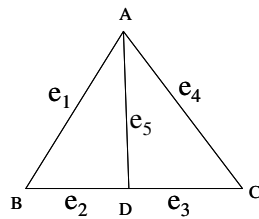
$Pe_1Qe_2Re_3Pe_4Se_5Re_6Te_7P$ is an Euler circuit. Therefore, this graph is a an Euler graph.

Fig 1.41



Consider the graph shown in fig.1.41. We observe that, in this graph, every sequence of edges which starts and ends with the same vertex and which includes all edges will contain at least one repeated edge. Thus, the graph has no Euler circuits. Hence this graph is not an Euler graph.

Fig. 1.42



It may be seen that the trail $Ae_1Be_2De_3Ce_4Ae_5D$ in the graph in fig 1.42 is an Euler trail. This graph therefore a Semi – Euler Graph.

Example 1: Show that the following graph contains an Euler Circuits

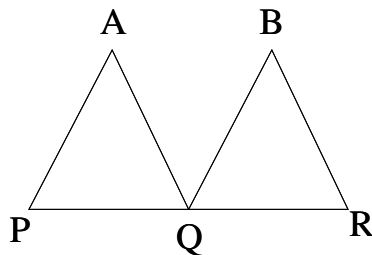
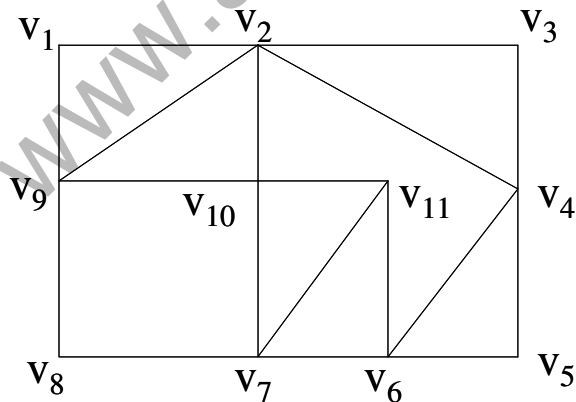


Fig.1.43

Solution: The graph contains an Euler Circuit PAQBRQP

Example 2: find an Euler circuit in the graph shown below.

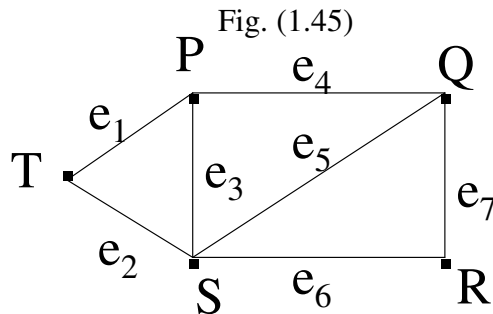
Fig.1.44



Solution:

$V_1V_2V_9V_{10}V_2V_{11}V_7V_{10}V_{11}V_6V_4V_2V_3V_4V_5V_6V_7V_8V_9V_1$

Example 3: show that the following graph contains an Euler trail.



Solution: the graph contains $Pe_1Te_2Se_3Pe_4Qe_5Se_6Re_7Q$ as an **Euler trail**.

ISOMORPHISM :

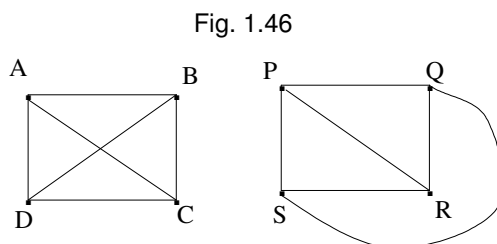
Consider two graphs $G = (V, E)$ and $G' = (V', E')$ suppose there exists a function $f : V \rightarrow V'$ such that (i) f is a one to one correspondence and (ii) for all vertices A, B of G $\{A, B\}$ is an edge of G if and only if $\{f(A), f(B)\}$ is an edge of G' , then f is called as **isomorphism** between G and G' , and we say that G and G' are **isomorphic graphs**.

In other words, two graphs G and G' are said to be isomorphic (to each other) if there is a one to one correspondence between their vertices and between their edges such that the adjacency of vertices is preserved such graphs will have the same structures, differing only in the way their vertices and edges are labelled or only in the way they are represented geometrically for any purpose, we regard them as essentially the same graphs.

When G and G' are isomorphic we write $G \cong G'$

Where a vertex A of G corresponds to the vertex $A' = f(A)$ of G' under a one to one correspondence $f : G \rightarrow G'$, we write $A \leftrightarrow A'$. Similarly, we write $\{A, B\} \leftrightarrow \{A', B'\}$ to mean that the edge AB of G and the edge $A'B'$ of G' correspond to each other, under f .

For example, look at the graphs shown in fig1.46



Consider the following one to one correspondence between the vertices of these two graphs.

$$A \leftrightarrow P, B \leftrightarrow Q, C \leftrightarrow R, D \leftrightarrow S$$

Under this correspondence, the edges in two graphs correspond with each other as indicated below:

$$\{A, B\} \leftrightarrow \{P, Q\}, \{A, C\} \leftrightarrow \{P, R\}, \{A, D\} \leftrightarrow \{P, S\}$$

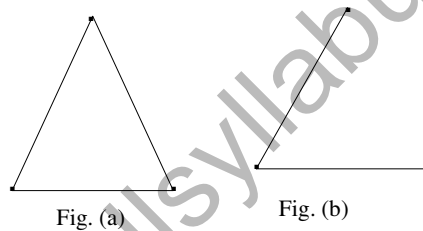
$$\{B, C\} \leftrightarrow \{Q, R\}, \{B, D\} \leftrightarrow \{Q, S\}, \{C, D\} \leftrightarrow \{R, S\},$$

We check that the above indicated one to one correspondence between the

Vertices / edges of the two graphs. Preserves the adjacency of the vertices. The existence of this correspondence proves that the two graphs are isomorphic (note that both the graphs represent the complete graph K_4).

Next, consider the graphs shown in figures 1.47 (a) and 1.47(b)

Fig. 1.47



We observe that the two graphs have the same number of vertices but different number of edges. Therefore, although there can exist one-to-one correspondence between the vertices, there cannot be a one-to-one correspondence between the edges. The two graphs are therefore not isomorphic.

From the definition of isomorphism of graphs, it follows that if two graphs are isomorphic, then they must have

1. The same number of vertices.
2. The same number of edges.
3. An equal number of vertices with a given degree.

These conditions are necessary but not sufficient. This means that two graphs for which these conditions hold need not be isomorphic.

In particular, two graphs of the same order and the same size need not be isomorphic. To see this, consider the graphs shown in figures 1.48(a) and (b).

Fig.1.48(a)

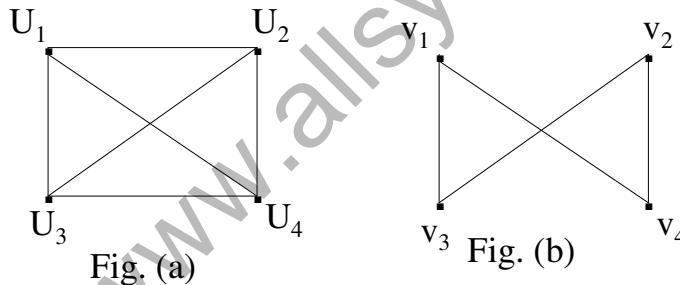


We note that both graphs are of order 4 and size 3. But the two graphs are not isomorphic. Observe that there are two pendant vertices in the first graph where as there are three pendant vertices in the second graph. As such, under any one-to-one correspondence between the vertices and the edges of the two graphs, the adjacency of vertices is not preserved

Example 1:

Prove that the two graphs shown below are isomorphic.

Fig.1.49



Solution: We first observe that both graphs have four vertices and four edges. Consider the following one – to- one correspondence between the vertices of the graphs.

$$u_1 \leftrightarrow v_1, u_2 \leftrightarrow v_4, u_3 \leftrightarrow v_3, u_4 \leftrightarrow v_2.$$

This correspondence give the following correspondence between the edges.

$$\{u_1, u_2\} \leftrightarrow \{v_1, v_4\}, \{u_1, u_3\} \leftrightarrow \{v_1, v_3\}$$

$$\{u_2, u_4\} \leftrightarrow \{v_4, v_2\}, \{u_3, u_4\} \leftrightarrow \{v_3, v_2\}.$$

These represent one-to-one correspondence between the edges of the two graphs under which the adjacent vartices in the first graph correspond to adjacent vertices in the second graph and vice-versa.

Example 2: Show that the following graphs are not isomorphic.

Fig. 1.50

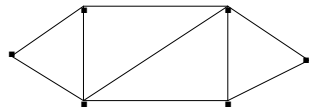


Fig. (a)

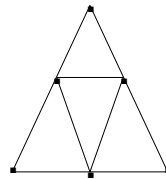


Fig. (b)

Solution: We note that each of the two graphs has 6 vertices and nine edges. But, the first graph has 2 vertices of degree 4 whereas the second graph has 3 vertices of degree 4. Therefore, there cannot be any one-to-one correspondence between the vertices and between the edges of the two graphs which preserves the adjacency of vertices. As such, the two graphs are not isomorphic.

UNIT II

Introduction the Graph Theory Contd.: Planner Graphs, Hamilton Paths and Cycles, Graph Colouring and Chromatic Polynomials.

PLANAR GRAPHS:

It has been indicated that a graph can be represented by more than one geometrical drawing. In some drawing representing graphs the edges intersect (cross over) at points which are not vertices of the graph and in some others the edges meet only at the vertices. A graph which can be represented by at least one plane drawing in which the edges meet only at vertices is called a '**planar graph**'

On the other hand, a graph which cannot be represented by a plane drawing in which the edges meet only at the vertices is called a **non planar graph**.

In other words, a non planar graph is a graph whose every possible plane drawing contains at least two edges which intersect each other at points other than vertices.

Example 1

Show that (i) a graph of order 5 and size 8, and (ii) a graph of order 6 and size 12, are planar graphs.

Solution: A graph of order 5 and size 8 can be represented by a plane drawing

Fig. 2.1

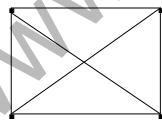


Fig. (a)

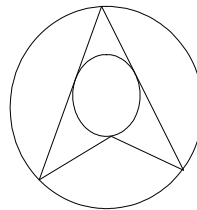


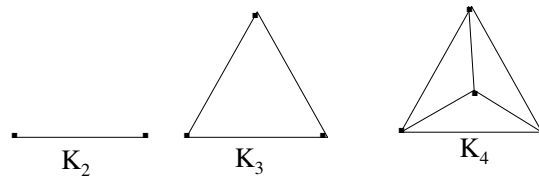
Fig. (b)

In which the edges of the graph meet only at the vertices, as shown in fig. 2.1 (a) therefore, this graph is a planar graph. Similarly, fig. 2.1(b) shows that a graph of order 6 and size 12 is a planar graph.

Example 2:

Show that the complete graphs K_2, K_3 and K_4 are planar graphs.

Fig. 2.2

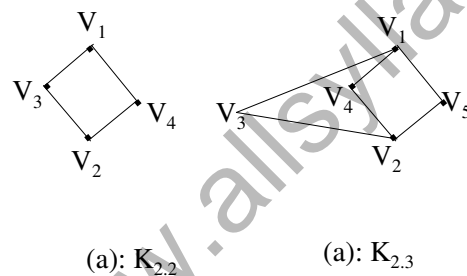


Solution: the diagrams in fig 2.2 represent the graphs K_2, K_3, K_4 . In none of these diagrams, the edge meet at points other than the vertices. Therefore K_2, K_3, K_4 are all planar graphs.

Example 3:

Show that the bipartite graphs $K_{2,2}$ and $K_{2,3}$ are planar graphs.

Fig. 2.3



Solution: In $K_{2,2}$, the vertex set is made up of two bipartites V_1, V_2 , with V_1 containing two vertices say V_1, V_2 and V_2 containing two vertices, say V_3, V_4 , and there is an edge joining every vertex in V_1 with every vertex in V_2 and vice-versa. Fig 2.3(a) represents this graph. In this fig. the edges meet only at the vertices therefore, $K_{2,2}$ is a planar graph.

In $K_{2,3}$ the vertex set is made up of two bipartites V_1 and V_2 , with V_1 containing two vertices, say V_1, V_2 , and V_2 containing three vertices, say V_3, V_4, V_5 and there is an edge joining every vertex in V_1 with every vertex in V_2 and Vice Versa. Fig. 2.3(b) represents this graph. In this figure the edges meet only at the vertices, therefore $K_{2,3}$ is a planar graph.

Example4:

Show that the complete graph K_5 (viz., the Kuratowski's first graph) is a non planar graph.

Solution:

We first recall that in the complete graph K_5 there are 5 vertices and there is an edge between every pair of vertices, totaling to 10 edges. (see fig. Ref. complete graph). This fig is repeated below with the vertices named as V_1, V_2, V_3, V_4, V_5 and the edges named $e_1, e_2, e_3, \dots, e_{10}$

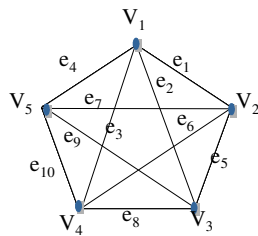


Fig. 2.4

In the above drawing of K_5 , the five edges $e_1, e_5, e_8, e_{10}, e_4$ form a pentagonal cycle and the remaining five edges e_2, e_3, e_6, e_7, e_9 are all inside this cycle and intersect at points other than the vertices.

Let us try to draw a diagram of K_5 in which the edges meet only at the vertices. In the pentagonal cycle present in fig (2.4) the edges meet only at the vertices. Let us start our new drawing of K_5 with this cycle: the cycle is shown in fig. 2.5 (a)

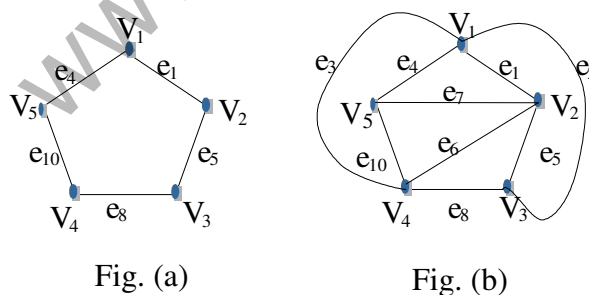


Fig. 2.5

Fig. (a)

Fig. (b)

Consider the edge $e_7 = \{V_2V_5\}$. This edge can be drawn either inside or outside the pentagonal cycle. Suppose we draw it inside, as shown in fig. 2.5 (b) the other case is similar now, consider the edges $e_2 = \{V_1V_3\}$ & $e_3 = \{V_1V_4\}$. If we draw these edges also inside the pentagon, they will intersect e_7 , that is, they cross e_7 at points, which are not vertices, therefore, let us draw of them outside: see fig. 2.5 (b).

Next consider the edge $e_6 = \{V_2,V_4\}$ if we draw this edge outside the pentagon intersects the edge e_2 ; see fig 2.5(b) therefore let us draw e_6 inside the pentagon.

Lastly, consider the edge $e_9 = \{V_3,V_5\}$ If we draw this edge outside the pentagon, it intersects the edge e_3 , and if we draw it inside, it intersects the edge e_6 .

This demonstrates that in every possible plane drawing of K_5 at least two edges of K_5 intersect at a point which is not a vertex of K_5 .

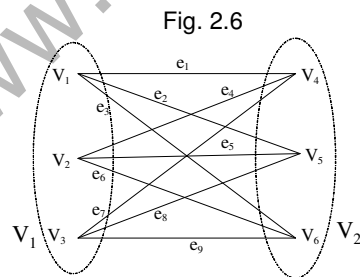
This proves that K_5 is a non planar graph.

Example 5:

Show that the complete bipartite graph $K_{3,3}$ (namely the Kuratowski's second graph) is a non-planar graph.

Solution:

by definition, $K_{3,3}$ is a graph with 6 vertices and 9 edges, in which the vertex set is made up of two bipartites V_1 and V_2 each containing three vertices such that every vertex in V_1 is joined to every vertex in V_2 by an edge and vice-versa.



Let us name the vertices in V_1 as v_1, v_2, v_3 and the vertices in V_2 as v_4, v_5, v_6 . Also let the edges be named as $e_1, e_2, e_3, \dots, e_9$.

A diagram of the graph is shown in fig (2.6). In this diagram of $K_{3,3}$, the six edges $e_1 = \{v_1, v_4\}$, $e_4 = \{v_2, v_4\}$, $e_5 = \{v_2, v_5\}$, $e_8 = \{v_3, v_5\}$, $e_9 = \{v_3, v_6\}$ and $e_3 = \{v_6, v_1\}$ form a hexagonal cycle and the remaining three edges e_2, e_6, e_7 either intersect these edges or intersect among themselves at points other than the vertices.

Let us try to draw a diagram of $K_{3,3}$ in which no two of its edges intersect. The hexagonal cycle present in fig.2.6 does not contain any mutually intersecting edges. Let us start our new drawing of $K_{3,3}$ with this cycle. This cycle is exhibited separately in fig. 2.7 (a)

Fig. 2.7

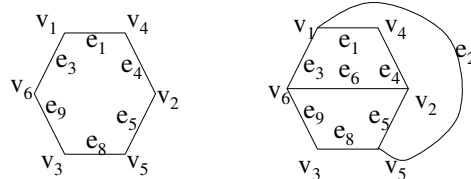


Fig. (a)

Fig. (b)

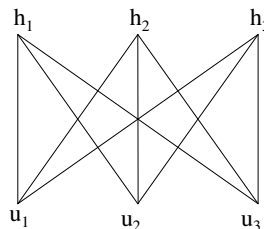
Consider three edge $e_6 = \{v_2, v_6\}$ this edge can be drawn either inside the hexagonal cycle or outside it. Let us draw it inside (as shown in fig.2.7 (b) the other case is similar. Now consider the edge $e_2 = \{v_1, v_5\}$. If we draw this edge the hexagon, it intersects the edges e_6 . Therefore, let us draw it outside the hexagon see fig. 2.7 (b)

Next consider the edge $e_7 = \{v_3, v_4\}$. If this edge is drawn inside the hexagon, it intersects the edge e_6 , and if it is drawn outside the hexagon, it intersects the edge e_2 . This demonstrates that in every possible plane drawing of $K_{3,3}$, at least two edges of $K_{3,3}$ intersect at a point which is not a vertex of $K_{3,3}$. this proves that $K_{3,3}$ is a non planar graph.

Example 6

Suppose there are three houses and three utility points (electricity, water sewerage, say) which are such that each utility point is joined to each house. Can the lines of joining be such that no two lines cross each other ?

Fig. 2.8



Solution:

Consider the graph in which the vertices are the three houses (h_1, h_2, h_3) and the three utility points (u_1, u_2, u_3). Since each house is joined to each utility point. The graph has to be $K_{3,3}$ (see fig. 2.8). This graph is non-planar and therefore, in its plane drawing, at least two of its edges cross each other. As such, it is not possible to have the lines joining the houses and the utility points such that no two lines cross each other.

HAMILTON CYCLES AND HAMILTON PATHS

Let G be a connected graph. If there is a cycle in G that contains all the vertices of G , then that cycle is called a '**Hamilton Cycle**' in G .

A Hamilton cycle in a graph of n vertices consists of exactly n edges, because, a cycle with n vertices has n edges.

By definition, a Hamilton cycle in Graph G must include all vertices in G , This does not mean that it should include all edges of G .

A graph that contains a Hamilton cycle is called a **Hamilton graph** (or Hamiltonian graph).

For example, in the graph shown in fig. (2.7), the cycle shown in thick lines is a Hamilton cycle. (observe that this cycle does not include the edge BD). the graph is therefore a Hamilton graph.

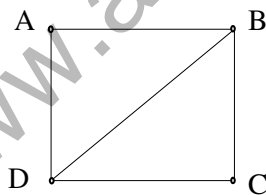


Fig.2.7

A path (if any) in a connected graph which includes every vertex (but not necessarily every edge) of the graph is called a Hamilton / Hamiltonian path in the graph.

For example: In the graph shown in fig (2.8), The path shown in thick lines is a Hamilton path.

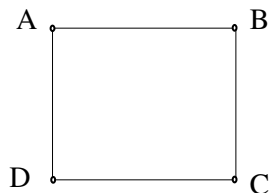


Fig. 2.8

In the graph shown in fig. (2.9), the path ABCFEDGHI is a Hamilton path. We check that this graph does not contain a Hamilton cycle.

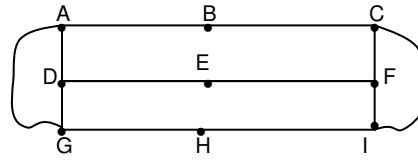


fig. (2.9)

Since a Hamilton path in a graph G meets every vertex of G , the length of a Hamilton path (if any) in a connected graph of n vertices is $n-1$ (a path with n vertices has $n-1$ edges)

Theorem 1:

If in a simple connected graph with n vertices (where $n \geq 3$) The sum of the degrees of every pair of non-adjacent vertices is greater than or equal to n , then the graph is Hamiltonian.

Theorem 2:

If in a simple connected graph with n vertices (where $n \geq 3$) the degree of every vertex is greater than or equal to $n/2$. then the graph is Hamiltonian .

Proof: If in a simple connected graph with n vertices, the degree of each vertex is greater than or equal to $n/2$. then the sum of the degrees of every pair of adjacent or non-adjacent vertices is greater than or equal to n , therefore, the graph is Hamiltonian (by Them 1).

Example 1:

Prove that the complete graph K_n where $n \geq 3$, is a hamilton graph.

Solution: In K_n , the degree of every vertex is $n-1$, if $n \geq 3$, we have $n-2 > 0$, or $2n-2 > n$, or $(n-1) > n/2$.

Thus, in K_n , where $n \geq 3$, the degree of every vertex is greater than $n/2$. Hence K_n is Hamiltonian by Them. 2.

Example 2:

Show that every simple K - Regular graph with $2K-1$ vertices is Hamiltonian.

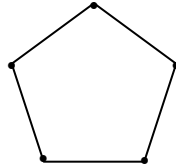
Solution: In a K - Regular graph , the degree of every vertex is K , and $K > K - 1/2 = 1/2 (2K - 1) = 1/2 n$. Where $n = 2K-1$ is the number of vertices, therefore, by Them. 2, the graph considered is Hamiltonian if it is simple.

Example 3:

Disprove the converses of theorems 1 and 2.

Solution: Consider a 2 – Regular graph with $n=5$, vertices, shown in fig. (2.10)

Fig. 2.10



Evidently, this graph is Hamiltonian. But the degree of every vertex is 2 which is less than $n/2$ and the sum of the degrees of every pair of vertices is 4 which is less than n . Thus, the converses of theorems 1 & 2 are not necessarily true.

Example 4:

Let G be a simple graph with n vertices and m edges where m is at least 3. if $m \geq \frac{1}{2}(n-1)(n-2)+2$. Prove that G is Hamiltonian. Is the converse true?

Solution :

Let u & v be any two non-adjacent vertices in G . Let x & y be their respective degrees. If we delete u, v from G , we get a subgraph with $n-2$ vertices. If this subgraph has q edges, then $q \leq \frac{1}{2}(n-2)(n-3)$. [in a simple graph of order n , the number of edges is $\leq \frac{1}{2}n(n-1)$] since u and v are non adjacent.

$m = q + x + y$, Thus

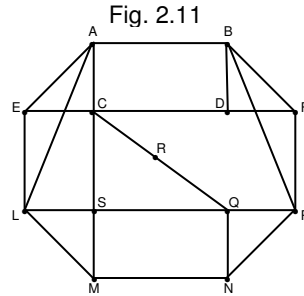
$$x + y = m - q$$

$$\geq \left\{ \frac{1}{2}(n-1)(n-2)+2 \right\} - \left\{ \frac{1}{2}(n-2)(n-3) \right\}$$

$$= n$$

Therefore, by Theorem 1, the graph is Hamiltonian. The converse of the result just proved is not always true. Because, a 2- Regular graph with five vertices shown in fig (2.10) is Hamiltonian but the inequality does not hold.

Example 5: Show that the graph shown in fig (2.11) is a Hamilton graph.

**Solution:**

By examining the given graph, we notice that in the graph there is a cycle AELSMNPQRCDFA which contains all the vertices of the graph. This cycle is a Hamiltonian cycle. Since the graph has a Hamiltonian cycle in it, the graph is a Hamiltonian graph.

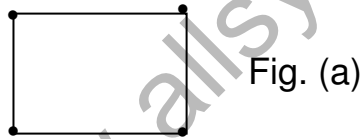
Example 6:

Exhibit the following.

(a): A graph which has both an Euler Circuit and a Hamilton cycle.

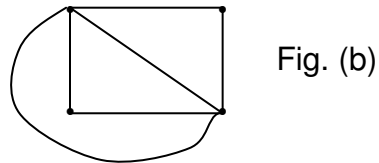
Solution:

The graph shown is the required graph.



(b) : A graph which has an Euler circuit but no Hamilton cycle.

Solution: The graph shown is the required graph.



(C) A graph which has a Hamilton cycle but no Euler Circuit.

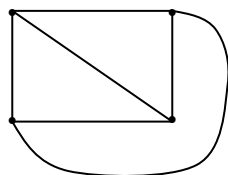


Fig. (c)

(d): A graph which has neither a Hamilton cycle nor an Euler circuit.



Fig. (d)

The following theorem contains useful information on the existence of Hamilton cycle in the complete graph K_n .

Theorem 3: In the complete graph with n vertices, where n is an odd number ≥ 3 , there are $(n-1)/2$ edge - disjoint Hamiltonian cycles.

Proof:

Let G be a complete graph with n vertices, where n is odd and ≥ 3 . Denote the vertices of G by $1, 2, 3, \dots, n$ and Represent them as points as shown in fig. (2.12)

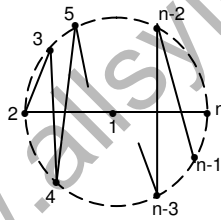


Fig. 2.12

We note that the polygonal pattern of edges from vertex 1 to vertex n as depicted in the fig is a cycle that includes all the vertices of G . This cycle is therefore a Hamilton cycle. This representation demonstrates that G has at least one Hamilton cycle. (In the fig (2.12)), the vertex 1 is at the centre of a circle and the other vertices are on its circumference. The circle is dotted.

Now, rotate the polygonal pattern clockwise by $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_k$ degrees where $\alpha_1 = 360^\circ/n-1, \alpha_2 = 2 \cdot 360^\circ/n-1, \alpha_3 = 3 \cdot 360^\circ/n-1, \dots, \alpha_k = (n-3)/2 \cdot 360^\circ/n-1$

Each of these $K = (n-3)/2$ rotations gives a Hamilton cycle that has no edge in common with any of the preceding ones. Thus, there exists $k = (n-3)/2$, new Hamilton cycles, all edge - disjoint from the one shown in fig (2.12) and also edge - disjoint among themselves thus, in G , there are exactly.

$$1+K = 1 + (n-3)/2 = 1/2 (n-1)$$

Mutually edge -disjoint Hamilton cycle.

This completes the proof of the theorem.

Example 7:

How many edge - disjoint Hamilton cycles exist in the complete graph with seven vertices? Also, draw the graph to show these Hamilton cycles.

Solution:

According to theorem 3, the complete graph K_n has $(n-1)/2$ edge - disjoint Hamilton cycles when $n \geq 3$ and n is odd. When $n = 7$, their number is $(7-1)/2 = 3$. As indicated in the proof of Theorem 3 .

One of these Hamilton cycles appears as shown in fig (2.13)

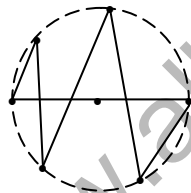


Fig. 2.13

The other two cycles are got by rotating the above shown cycle clock wise through angles.

$$\alpha_1 = 360^\circ/7-1, = 60^\circ, \text{ and } \alpha_2 = 2(360^\circ)/7-1, = 120^\circ$$

TRAVELING -SALESMAN PROBLEM :

A problem closely related to the question of Hamiltonian circuits is the traveling sales man problem, stated as follows: A sales man is required to visit a number of cities during a trip, given the distances between the cities, in what order should be travel so as to visit every city precisely once and return home, with the minimum mileage traveled ?

Representing the cities by vertices and the roads between them by edges, we get a graph. In this graph, with every edge e_i there is associated a real number (the distance in miles, say), $w(e_i)$ such a graph is called a weighted graph; $w(e_i)$ being the weight of edge e_i .

In our problem, if each of the cities has a road to every other city, we have a 'complete weighted graph'. This graph has numerous Hamiltonian circuits, and we are to pick the one that has the smallest sum of distances (or weights)

The total number of different (not edge - disjoint, of course) Hamiltonian circuits in a complete graph of n vertices can be shown to be $(n-1)!/2$.

This follows from the fact that starting from any vertex we have $n-1$ edges to choose from the first vertex, $n-2$ from the second, $n-3$ from the third, and so on. These being independent choices.

We get $(n-1)!$ possible number of choices. This number is, however, divided by 2, because each Hamiltonian circuit has been counted twice.

Theoretically, the problem of the traveling salesman can always be solved by enumerating all $(n-1)!/2$ Hamiltonian circuits, calculating the distance traveled in each, and then picking the shortest one. However for a large value of n , the labor involved is too great even for a digital computer (try solving it for the 50 state capitals in the united states: $n = 50$).

The problem is to prescribe a manageable algorithm for finding the shortest route. No efficient algorithm for problems of arbitrary size has yet been found, although many attempts have been made. Since this problem has applications in operations research, some specific large - scale examples have been worked out. There are also available several heuristic methods of solution that give a route very close to the shortest one.

SUMMARY:

In this chapter we discussed the subgraph – a graph that is part of another graph, walks, path, circuits, Euler lines, Hamiltonian paths, and Hamiltonian circuits in a graph G are its subgraphs with special properties. A given graph G can be characterized and studied in terms of the presence or absence of these sub graphs. Many physical problems can be represented by graphs and solved by observing the relevant properties of the corresponding graphs.

Various types of walks

Discussed in this chapter are summarized in fig (2.14). The arrows point in the direction of increasing restriction.

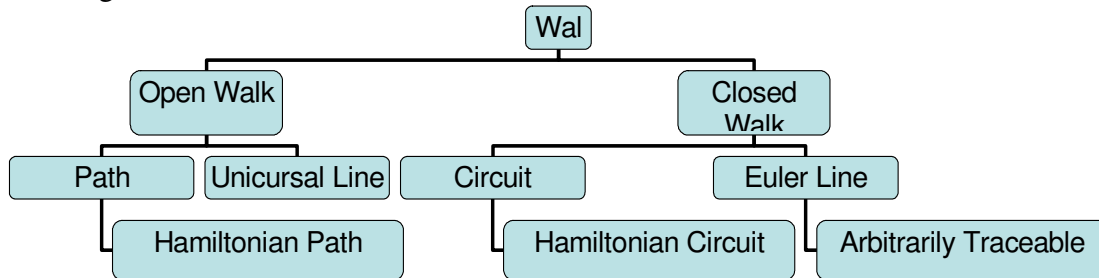


Fig. 2.14 Different Types of Walks

GRAPH COLORING:

Given a planar or non-planar graph G , if we assign colors (colours) to its vertices in such a way that no two adjacent vertices have (receive) the same color, then we say that the graph G is Properly colored.

In other words, proper coloring of a graph means assigning colors to its vertices such that adjacent vertices have different colors.

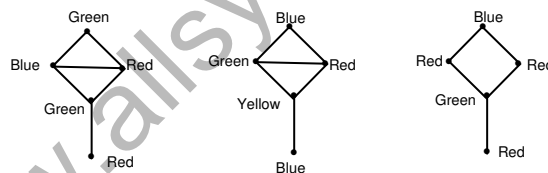


Fig. 2.15

In fig. (2.15), the first two graphs are properly colored whereas the third graph is not properly colored.

By Examining the first two graphs in fig (2.15) which are properly colored, we note the following

- i) A graph can have more than one proper coloring.
- ii) Two non-adjacent vertices in a properly colored graph can have the same color.

CHROMATIC NUMBER:

A graph G is said to be K -colorable if we can properly color it with K (number of) colors.

A graph G which is K -colorable but not $(K-1)$ -colorable is called a ' **K -Chromatic graph**'.

In other words, a K -Chromatic graph is a graph that can be properly colored with K colors but not with less than K colors.

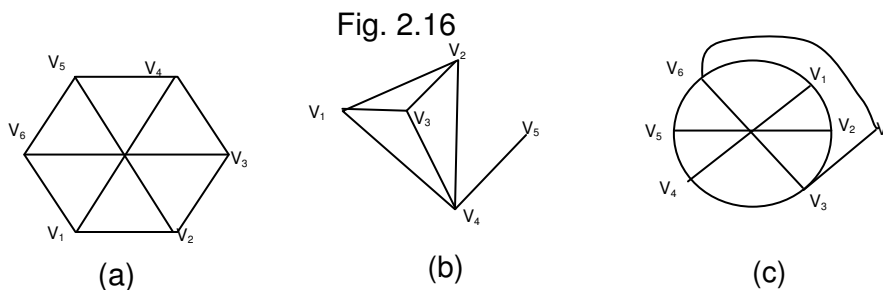
If a graph G is K -Chromatic, then K is called the chromatic number of G . Thus, the chromatic number of a graph is the minimum number of colors with which the graph can be properly colored. The chromatic number of a graph G is usually denoted by $\chi(G)$.

SOME RESULTS:

- i) A graph consisting of only isolated vertices (ie., Null graph) is 1-Chromatic (Because no two vertices of such a graph are adjacent and therefore we can assign the same color to all vertices).
- ii) A graph with one or more edges is at least 2-chromatic (Because such a graph has at least one pair of adjacent vertices which should have different colors).
- iii) If a graph G contains a graph G_1 as a subgraph, then

$$\chi(G) \geq \chi(G_1).$$
- iv. If G is a graph of n vertices, then $\chi(G) \leq n$.
- v. $\chi(K_n) = n$, for all $n \geq 1$. (Because, in K_n , every two vertices are adjacent and as such all the n vertices should have different colors)
- vi. If a graph G contains K_n as a subgraph, then $\chi(G) \geq n$.

Example 1: Find the chromatic number of each of the following graphs.



Solution :

i) For the graph (a), let us assign a color α to the vertex V_1 , then for a proper coloring, we have to assign a different color to its neighbors V_2, V_4, V_6 , since V_2, V_4, V_6 are mutually non-adjacent vertices, they can have the same color as V_1 , namely α .

Thus, the graph can be properly colored with at least two colors, with the vertices V_1, V_3, V_5 having one color α and V_2, V_4, V_6 having a different color β . Hence, the chromatic number of the graph is 2.

ii) For the graph (b), let us assign the color α to the vertex V_1 . Then for a proper coloring its neighbours V_2, V_3 & V_4 cannot have the color α .

Further more, V_2, V_3, V_4 must have different colors, say β, γ, δ . Thus, at least four colors are required for a proper coloring of the graph.

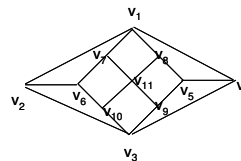
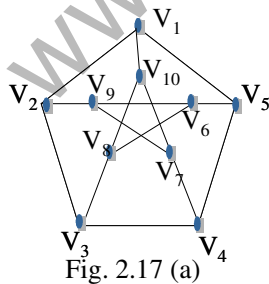
Hence the chromatic number of the graph is 4.

iii) For the graph (c), we can assign the same color, say α , to the non-adjacent vertices V_1, V_3, V_5 .

Then the vertices V_2, V_4, V_6 consequently V_7 and V_8 can be assigned the same color which is different from both α and β . Thus, a minimum of three colors are needed for a proper coloring of the graph. Hence its chromatic number is 3.

Example 2: Find the chromatic numbers of the following graphs.

Fig. 2.17

**Solution (i):**

We note that the graph (a) is the Peterson graph. By observing the graph, we note that the vertices V_1, V_3, V_6 and V_7 can be assigned the same color, say α . Then the vertices V_2, V_4, V_8 and V_{10} can be assigned the same color, β (other than α). Now, the vertices V_5 and V_9 have to be assigned colors other than α and β ; they can have the

same color γ . Thus, a minimum of three colors are required for a proper coloring of this graph. Hence, the chromatic number of this graph is 3.

Solution (ii) :

By observing the graph (b), (this graph is called the **Herschler graph**), we note that the vertices V_1, V_3, V_5, V_6 and V_{11} can be assigned the same color α and all the remaining vertices: V_2, V_4, V_7, V_8, V_9 and V_{10} can be assigned the same color β (other than α). Thus two colors are sufficient

(one color is not sufficient) for proper coloring of the graph. Hence its chromatic number is 2.

Example (3):

Prove that a graph of order n (≥ 2) consisting of a single cycle is 2-chromatic if n is even and 3-chromatic if n is odd.

Solution:

The graph being considered is shown as below.

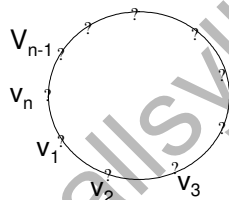


Fig. 2.18

Obviously, the graph cannot be properly colored with a single color. Assign two colors alternatively to the vertices, starting with V_1 . Then, the odd vertices, V_1, V_3, V_5 etc., will have a color α and the even vertices V_2, V_4, V_6 will have a different color β . Suppose n is even, then the vertex V_n is an even vertex and therefore will have the color β , and the graph gets properly colored therefore, the graph is 2-chromatic

Suppose n is odd, then the vertex V_n is an odd vertex and therefore will have the color α and the graph is not properly colored (because, then the adjacent vertices V_n and V_1 will have the same color α). To make it properly colored, it is enough if V_n is assigned a third color, γ . Thus, in this case, the graph is 3-chromatic.

Example 4:

Prove that a graph G is 2-chromatic if and only if it is non-null bipartite graph.

Solution:

Suppose a graph G is 2-chromatic. Then it is non-null and some vertices of G have one color, say α and the rest of the vertices have another color, say β . Let V_1 be the set of vertices having color α and V_2 be the set of vertices having color β . Then $V_1 \cup V_2 = V$. The vertex set of G , and $V_1 \cap V_2 = \Phi$. Also, no two vertices of V_1 can be adjacent and no two vertices of V_2 can be adjacent. As such, every edge in G has one end in V_1 and the other end in V_2 . Hence G is bipartite graph.

Conversely, suppose G is a non-null bipartite graph. Then the vertex set of G has two bipartites V_1 and V_2 such that every edge in G has one end in V_1 and another end in V_2 . Consequently, G cannot be properly colored with one color; because then vertices in V_1 and V_2 will have the same color and every edge has both of its ends of the same color. Suppose we assign a color α to all vertices in V_1 and a different color β to all vertices in V_2 . This will make a proper coloring of V . Hence G is 2-Chromatic.

Example 5 :

If $\Delta(G)$ is the maximum of the degrees of the vertices of a graph G , then prove that $\chi(G) \leq 1 + \Delta(G)$ (i)

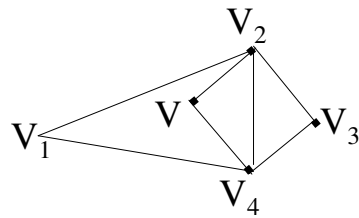
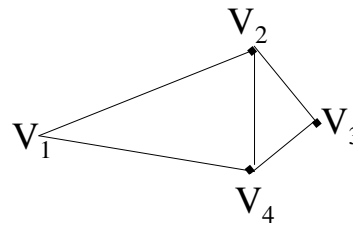
Solution:

Suppose G contains $n = 2$ vertices, then the degrees of both the vertices is 1, so that $\Delta(G) = 1$, also $\chi(G) = 2$. Hence $\chi(G) = 1 + \Delta(G)$.

Thus, the required inequality (i) is verified for $n=2$.

Assume that the inequality is true for all graphs with K - vertices. Consider a graph G' with $K + 1$ vertices. If we remove any vertex v from G' then the resulting graph H will have K vertices and $\Delta(H) \leq \Delta(G')$. since H has K vertices, the inequality (i) holds for H (by the assumption made). Therefore, $\chi(H) \leq 1 + \Delta(H)$. since $\Delta(H) \leq \Delta(G')$, this yields $\chi(H) \leq 1 + \Delta(G')$

Now, a proper coloring of G' can be achieved by retaining the colors assigned to the vertices in H and by assigning a color to v that is different from the colors assigned to the vertices adjacent to it.

Fig. (a): G' Fig. (b): H

The color to be assigned to V can be one of the colors already assigned to a vertex in H that is not adjacent to V . Thus, a proper

Coloring of G' can be done without the use of a new color.

Hence $\chi(G') = \chi(H) \leq 1 + \Delta(G')$.

Thus, if the inequality (1) holds for all graphs with K vertices, it holds for a graph with $K + 1$ vertices. Hence, by induction, it follows that the inequality (1) holds for all graphs.

EULER'S FORMULA

If G is a planar graph, then G can be represented by a diagram in a plane. In which the edges meet only at the vertices. Such a diagram divides the plane into a number of parts called **regions (or faces)**, of which exactly one part is unbounded. The number of edges that form the boundary of a region is called the **degree** of that region.

For example, in the diagram of a planar graph shown in fig. (2.20) the diagram divides the plane into 6 regions $R_1, R_2, R_3, R_4, R_5, R_6$. We observe that each of the regions R_1 to R_5 is bounded and the region R_6 is unbounded. That is, R_1 to R_5 are in the interior of the graph while R_6 is in the **Exterior**.

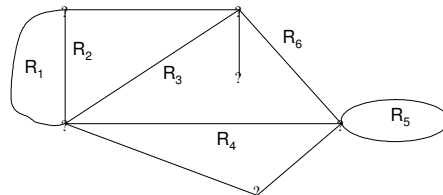


Fig. 2.20

We further observe that, the fig (2.20) the boundary of the region R_1 is made up of two edges. Therefore, the degree of R_1 is 2. We write this as $d(R_1) = 2$. The boundary of each of the regions R_2 and R_4 is made up of 3 edges; therefore, $d(R_2) = d(R_4) = 3$. The boundary of the region R_3 consists of 4 edges of which one is pendant edge.

Therefore, $d(R_3) = 5$. The region R_5 is bounded by a single edge (loop) therefore, $d(R_5) = 1$. The boundary of the exterior region R_6 consists of six edges; therefore, $d(R_6) = 6$. We note that

$$d(R_1) + d(R_2) + d(R_3) + d(R_4) + d(R_5) + d(R_6) = 20.$$

Which is twice the number of edges in the graph. This property is analogous to the handshaking property and is true for all planar graphs.

It should be pointed out that the regions are determined by a diagram of a planar graph and not by the graph itself. This means that if we change the diagram of the graph, the regions determined by the new diagram will be generally different from those determined by the old one in the sense that the unbounded region in the old diagram need not be unbounded in the new diagram. However, the interesting fact is **that the total number of regions in the two diagrams remains the same.**

The proof of this fact is contained in the following **Euler's fundamental theorem** on planar graphs.

Theorem:

A connected planar graph G with n vertices and m edges has exactly $m - n + 2$ regions in all of its diagrams.

Proof:

Let r denote the number of regions in a diagram of G . The theorem states that,
 $r = m - n + 2$, or $n - m + r = 2$ (1)

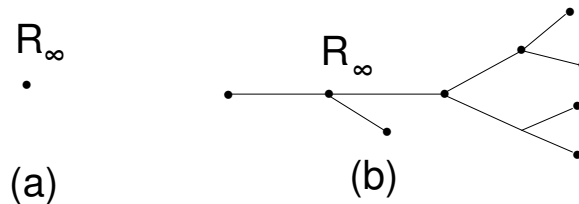
We give the proof by induction on m .

If $m = 0$, then n , must be equal to 1. Because, if $n > 1$, then G will have at least two vertices and there must be an edge connecting them (because G is connected), so that $m \neq 0$, which is a contradiction.

If $n = 1$, a diagram of G determines only one region – the entire plane region (as shown in fig 2.21 (a)).

Thus , if $m = 0$, then $n = 1$ and $r = 1$, so that $n-m +r = 2$. This verifies the theorem for $m = 0$.

Fig. 2.21



Now, assume that the theorem holds for all graphs with $m=k$ number of edges, where k is a non-negative integer.

Consider a graph G_{k+1} with $k + 1$ edges and n vertices. First, suppose that G_{k+1} has no cycles in it. Then a diagram of G_{k+1} will be of the form shown in fig. 2.21 (b) in which the number of vertices will be exactly one more than the number of edges, and the diagram will determine only one region—the entire plane region (as in fig. 2.21 (b)). Thus for G_{k+1} , we have, in this case, $n = (k+1)+1$ and $r = 1$, so that

$$n - (k+1) + r = 2.$$

This means that the result (i) is true when $m=k+ 1$ as well, if G_{k+1} contains no cycles in it.

Next, suppose G_{k+1} contains at least one cycle. Let r be the number of regions which a diagram of G_{k+1} determine. Consider an edge 'e' in a cycle and remove it from G_{k+1} . The resulting graph, $G_{k+1} - e$, will have n vertices and $(k+1)-1=k$ edges, and its diagram will determine $r-1$ regions. Since $G_{k+1} - e$ has k edges, the theorem holds for this graph (by the induction assumption made).

That is we have

$$r - 1 = k - n + 2, \text{ or } n - (k + 1) + r = 2$$

This means that in this case also the result (1) is true when $m = k + 1$ as well.

Hence, by induction, it follows that the result (1) is true for all **non- negative** integers m . This completes the proof of the theorem.

Corollary 1 :

If G is connected simple planar graph with n (≥ 3) vertices, m (> 2) edges and r regions, then (i) $m \geq \frac{3}{2}r$ and (ii) $m \leq 3n - 6$.

Proof:

Since the graph G is simple, it has no multiple edges and no loops. As such, every region must be bounded by three or more edges. Therefore, the total number of edges that bound all the regions is greater than or equal to $3r$. On the other hand, an edge is in the boundary of at most two regions. Therefore, the total number of edges that bound all regions is less than or equal to $2m$. Thus, $3r \leq 2m$. or $m \geq \frac{3}{2}r$

This is required result (i) .

Now, substituting for r from Euler's formula in the result just proved, we get $m \geq \frac{3}{2}(m - n + 2)$

Which simplifies to $m \leq 3n - 6$. This is required result (ii)

Corollary 2:

Kuratowski's first graph, K_5 , is non-planar.

Proof:

The graph K_5 is simple, connected and has $n = 5$ vertices and $m = 10$ edges; refer to figure Kuratowski's first graph. If this graph is planar, then by result (ii) of Corollary 1, we should have $m \leq 3n - 6$; that is $10 \leq 15 - 6$, which is not true. Therefore, K_5 is non-planar

Corollary 3:

Kuratowski's second graph, $K_{3,3}$, is non-planar.

Proof: We first note that $K_{3,3}$ is simple, connected and has $n = 6$ vertices and $m = 9$ edges; see fig Kuratowski's second graph.

Suppose $K_{3,3}$ is planar. By examining the figure Kuratowski's graph, we note that $K_{3,3}$ has no cycles of length 3. Therefore by result (iii) of Corollary 1, we should have $m \leq 2n - 4$; that is, $9 \leq 12 - 4$, which is not true. Hence, $K_{3,3}$ is non-planar.

Corollary 4:

Every connected simple planar graph G contains a vertex of degree less than 6.

Proof:

Suppose every vertex of G is of degree greater than or equal to 6. Then, if d_1, d_2, \dots, d_n are the degrees of the n vertices of G , we have $d_1 \geq 6, d_2 \geq 6, \dots, d_n \geq 6$.

Adding these, we get

$$d_1 + d_2 + \dots + d_n \geq 6n.$$

By handshaking property, the left hand side of this inequality is equal to $2m$, where m is the number of edges in G , thus, $2m \geq 6n$, or $3n \leq m$.

On the other hand, by the result (ii) of corollary 1, (Result (ii) is $m \leq 3n - 6$).

We should have $m \leq 3n - 6$. Thus, $3n \leq m \leq 3n - 6$. This cannot be true.

Therefore, G must have a vertex of degree less than 6.

Example 1:

Verify Euler's formula for the planar graph shown in figure 2.20.

Solution:

The given graph has $n=6$ vertices, $m=10$ edges and $r=6$ regions. Thus,

$$n - m + r = 6 - 10 + 6 = 2.$$

The Euler's formula is thus verified for the given graph.

Example 2:

Verify Euler's formula for the planar graphs shown below:

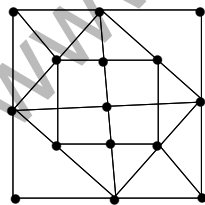


Fig. (a)

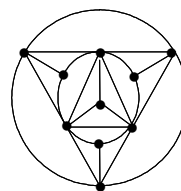


Fig. (b)

Fig. 2.22

Solution:

We observe that the first of the given graphs has $n = 17$ vertices, $m = 34$ edges and $r = 19$ regions. Thus, $n - m + r = 17 - 34 + 19 = 2$.

In the second of the given graphs, there are $n = 10$ vertices, $m = 24$ edges and $r = 16$ regions, so that $n - m + r = 10 - 24 + 16 = 12$.

Thus, for both of the given graphs, Euler's formula is verified.

Example 3:

For the diagram of a planar graph shown below, find the degrees of regions and verify that the sum of these degrees is equal to twice the number of edges

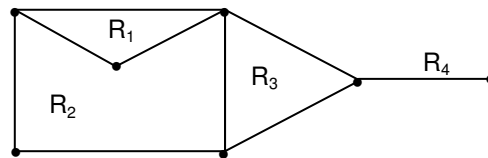


Fig. 2.23

Solution:

The diagram has 9 edges and 4 regions. The region R_1 is bound by three edges. Therefore, $d(R_1)=3$. Similarly, $d(R_2)=5$, $d(R_3)=3$.

The infinite region R_4 is bound by 5 edges plus a pendant edge. Therefore, $d(R_4)=7$. (Recall that while determining the degree of a region, a pendant edge is counted twice).

Accordingly,

$$\begin{aligned} d(R_1) + d(R_2) + d(R_3) + d(R_4) &= 18 \\ &= \text{twice the no. of edges.} \end{aligned}$$

Example 4:

A connected planar graph has 9 vertices with degrees 2,2,3,3,3,4,5,6,6. Find the number of regions of G .

Solution:

The given graph has $n = 9$ vertices. Let m be the number of edges and r be the number of regions.

Therefore by the Handshaking property, we have

$$\begin{aligned} 2m &= \text{sum of degrees of vertices} \\ &= 2+2+3+3+3+4+5+6+6 \\ &= 34. \end{aligned}$$

Therefore, $m = 17$.

By using Euler's formula, we find that

$$\begin{aligned} r &= m - n + 2. \\ &= 17 - 9 + 2 = 10 \end{aligned}$$

Thus, the given graph has 10 regions.

Example 5:

Show that every connected simple planar graph G with less than 12 vertices must have a vertex of degree ≤ 4 .

Solution:

Suppose every vertex of G has degree greater than 4. Then, if $d_1, d_2, d_3, d_4, \dots, d_n$ are the degrees of n vertices of G , we have

$$d_1 \geq 5, d_2 \geq 5, \dots, d_n \geq 5 \text{ so that,}$$

$$d_1 + d_2 + d_3 + d_4 + \dots + d_n \geq 5n, \text{ or } 2m \geq 5n, \text{ by hand shaking property,}$$

$$\text{or } 5n/2 \leq m \dots \dots \dots (i)$$

On the other hand, Corollary 1 requires $m \leq 3n - 6$. Thus, we should have, in view of (i), $5n/2 \leq 3n - 6$ or $n \geq 12 \dots \dots \dots (ii)$

Thus, if every vertex of G has degree greater than 4, then G must have at least 12 vertices. Hence, if G has less than 12 vertices, it must have a vertex of degree ≤ 4 .

Example 6:

Show that if a planar graph G of order n and size m has r regions and k components, then $n - m + r = k + 1$.

Solution:

Let H_1, H_2, \dots, H_k be the k components of G . Let the number of vertices, the number of edges and the number of non - exterior regions in H_i be n_i, m_i, r_i respectively, $i = 1, 2, \dots, k$. the exterior region is the same for all components.

$$\text{Therefore, } \sum n_i = n, \quad \sum m_i = m, \quad \sum r_i = r - 1.$$

If the exterior region is not considered, then the Euler's formula applied to H_i yields

$$n_i - m_i + r_i = 1.$$

On summation (from $i = 1$ to $i = k$), this yields

$$n - m + (r - 1) = k, \text{ or } n - m + r = k + 1.$$

2.5.1 Chromatic Polynomials:

Given a connected graph G & λ number of different colors, let us take up the problem of finding the number of different ways of properly coloring G with these λ colors.

First, consider the null graph N_n with n vertices. In this graph, no two vertices are adjacent. Therefore, a proper coloring of this graph can be done by assigning a single color to all the vertices. Thus, if there are λ number of colors, each vertex of the graph has λ possible choices of colors assigned to it, and as such the graph can be properly colored in λ^n different ways

Next consider the complete graph K_n . In this graph, every two vertices are adjacent, and as such there must be at least n colors for a proper coloring of the graph. If the number of different colors available is λ , then the number of ways of properly coloring K_n is

- (i) Zero if $\lambda < n$,
- (ii) One if $\lambda = n$,
- (iii) Greater than 1 if $\lambda > n$.

Let $v_1, v_2, v_3, \dots, v_n$ be the vertices of K_n and suppose $\lambda > n$.

For a proper coloring of K_n , the vertex v_1 can be assigned any of the λ colors, the vertex v_2 can be assigned any of the remaining $\lambda - 1$ colors, the vertex v_3 can be assigned any of the remaining $\lambda - 2$ colors and finally the vertex v_n can be assigned any of the $\lambda - n + 1$ colors. Thus, K_n can be properly colored in $\lambda(\lambda - 1)(\lambda - 2) \dots (\lambda - n + 1)$ different ways if $\lambda > n$.

Lastly, consider the graph L_n which is a path consisting of n vertices $v_1, v_2, v_3, \dots, v_n$ shown below:

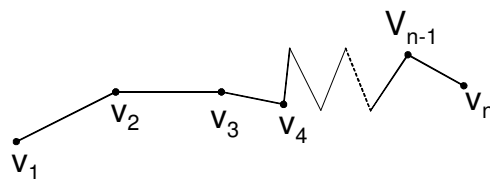


Figure 2.44

This graph cannot be properly colored with one color, but can be properly colored with 2 colors – by assigning one color to v_1, v_3, v_5, \dots and another color to v_2, v_4, v_6, \dots . Suppose there are $\lambda \geq 2$ number of colors available. Then, for a proper coloring of the graph, the vertex v_1 can be assigned any one of the λ colors and each of the remaining vertices can be assigned any one of $\lambda-1$ colors.

(Bear in mind that alternative vertices can have the same color). Thus, the graph L_n can be properly colored in $\lambda(\lambda-1)^{n-1}$ different ways.

The number of different ways of properly coloring a graph G with λ number of colors is denoted by $P(G, \lambda)$. Thus, from what is seen in the above three illustrate examples, we note that

$$(i) \quad P(N_n, \lambda) = \lambda^n,$$

$$(ii) \quad P(K_n, \lambda) = 0 \text{ if } \lambda < n,$$

$$P(K_n, n) = 1 \text{ if } \lambda = n, \text{ and}$$

$$P(K_n, \lambda) = \lambda (\lambda - 1)(\lambda - 2) \dots (\lambda - n + 1) \text{ if } \lambda > n,$$

$$(iii) \quad P(L_n, \lambda) = \lambda (\lambda - 1)^{n-1} \text{ if } \lambda \geq 2,$$

We observe that in each of the above cases, $P(G, \lambda)$ is a polynomial. Motivated by these cases, we take that $P(G, \lambda)$ is polynomial for all connected graph G . This polynomial is called the **Chromatic Polynomial**.

It follows that if a graph G is made up of n parts, G_1, G_2, \dots, G_n , then $P(G, \lambda)$ is given by the following

PRODUCT RULE:

$$P(G, \lambda) = P(G_1, \lambda) \cdot P(G_2, \lambda) \cdot \dots \cdot P(G_n, \lambda)$$

In particular, If G is made up of two parts G_1 and G_2 , then we have $P(G, \lambda) = P(G_1, \lambda) \cdot P(G_2, \lambda)$ so that

$$P(G_2, \lambda) = P(G, \lambda) / P(G_1, \lambda)$$

DECOMPOSITION THEOREM:

Let G be a graph and $e = \{a, b\}$ be an edge of G . Let $G_e = G - e$ be that subgraph of G which is obtained by deleting e from G without deleting vertices a and b . Suppose we construct a new graph G_e' by coalescing (identifying / merging) the vertices a and b in G_e . Then G_e' is subgraph of G_e as well as G .

The process of obtaining G_e and G_e' from G is illustrated in Figure 2.45.

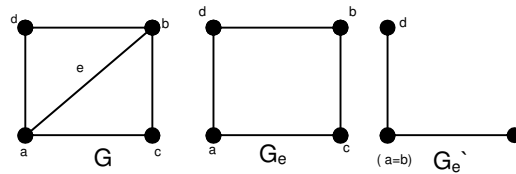


Figure 2.45

The following theorem called the Decomposition theorem for chromatic polynomials given an expression for $P(G, \lambda)$ in terms of $P(G_e, \lambda)$ and $P(G'_e, \lambda)$ for a connected graph G .

Theorem 1:

If G is a connected graph and $e = \{a, b\}$ is an edge of G , then

$$P(G_e, \lambda) = P(G, \lambda) + P(G'_e, \lambda)$$

Proof: In a proper coloring of G_e , the vertices a and b can have the same color or different colors. In every proper coloring of G , the vertices a and b have different colors and in every proper coloring of G'_e these vertices have the same color. Therefore, the number of proper colorings of G_e is the sum of the number of proper colorings of G and the number of proper colorings of G'_e . That is, $P(G_e, \lambda) = P(G, \lambda) + P(G'_e, \lambda)$

This completes the proof of the theorem.

MULTIPLICATION THEOREM

The following theorem gives an expression for $P(G, \lambda)$ for a special class of graphs.

Theorem 2: If a graph G has sub graphs G_1 and G_2 such that $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = K_n$ for some positive integer n , then

$$P(G, \lambda) = P(G_1, \lambda) \cdot P(G_2, \lambda) / \lambda^{(n)}$$

$$\text{Where } \lambda^{(n)} = \lambda(\lambda - 1)(\lambda - 2) \dots (\lambda - n + 1)$$

Given $\lambda > n$ number of different colors, there are $\lambda^{(n)} = \lambda(\lambda - 1)(\lambda - 2) \dots (\lambda - n + 1)$ number of proper colorings of K_n . For each of these $\lambda^{(n)}$ proper colorings of K_n , the product rule yields $P(G_1, \lambda) / \lambda^{(n)}$ ways of properly coloring the remaining vertices of G_1 . Similarly, there are $P(G_2, \lambda) / \lambda^{(n)}$ ways of properly coloring the remaining vertices of G_2 . As such

$$\begin{aligned}
 P(G, \lambda) &= P(K_n, \lambda) \cdot P(G_1, \lambda) / \lambda^{(n)} \cdot P(G_2, \lambda) / \lambda^{(n)} \\
 &= \lambda^{(n)} \cdot P(G_1, \lambda) / \lambda^{(n)} \cdot P(G_2, \lambda) / \lambda^{(n)} \\
 &= P(G_1, \lambda) \cdot P(G_2, \lambda) / \lambda^{(n)}
 \end{aligned}$$

This completes the proof of the theorem.

Example 1: Find the chromatic polynomial for the graph shown in Figure 2.46. What is its chromatic number ?

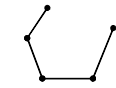


Figure 2.46

We observe that the given graph G is a path of length $n = 5$, namely L_5 . Therefore, its chromatic polynomial is

$$P(G, \lambda) = \lambda(\lambda-1)^{n-1} = \lambda(\lambda-1)^4$$

Next, we note that the chromatic number of the graph is $\chi(G)=2$. (Because, this graph cannot be properly colored with one color but can be properly colored with 2 colors by assigning two colors to the alternative vertices).

Example 2:

Find the chromatic number and the chromatic polynomial for the graph $K_{1,n}$. We note that $K_{1,n}$ is the complete bipartite graph wherein one bipartite of the vertex set has only one vertex, say v , and the other bipartite has n vertices, say v_1, v_2, \dots, v_n . A proper coloring of this graph cannot be done with just one color and but can be done with two colors – by assigning one color to v and another color to all of v_1, v_2, \dots, v_n . Thus, the chromatic number of this graph is 2.

If λ colors are available, then the vertex v can be colored in λ ways and each of the vertices v_1, v_2, \dots, v_n can be colored in $\lambda-1$ ways. Therefore, the number of ways of properly coloring the graph is $\lambda(\lambda-1)^n$. This is the chromatic polynomial for the graph.

Example 3:

(a) consider the graph $K_{2,3}$ shown in Figure 2.47. Let λ denote the number of colors available to properly color the vertices of this graph. Find:

- (i) how many proper colorings of the graph have vertices a, b colored the same.
- (ii) how many proper colorings of the graph have vertices a, b colored with different colors.

(iii) The chromatic polynomial of the graph.

(b) For the graph $K_{2,n}$ what is the chromatic polynomial?

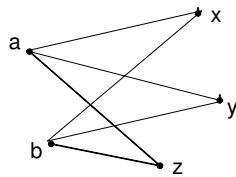


Figure 2.47

(a): (i) If the vertices a and b are to have the same color, then there are λ choices for coloring the vertex a and only one choice for the vertex b (or vice versa). Consequently, there are $\lambda-1$ choices for each of the vertices x, y, z . Hence, the number of proper colorings (in this case) is $\lambda (\lambda-1)^3$

(ii) If the vertices a and b are to have different colors, then there are λ choices for coloring the vertex a and $\lambda-1$ choices for the vertex b (or vice versa). Consequently, there are $\lambda-2$ choices for each of the vertices x, y, z . Hence the number of proper colorings (in this case) is $\lambda (\lambda-1) (\lambda-2)^3$.

(iii) Since the two cases of the vertices a and b have the same color or different colors are exhaustive and mutually exclusive, the chromatic polynomial of the graph is

$$P(K_{2,3}, \lambda) = \lambda (\lambda-1)^3 + \lambda(\lambda-1) (\lambda-2)^3.$$

(b): Let $V_1 = \{a, b\}$ and $V_2 = \{x_1, x_2, x_3, \dots, x_n\}$ be the two bipartites of $K_{2,n}$. Then, if a and b are to have the same color, the number of proper colorings of $K_{2,n}$ is $\lambda (\lambda-1)^n$ as in case (i) above, If a and b are to have different colors, the number of proper colorings is $\lambda(\lambda-1)(\lambda-2)^n$,

as in case (ii) above. Consequently, the chromatic polynomial for $K_{2,n}$ is

$$P(K_{2,n}, \lambda) = \lambda (\lambda-1)^n + \lambda(\lambda-1) (\lambda-2)^n .$$

Example 4: Find the chromatic polynomial for the cycle C_4 of length 4.

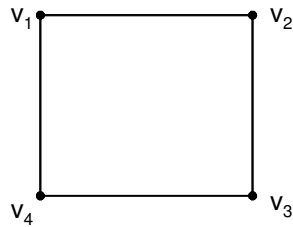


Figure 2.48

A cycle of length 4, namely C_4 , is shown in Figure 2.48. Let us redesignate it as G and denote the edge $\{v_2, v_3\}$ as e . Then the graph G_e and G_e' would be as shown below..

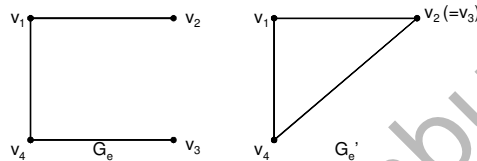


Fig. 2.49

We note that the graph G_e is a path with 4 vertices. Therefore, $P(G_e, \lambda) = \lambda (\lambda-1)^3$. Also, the graph G_e' is the graph K_3 . Therefore $P(G_e', \lambda) = \lambda(\lambda-1)(\lambda-2)$. Accordingly, using the decomposition theorem, we find that

$$\begin{aligned} P(C_4, \lambda) &= P(G, \lambda) = P(G_e, \lambda) - P(G_e', \lambda) \\ &= \lambda (\lambda-1)^3 - \lambda (\lambda-1) (\lambda-2) \\ &= \lambda^4 - 4 \lambda^3 + 6 \lambda^2 - 3 \lambda . \end{aligned}$$

This is the chromatic polynomial for the given cycle.

Example 5: Find the chromatic polynomial for the graph shown below. If 5 colors are available, in how many ways can the vertices of this graph be properly colored?.

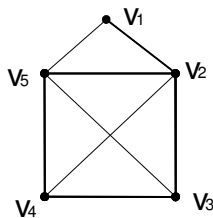


Figure 2.50

Let us denote the given graph by G and the edge $\{v_1, v_2\}$ by e . Then the graph G_e and G_e' would be as shown in Figure 2.51.

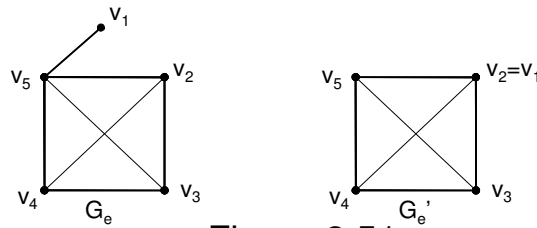


Figure 2.51

Let us redesignate the graph G_e as H and denote the edge $\{v_1, v_5\}$ as f . Then the graph H_f and H_f' would appear as shown below:

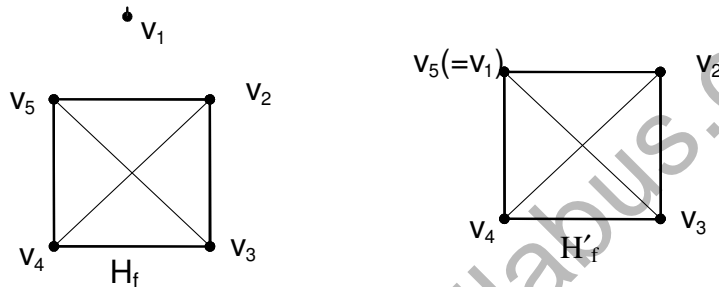


Figure 2.52

Applying the decomposition theorem to the graphs G and H we note that

$$\begin{aligned} P(G, \lambda) &= P(G_e, \lambda) - P(G_e', \lambda) \\ &= P(H, \lambda) - P(G_e', \lambda) \\ &= \{ P(H_f, \lambda) - P(H_f', \lambda) \} - P(G_e', \lambda) \text{----- (1)} \end{aligned}$$

We observe that both of the graphs G_e' and H_f' are the graph K_4 and the graph H_f is a disconnected graph having N_1 - null graph of order 1 consisting of the single vertex v_1) and K_4 as components. Accordingly,

$$P(G_e', \lambda) = P(H_f', \lambda) = P(K_4, \lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda-3)$$

$$\begin{aligned} \text{And } P(H_f, \lambda) &= P(N_1, \lambda) \cdot P(K_4, \lambda) \\ &= \lambda \cdot \lambda(\lambda-1)(\lambda-2)(\lambda-3). \end{aligned}$$

Consequently, expression (i) gives

$$\begin{aligned} P(G, \lambda) &= \lambda \cdot \lambda(\lambda-1)(\lambda-2)(\lambda-3) - 2\lambda(\lambda-1)(\lambda-2)(\lambda-3) \\ &= \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-2) \\ &= \lambda(\lambda-1)(\lambda-2)^2(\lambda-3). \end{aligned}$$

This is the chromatic polynomial for the given graph.

For $\lambda = 5$, this polynomial gives

$$P(G, \lambda) = 5 \times 4 \times 3^2 \times 2 = 360.$$

This means that if 5 colors are available, the vertices of the graph can be properly colored in 360 different ways.

Example 6: Use the multiplication theorem to find $P(G, \lambda)$ for the graph shown in Figure (2.50).

The graph G in figure 2.50 can be regarded as the union of the graphs G_1 and G_2 shown in figures 2.53 (a) and 2.53(b).

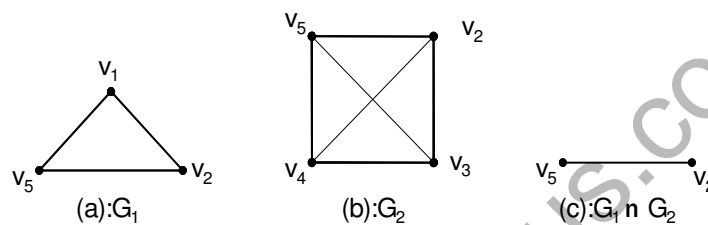


Fig.2.53

Then $G_1 \cap G_2 = \{v_5, v_2\}$ Shown in Figure 2.53 (c).

WE note that G_1 is the same as K_3 , G_2 is the same as K_4 and $G_1 \cap G_2$ is the same as K_2 . Hence, using the multiplication theorem (Theorem 2), we get

$$\begin{aligned} P(G, \lambda) &= P(G_1, \lambda) \cdot P(G_2, \lambda) / \lambda^{(2)} \\ &= P(K_3, \lambda) \cdot P(K_4, \lambda) / \lambda^{(2)} \\ &= \lambda(\lambda-1)(\lambda-2) \cdot \lambda(\lambda-1)(\lambda-2)(\lambda-3) / \lambda(\lambda-1) \\ &= \lambda(\lambda-1)(\lambda-2)^2(\lambda-3) \end{aligned}$$

As the chromatic polynomial for the give G . (This result agrees with the result proved in example 5)

Example 7: Find the chromatic polynomial for the graph shown below:

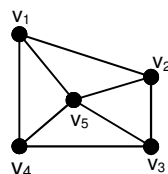


Figure 2.54

Let us denote the given graph by G and the edge $\{v_1, v_5\}$ as e . Then the graph G_e and G_e' would be as shown below.

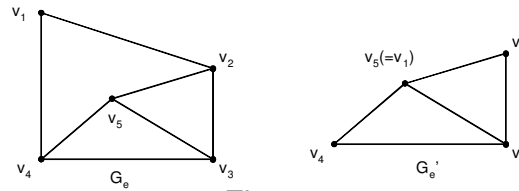


Figure 2.55

Let us redesignate G_e as H and denote the edge $\{v_5, v_2\}$ by f . Then the graphs H_f and H_f' are as shown below.

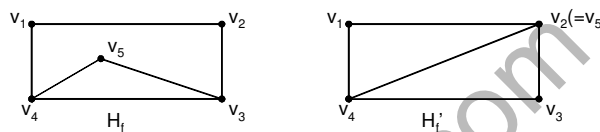


Figure 2.56

Now, we note that H_f' is the union of the cycles $v_1v_4v_2v_1$ and $v_2v_3v_4v_2$ each of which is the same as K_3 , and that the intersection of these cycles is the edge $\{v_4, v_2\}$ which is the same as K_2 . Therefore, by the multiplication theorem, we have

$$P(H_f', \lambda) = P(K_3, \lambda) \cdot P(K_3, \lambda) / \lambda^{(2)} \quad \text{--- (i)}$$

Similarly,

$$P(G_e', \lambda) = P(K_3, \lambda) \cdot P(K_3, \lambda) / \lambda^{(2)} \quad \text{--- (ii)}$$

Next, we note that H_f is the union of the cycles $v_1v_2v_3v_4v_1$ and $v_5v_3v_4v_5$ and that the intersection of these cycles is the edge $\{v_4, v_3\}$. The first of these cycles is C_4 , the second cycle is K_3 and the edge $\{v_4, v_3\}$ is K_2 . Therefore, by the multiplication theorem, we have

$$P(H_f, \lambda) = P(C_4, \lambda) \cdot P(K_3, \lambda) / \lambda^{(2)} \quad \text{--- (iii)}$$

Now, by using the decomposition theorem and the fact that $H \cong G_e$, we get.

$$\begin{aligned} P(G, \lambda) &= P(G_e, \lambda) - P(G_e', \lambda) \\ &= P(H, \lambda) - P(G_e', \lambda) \\ &= P(H_f, \lambda) - P(H_f', \lambda) - P(G_e', \lambda) \\ &= 1/\lambda^{(2)} \{ P(C_4, \lambda) \cdot P(K_3, \lambda) - 2P(K_3, \lambda)P(K_3, \lambda) \}, \\ &\quad \text{using (i) - (iii)} \\ &= P(K_3, \lambda) / \lambda^{(2)} \{ P(C_4, \lambda) - 2P(K_3, \lambda) \} \end{aligned}$$

Using the result of Example 4 and the expressions for $P(K_3, \lambda)$ & $\lambda^{(2)}$ this becomes

$$\begin{aligned}
 P(G, \lambda) &= \lambda(\lambda-1)(\lambda-2) / \lambda(\lambda-1) \{ \lambda \{ \lambda(\lambda-1)^3 - (\lambda-1)(\lambda-2) \} - 2\lambda(\lambda-1)(\lambda-2) \} \\
 &= \lambda(\lambda-1)(\lambda-2) \{ (\lambda-1)^2 - 3(\lambda-2) \} \\
 &= \lambda(\lambda-1)(\lambda-2)(\lambda^2 - 5\lambda + 7).
 \end{aligned}$$

Example 8: Let $G = G(V, E)$ be a graph with $a, b \in V$ but $\{a, b\} = e \notin E$. Let Ge^+ denote the graph obtained by including e into G and Ge^{++} denote the graph obtained by coalescing (merging) the vertices a and b . Prove that

$$P(G, \lambda) = P(Ge^+, \lambda) + P(Ge^{++}, \lambda)$$

Hence find the chromatic polynomial for the graph shown in figure 2.57.

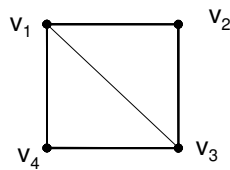


Figure 2.57

Let us redesignate Ge^+ as H . Then, from the definitions of Ge^+ and Ge^{++} , we find that $H_e = G$ and $H_{e'} = Ge^{++}$. Now, applying the decomposition theorem to H , we get

$$P(H_e, \lambda) = P(H, \lambda) + P(H_{e'}, \lambda)$$

This is the same as

$$P(G, \lambda) = P(Ge^+, \lambda) + P(Ge^{++}, \lambda)$$

Which is the required result.

For the graph G shown in figure 2.57, if $e = \{v_2, v_4\}$, the graphs Ge^+ and Ge^{++} are as shown below:

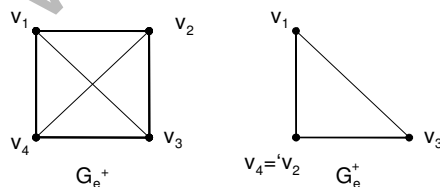


Figure 2.58

We note that Ge^+ is K_4 and Ge^{++} is K_3 . Therefore,

$$P(Ge^+, \lambda) = P(K_4, \lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda-3)$$

$$\text{and } P(Ge^{++}, \lambda) = P(K_3, \lambda) = \lambda(\lambda-1)(\lambda-2)$$

Accordingly, the chromatic polynomial for the given graph is

$$\begin{aligned} P(G, \lambda) &= P(Ge^+, \lambda) + P(Ge^{++}, \lambda) \\ &= \lambda(\lambda-1)(\lambda-2)(\lambda-3) + \lambda(\lambda-1)(\lambda-2) \\ &= \lambda(\lambda-1)(\lambda-2)^2. \end{aligned}$$

Example 9: Prove the following:

- (a) for any graph G , the constant term in $P(G, \lambda)$ is zero.
 (b) For any graph $G = G(V, E)$ with $|E| \geq 1$, the sum of the coefficients in $P(G, \lambda)$ is zero.

Solution:

Let $P(G, \lambda) = a_0 + a_1\lambda + a_2\lambda^2 + \dots + a_r\lambda^r$. Then

$$P(G, 0) = a_0 \text{ \& } P(G, 1) = a_0 + a_1 + a_2 + \dots + a_r.$$

(a) For any graph G , $P(G, 0)$ represents the number of ways of properly coloring G with zero number of colors. Since a graph cannot be colored with no color on hand, it follows that $P(G, 0) = 0$: that is $a_0 = 0$.

(b) For any graph G , $P(G, 1)$ represents the number of ways of properly coloring G with 1 color. If G has at least one edge, G cannot be properly colored with 1 color. This means that, for $G = G(V, E)$ with $|E| \geq 1$, we have

$$P(G, 1) = 0, \text{ that is, } a_0 + a_1 + a_2 + \dots + a_r = 0.$$

Exercises

01. Determine the chromatic polynomials for the graphs shown below:.

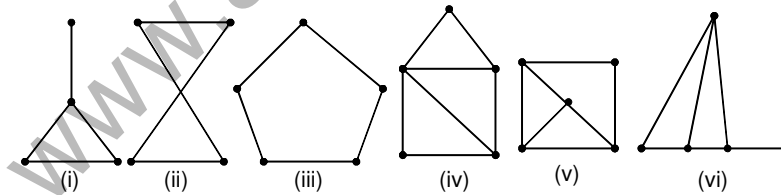


Figure 2.59

Ans 1. $\lambda(\lambda-1)^2(\lambda-2)$.

Ans 2. $\lambda(\lambda-1)^2(\lambda-2)^2$.

Ans 3. $\lambda(\lambda-1)(\lambda-2)(\lambda^2 - 2\lambda + 2)$.

Ans 4. $\lambda(\lambda-1)(\lambda-2)^3$

Ans 5. $\lambda(\lambda-1)(\lambda-2)(2\lambda-5)$.

Ans 6. $\lambda(\lambda-1)^2(\lambda-2)^2$.

02. If 4 colors are available, in how many different ways can the vertices of each graph in Figure 2.59 be properly colored?

Ans: (i) 72 (ii) 144 (iii) 240 (iv) 96 (v) 72 (vi) 144

03. For $n \geq 3$, Let G_n be the graph obtained by deleting one edge from K_n . Determine $P(G_n, \lambda)$ and $\chi(G_n)$.

04. If C_n denotes a cycle of length $n \geq 3$, prove that $P(C_n, \lambda) = (\lambda-1)_n + (-1)^n(\lambda-1)$

05. If C_n denotes a cycle of length $n \geq 4$, prove that $P(C_n, \lambda) + P(C_{n-1}, \lambda) = \lambda(\lambda-1)^{n-1}$

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