Lectures on
The Riemann Zeta–Function

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The aim of these lectures is to provide an intorduction to the theory of the Riemann Zeta-function for students who might later want to do research on the subject. The Prime Number Theorem, Hardy's theorem on the Zeros of $\zeta(s)$, and Hamburger's theorem are the principal results proved here. The exposition is self-contained, and required a preliminary knowledge of only the elements of function theory.

Contents

Lecture 1

The Maximum Principle

Theorem 1. *If D is a domain bounded by a contour C for which Cauchy's theorem is valid, and f is continuous on C regular in D, then "$|f| \leq M$ on C" implies "$|f| \leq M$ in D", and if $|f| = M$ in D, then f is a constant.* 1

Proof. (a) Let $z_o \in D$, n a positive integer. Then

$$|f(z_o)|^n = \Big|\frac{1}{2\pi i}\int\limits_C \frac{\{f(z)\}^n dz}{z - z_o}\Big| \leq \frac{l_c \cdot M^n}{2\pi\delta},$$

where l_c = length of C, δ = distance of z_o from C. As $n \to \infty$

$$|f(z)| \leq M.$$

(b) If $|f(z_o)| = M$, then f is a constant. For, applying Cauchy's integral formula to $\dfrac{d}{dz}[\{f(z)\}^n]$, we get

$$|n\{f(z_o)\}^{n-1} \cdot f'(z_o)| = \Big|\frac{1}{2\pi i}\int\limits_C \frac{f^n dz}{(z - z_o)^2}\Big| \leq \frac{l_C M^n}{2\pi\delta^2} \quad ,$$

so that

$$|f'(z_o)| \leq \frac{l_c M}{2\pi\delta^2} \cdot \frac{1}{n} \to 0, \text{ as } n \to \infty$$

Hence

$$|f'(z_o)| = 0.$$

2 (c) If $|f(z_o)| = M$, and $|f'(z_o)| = 0$, then $f''(z_o) = 0$, for

$$\frac{d^2}{dz^2}[\{f(z)\}^n] = n(n-1)\{f(z)\}^{n-2}\{f'(z)\}^2 + n\{f(z)\}^{n-1} f''(z).$$

At z_o we have

$$\frac{d^2}{dz^2}[\{f(z)\}^n]_{z=z_o} = n f^{n-1}(Z_0) f''(z_0),$$

so that

$$|nM^{n-1} f''(z_0)| = \Big|\frac{2!}{2\pi i}\int_C \frac{\{f(z)\}^n dz}{(z-z_o)^3}\Big|$$
$$\leq \frac{2! l_c}{2\pi\delta^3} M^n,$$

and letting $n \to \infty$, we see that $f''(z_o) = 0$. By a similar reasoning we prove that all derivatives of f vanish at z_0 (an arbitrary point of D). Thus f is a constant.

□

Remark. The above proof is due to Landau [12, p.105]. We shall now show that the restrictions on the nature of the boundary C postulated in the above theorem cab be dispensed with.

Theorem 2. *If f is regular in a domain D and is not a constant, and $M = \max_{z\in D} |f|$, then*

$$|f(z_o)| < M, \quad z_o \in D.$$

For the proof of this theorem we need a

Lemma. *If f is regular in $|z - z_0| \leq r$, $r > 0$, then*

$$|f(z_0)| \leq M_r,$$

3 *where* $M_r = \max\limits_{|z - z_o| = r} |f(z)|$, *and* $|f(z_o)| = M_r$ *only if* $f(z)$ *is constant for* $|z - z_o| = r$.

Proof. On using Cauchy's integral formula, we get

$$\begin{aligned} f(z_o) &= \frac{1}{2\pi i} \int\limits_{|z - z_o| = n} \frac{f(z)}{z - z_0} dz \\ &= \frac{1}{2\pi} \int\limits_0^{2\pi} f(z_o + re^{i\theta}) d\theta. \end{aligned} \tag{1}$$

Hence

$$|f(z_o)| \leq M_r.$$

Further, if there is a point ζ (such that) $|\zeta - z_o| = r$, and $|f(\zeta)| < M_r$, then by continuity, there exists a neighbourhood of ζ, on the circle $|z - z_o| = r$, in which $|f(z)| \leq M_r - \varepsilon$, $\varepsilon > 0$ and we should have $|f(z_o)| < M_r$; so that "$|f(z_o)| = M_r$" implies "$|f(z)| = M_r$ everywhere on $|z - z_o| = r$". That is $|f(z_o + re^{i\theta})| = |f(z_o)|$ for $0 \leq \theta \leq 2\pi$, or

$$f(z_o + re^{i\theta}) = f(z_o) e^{i\varphi}, \quad 0 \leq \varphi \leq 2\pi$$

On substituting this in (1), we get

$$1 = \frac{1}{2\pi} \int\limits_0^{2\pi} \cos \varphi d\theta$$

Since φ is a continuous function of θ and $\cos \varphi \leq 1$, we get $\cos \varphi \cdot 1$ for all θ, i.e. $\varphi = 0$, hence $f(s)$ is a constant. □

Remarks. The Lemma proves the maximum principle in the case of a 4
circular domain. An alternative proof of the lemma is given below [7, Bd 1, p.117].

Aliter. If $f(z_0 + re^{i\theta}) = \phi(\theta)$ (complex valued) then

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} \phi(\theta) d\theta$$

Now, if a and b are two complex numbers, $|a| \leq M$, $|b| \leq M$ and $a \neq b$, then $|a + b| < 2M$. Hence if $\phi(\theta)$ is not constant for $0 \leq \theta < 2\pi$, then there exist two points θ_1, θ_2 such that

$$|\phi(\theta_1) + \phi(\theta_2)| = 2M_r - 2\varepsilon, \text{ say, where } \varepsilon > 0$$

On the other hand, by regularity,

$$\left.\begin{aligned} |\phi(\theta_1 + t) - \phi(\theta_1)| < \varepsilon/2, \text{ for } 0 < t < \delta \\ |\phi(\theta_2 + t) - \phi(\theta_2)| < \varepsilon/2, \text{ for } 0 < t < \delta \end{aligned}\right\}$$

Hence

$$|\phi(\theta_1 + t) + \phi(\theta_2 + t)| < 2M_r - \varepsilon, \text{ for } 0 < t < \delta.$$

Therefore

$$\begin{aligned} \int_0^{2\pi} \phi(\theta) d\theta &= \int_0^{\theta_1} + \int_{\theta_1}^{\theta_1+\delta} + \int_{\theta_1+\delta}^{\theta_2} + \int_{\theta_2}^{\theta_2+\delta} + \int_{\theta_2+\delta}^{2\pi} \\ &= \left[\int_0^{\theta_1} + \int_{\theta_1+\delta}^{\theta_2} + \int_{\theta_2+\delta}^{2\pi} \right] \phi(\theta) d\theta + \int_0^{\delta} [\phi(\theta_1 + t) + \phi(\theta_2 + t)]\, dt \end{aligned}$$

Hence

$$\left| \int_0^{2\pi} \phi(\theta)\, d\theta \right| \leq M_r(2\pi - 2\delta) + (2M_r - \varepsilon)\delta = 2\pi M_r - \varepsilon\delta$$

$$\left| \frac{1}{2\pi} \int_0^{2\pi} \phi(\theta) d\theta \right| < M_r$$

5 **Proof of Theorem 2.** Let $z_o \in D$. Consider the set G_1 of points z such that $f(z) \neq f(z_o)$. This set is not empty, since f is non-constant. It is a proper subset of D, since $z_o \in D$, $z_o \notin G_1$. It is open, because f is continuous in D. Now D contains at least one boundary point of G_1. For if it did not, $G_1' \cap D$ would be open too, and $D = (G_1 \cup G_1')D$ would be disconnected. Let z_1 be the boundary point of G_1 such that $z_1 \in D$. Then $z_1 \notin G_1$, since G_1 is open. Therefore $f(z_1) = f(z_o)$. Since $z_1 \in BdG_1$ and $z_1 \in D$, we can choose a point $z_2 \in G_1$ such that the neighbourhood of z_1 defined by

$$|z - z_1| \leq |z_2 - z_1|$$

lies entirely in D. However, $f(z_2) \neq f(z_1)$, since $f(z_2) \neq f(z_o)$, and $f(z_o) = f(z_1)$. Therefore $f(z)$ is not constant on $|z - z_1| = r$, (see (1) on page 3) for, if it were, then $f(z_2) = f(z_1)$. Hence if $M' = \max\limits_{|z-z_1|=|z_2-z_1|} |f(z)|$, then

$$|f(z_1)| < M' \leq M = \max_{z \in D} |f(z)|$$

$$\text{i.e.} \qquad |f(z_o)| < M$$

Remarks. The above proof of Theorem 2 [7, Bd I, p.134] does not make use of the principle of analytic continuation which will of course provide an immediate alternative proof once the Lemma is established.

Theorem 3. *If f is regular in a bounded domain D, and continuous in $\bar{D}$, then f attains its maximum at some point in Bd D, unless f is a constant.*

Since D is bounded, $\bar{D}$ is also bounded. And a continuous function on a compact set attains its maximum, and by Theorem 2 this maximum 6
cannot be attained at an interior point of D. Note that the continuity of $|f|$ is used.

Theorem 4. *If f is regular in a bounded domain D, is non-constant, and if for every sequence $\{z_n\}$, $z_n \in D$, which converges to a point $\xi \in$ Bd D, we have*

$$\limsup_{n \to \infty} |f(z_n)| \leq M,$$

then $|f(z)| < M$ for all $z \in D$. [17, p.111]

Proof. D is an F_σ, for define sets C_n by the property: C_n consists of all z such that $|z| \leq n$ and such that there exists an open circular neighbourhood of radius $1/n$ with centre z, which is properly contained in D. Then $C_n \subset C_{n+1}$, $n = 1, 2, \ldots$; and C_n is compact. □

Define

$$\max_{Z \in C_n} |f(z)| \equiv m_n.$$

By Theorem 2, there exists a $z_n \in Bd\ C_n$ such that $|f(z_n)| = m_n$. The sequence $\{m_n\}$ is monotone increasing, by the previous results; and the sequence $\{z_n\}$ is bounded, so that a subsequence $\{z_{n_p}\}$ converges to a limit $\xi \in Bd\ D$. Hence

$$\begin{aligned} & \overline{\lim} |f(z_{n_p})| \leq M \\ \text{i.e.} \quad & \overline{\lim}\ m_{n_p} \leq M \\ \text{or} \quad & m_n < M \text{ for all } n \\ \text{i.e.} \quad & |f(z)| < M. \quad z \in D. \end{aligned}$$

N.B. That $\xi \in Bd\ D$ can be seen as follows. If $\xi \in D$, then there
7 exists an N_o such that $\xi \in C_{N_o} \subset D$, and $|f(\xi)| < m_{N_o} \leq \lim m_n$, whereas we have $f(z_{n_p}) \to f(\xi)$, so that $|f(z_{n_p})| \to |f(\xi)|$, or $\lim m_n = |f(\xi)|$.

Corollaries. *(1) If f is regular in a bounded domain D and continuous in D, then by considering $e^{f(z)}$ and $e^{-if(z)}$ instead of $f(z)$, it can be seen that the real and imaginary parts of f attain their maxima on $Bd\ D$.*

(2) If f is an entire function different from a constant, then

$$M(r) \equiv \underset{|z|=r}{l.u.b}\ |f(z)|, \text{ and}$$
$$A(r) \equiv \underset{|z|=r}{l.u.b}\ \mathrm{Re}\{f(z)\}$$

are strictly increasing functions of r. Since $f(z)$ is bounded if $M(r)$ is, for a non-constant function f, $M(r) \to \infty$ with r.

(3) *If f is regular in D, and $f \neq 0$ in D, then f cannot attain a minimum in D.*

Further if f is regular in D and continuous in $\bar{D}$, and $f \neq 0$ in $\bar{D}$, and $m = \min\limits_{z \in Bd\ D} |f(z)|$, then $|f(z)| > m$ for all $z \in D$ unless f is a constant. [We see that $m > 0$ since $f \neq 0$, and apply the maximum principle to $1/f$].

(4) *If f is regular and $\neq 0$ in D and continuous in $\bar{D}$, and $|f|$ is a constant γ on Bd D, then $f = \gamma$ in D. For, obviously $\gamma \neq 0$, and, by Corollary 3, $|f(z)| \geq \gamma$ for $z \in D$, se that f attains its maximum in D, and hence $f = \gamma$.*

(5) *If f is regular in D and continuous in $\bar{D}$, and $|f|$ is a constant on Bd D, then f is either a constant or has a zero in D.* 8

(6) ***Definition:*** *If f is regular in D, then α is said to be a* boundary value *of f at $\xi \in$ Bd D, if for* every *sequence $z_n \to \xi \in$ Bd D, $z_n \in D$, we have*

$$\lim_{n \to \infty} f(z_n) = \alpha$$

(a) *It follows from Theorem 4 that if f is regular and non - constant in D, and* has boundary-values $\{\alpha\}$, *and $|\alpha| \leq H$ for each α, then $|f(z)| < H$, $z \in D$. Further, if $M' = \max\limits_{\alpha \in b.v} |\alpha|$ then $M' = M \equiv \max\limits_{z \in D} |f(z)|$*

For, $|\alpha|$ is a boundary value of $|f|$, so that $|\alpha| \leq M$, hence $M' \leq M$. On the other hand, there exists a sequence $\{z_n\}$, $z_n \in D$, such that $|f(z_n)| \to M$. We may suppose that $z_n \to z_o$ (for, in any case, there exists a subsequence $\{z_{n_p}\}$ with the limit z_o, say, and one can then operate with the subsequence). Now $z_o \notin D$, for otherwise by continuity $f(z_o) = \lim\limits_{n \to \infty} f(z_n) = M$, which contradicts the maximum principle. Hence $z_o \in$ Bd D, and M is the boundary-value of $|f|$ at z_o. Therefore $M \leq M'$, hence $M = M'$.

(b) *Let f be regular and non-constant in D, and have boundary-values $\{\alpha\}$ on Bd D. Let $m = \min\limits_{z \in D} |f|$; $m' = \min\limits_{\alpha \in b.v.} |\alpha| > 0$.*

Then, if $f \neq 0$ *in* D*, by Corollary (6)a, we get*

$$\max_{z \in D} \frac{1}{|f(z)|} = \frac{1}{m'} = \frac{1}{m};$$

9 *so that* $|f| > m = m'$ *in* D*. Thus if there exists a point* $z_o \in D$ *such that* $|f(z_o)| \leq m'$*, then* f *must have a zero in* D*, so that* $m' > 0 = m$*. We therefore conclude that,* if $m' > 0$, then $m' = m$ if and only if $f \neq 0$ in D [6, Bd. I, p.135].

Lecture 2

The Phragmen-Lindelof principle

We shall first prove a crude version of the Phragmen-Lindelof theorem 10
and then obtain a refined variant of it. The results may be viewed as extensions of the maximum-modulus principle to infinite strips.

Theorem 1. *[6, p.168] We suppose that*

(i) f *is regular in the strip* $\alpha < x < \beta$; f *is continuous in* $\alpha \leq x \leq \beta$

(ii) $|f| \leq M$ *on* $x = \alpha$ *and* $x = \beta$

(iii) f *is bounded in* $\alpha \leq x \leq \beta$

Then, $|f| \leq M$ *in* $\alpha < x < \beta$*; and* $|f| = M$ *in* $\alpha < x < \beta$ *only if* f *is a constant.*

Proof. (i) If $f(x + iy) \to O$ as $y \to \pm\infty$ uniformly in x, $\alpha \leq x \leq \beta$, then the proof is simple. Choose a rectangle $\alpha \leq x \leq \beta$, $|y| \leq \eta$ with η sufficiently large to imply $|f|(x \pm i\eta) \leq M$ for $\alpha \leq x \leq \beta$. Then, by the maximum-modulus principle, for any z_o in the interior of the rectangle,

$$|f(z_o)| \leq M.$$

(ii) If $|f(x+iy)| \not\to 0$, as $y \to \pm\infty$, consider the modified function $f_n(z) = f(z)e^{z^2/n}$. Now

$$|f_n(z)| \to 0 \text{ as } y \to \pm\infty \text{ uniformly in } x, \text{ and}$$
$$|f_n(z)| \leq Me^{c^2/n}$$

on the boundary of the strip, for a suitable constant c.

11 Hence

$$|f_n(z_o)| \leq Me^{c^2/n}$$

for an interior point z_o of the rectangle, and letting $n \to \infty$, we get

$$|f(z_o)| \leq M.$$

If $|f(z_o)| = M$, then f is a constant. For, if f were not a constant, in the neighbourhood of z_o we would have points z, by the maximum-modulus principle, such that $|f(z)| > M$, which is impossible. □

Theorem 1 can be restated as:

Theorem 1′**:** Suppose that

(i) f is continuous and bounded in $\alpha \leq x \leq \beta$

(ii) $|f(\alpha+iy)| \leq M_1$, $|f(\beta+iy)| \leq M_2$ for all y

Then

$$|f(x_o+iy_o)| \leq M_1^{L(x_o)} M_2^{1-L(x_o)}$$

for $\alpha < x_o < \beta$, $|y_o| < \infty$, where $L(t)$ is a linear function of t which takes the value 1 at α and 0 at β. If equality occurs, then

$$f(z) = cM_1^{L(z)} M_2^{1-L(z)}; \quad |c| = 1.$$

Proof. Consider

$$f_1(z) = \frac{f(z)}{M_1^{L(z)}\ M_2^{1-L(z)}}$$

12 and apply Theorem 1 to $f_1(z)$.

More generally we have [12, p.107] □

Theorem 2. *Suppose that*

(i) *f is regular in an open half-strip D defined by:*

$$z = x + iy, \quad \alpha < x < \beta, \quad y > \eta$$

(ii) $\overline{\lim\limits_{\substack{z\to\xi\\ in\ D}}}|f| \leq M,$ *for* $\xi \in Bd\ D$

(iii) $f = O\left[\exp\left\{e^{\frac{\theta\pi|z|}{\beta-\alpha}}\right\}\right], \theta < D$

uniformly in D.

Then, $|f| \leq M$ in D; and $|f| < M$ in D unless f is a constant.

Proof. Without loss of generality we can choose

$$\alpha = -\frac{\pi}{2}, \quad \beta = +\pi/2, \quad \eta = 0$$

Set

$$\begin{aligned} g(z) &= f(z).\exp(-\sigma e^{-ikz}) \\ &\equiv f(z).\{\omega(z)\}^{\sigma}, \text{ say,} \end{aligned}$$

where $\sigma > 0$, $\theta < k < 1$. Then, as $y \to +\infty$,

$$g = O\left[\exp\left\{e^{\theta|z|} - \sigma e^{ky}\cos\frac{k\pi}{2}\right]\right.$$

uniformly in x, and **13**

$$\begin{aligned} e^{\theta|z|} - \sigma e^{ky}\cos\frac{k\pi}{2} &\leq e^{\theta(y+\pi/2)} - \sigma e^{ky}\cos\frac{k\pi}{2} \\ &\to -\infty, \text{ as } y \to \infty, \end{aligned}$$

uniformly in x, since $\theta < k$. Hence

$$|g| \to 0 \text{ uniformly as } y \to \infty,$$

so that

$$|g| \leq M, \text{ for } y > y', \alpha < x < \beta.$$

Let $z_o \in D$. We can then choose H so that

$$|g(z)| \leq M$$

for $y = H$, $\alpha < x < \beta$, and we may also suppose that $H > y_0 = \text{im}(z_o)$.

Consider now the rectangle defined by $\alpha < x < \beta$, $y = 0$, $y = H$. Since $|\omega^\sigma| \leq 1$ we have

$$\varlimsup_{\substack{z \to \xi \\ \text{in } D}} |g| \leq M$$

for every point ξ on the *boundary of this rectangle*. Hence, by the maximum-modulus principle,

$$\begin{aligned} & |g(z_o)| < M, \text{ unless } g \text{ is a constant;} \\ \text{or} \qquad & |f(z_o)| < M|e^{\sigma e^{-ikz_o}}| \end{aligned}$$

14 Letting $\sigma \to 0$, we get

$$|f(z_o)| < M.$$

□

Remarks. The choice of ω in the above proof is suggested by the critical case $\theta = 1$ of the theorem when the result is *no longer true*. For take

$$\theta = 1, \quad \alpha = -\pi/2, \quad \beta = \pi/2, \quad \eta = 0, \quad f = e^{e^{-iz}}$$

Then

$$|e^{e^{-iz}}| = e^{\text{Re}(e^{-iz})} = e^{\text{Re}\{e^{y-ix}\}} = e^{e^y \cos x}$$

So

$$|f| = 1 \text{ on } x = \pm\pi/2, \quad \text{ and } f = e^{e^y} \text{ on } x = 0.$$

Corollary 1. *If f is regular in D, continuous on Bd D, $|f| \leq M$ on Bd D, and*

$$f = O\left(e^{e^{\frac{|z|\pi\theta}{\beta-\alpha}}}\right), \quad \theta < 1,$$

then $|f| \leq M$ in D.

Corollary 2. *Suppose that hypotheses (i) and (iii) of Theorem 2 hold; suppose further that f is continuous on Bd D,*

$$f = O(y^a) \text{ on } x = \alpha, \quad f = O(y^b) \text{ on } x = \beta. \text{ Then}$$
$$f = O(y^c) \text{ on } x = \gamma,$$

uniformly in γ, where $c = p\gamma + q$, and $px + q$ is the linear function which equals a at $x = \alpha$ and b at $x = \beta$.

Proof. Let $\eta > 0$. Define $\varphi(z) = f(z)\psi(z)$, where ψ is the single-valued branch of $(-zi)^{-(pz+q)}$ defined in D, and apply Theorem 2 to φ. We first observe that φ is regular in D and continuous in $\bar{D}$. Next **15**

$$\begin{aligned}
|\psi(z)| &= |(y - ix)^{-(px+q)-ipy}| \\
&= |y - ix|^{-(px+q)} \exp(py \operatorname{im} \frac{\log y - ix}{y}) \\
&= y^{-(px+q)} |1 + O\left(\frac{1}{y}\right)|^{O(1)} \exp\left[py\left\{-\frac{x}{y} + O\left(\frac{1}{y^2}\right)\right\}\right] \\
&= y^{-(px+q)} e^{-px} \{1 + O\left(\frac{1}{y}\right)\},
\end{aligned}$$

the O's being uniform in the $x's$.

Thus $\varphi = O(1)$ on $x = \alpha$ and $x = \beta$. Since $\psi = O(y^k) = O(|z|^k)$, we see that condition (iii) of Theorem 2 holds for φ if θ is suitably chosen. Hence $\varphi = O(1)$ uniformly in the strip, which proves the corollary. □

Theorem 3. *[12, p.108] Suppose that conditions (i) and (iii) of Theorem 2 hold. Suppose further that f is continuous in Bd D, and*

$$\overline{\lim_{y\to\infty}} |f| \leq M \text{ on } x = \alpha, \quad x = \beta$$

Then

$$\overline{\lim_{y\to\infty}} |f| \leq M \text{ uniformly in } \alpha \leq x \leq \beta.$$

Proof. f is bounded in $\bar{D}$, *by the previous theorem*. Let $\eta \geq 0$, $\sum > 0$. **16**
Let $H = H(\varepsilon)$ be the ordinate beyond which $|f| < M + \varepsilon$ on $x = \alpha$, $x = \beta$.

Let $h > 0$ be a constant. Then

$$\left|\frac{z}{z+hi}\right| < 1$$

in the strip. Choose $h = h(H)$ so large that

$$\left|f(z) \cdot \frac{z}{z+hi}\right| < M + \varepsilon \text{ on } y = H$$

(possible because $\left|f \cdot \dfrac{z}{z+hi}\right| \leq |f| < M$). Then the function

$$g(z) = f \cdot \frac{z}{z+hi}$$

satisfies the conditions of Theorem 2 (with $M + \varepsilon$ in place of M) in the strip *above* $y = H$. Thus

$$|g| \leq M + \varepsilon \text{ in this strip}$$

and so $\overline{\lim\limits_{y\to\infty}}|g| \leq M + \varepsilon$ uniformly.

That is,

$$\overline{\lim}|f| \leq M + \varepsilon,$$

since $\dfrac{z}{z+hi} \to 1$ uniformly in x. Hence

$$\overline{\lim}|f| \leq M.$$

□

17 **Corollary 1.** *If conditions (i) and (iii) of Theorem 2 hold, if f is continuous on Bd D, and if*

$$\overline{\lim}|f| \leq \begin{cases} a & \text{on } x = \alpha, \\ b & \text{on } x = \beta, \end{cases}$$

where $a \neq 0$, $b \neq 0$, then

$$\overline{\lim_{y\to\infty}}|f| \leq e^{px+q},$$

uniformly, p and q being so chosen that $e^{px+q} = a$ for $x = \alpha$ and $= b$ for $x = \beta$.

Proof. Apply Theorem 3 to $g = fe^{-(pz+q)}$ □

Corollary 2. *If* $f = O(1)$ *on* $x = \alpha$, $f = o(1)$ *on* $x = \beta$, *then* $f = o(1)$ *on* $x = \gamma$, $\alpha < \gamma \leq \beta$.

[if conditions (i) and (iii) of Theorem 2 are satisfied].

Proof. Take $b = \varepsilon$ in Corollary 1, and note that $e^{p\gamma+q} \to 0$ as $\varepsilon \to 0$ for fixed γ, provided p and q are chosen as specified in Corollary 1. □

Lecture 3

Schwarz's Lemma

Theorem 1. *Let f be regular in $|z| < 1$, $f(o) = 0$.* **18**
If, for $|z| < 1$, we have $|f(z)| \leq 1$, then

$$|f(z)| \leq |z|, \quad \textit{for} \quad |z| < 1.$$

Here, equality holds only if $f(z) \equiv c\, z$ and $|c| = 1$. We further have $|f'(o)| \leq 1$.

Proof. $\dfrac{f(z)}{z}$ is regular for $|z| < 1$. Given $o < r < 1$ choose ρ such that $r < \rho < 1$; then since $|f(z)| \leq 1$ for $|z| = \rho$, it follows by the maximum modulus principle that

$$\left|\frac{f(z)}{z}\right| \leq \frac{1}{\rho},$$

also for $|z| = r$. Since the L.H.S is independent of $\rho \to 1$, we let ρ and obtain $|f(z)| \leq |z|$, for $|z| < 1$.

If for z_0, ($|z_0| < 1$), we have $|f(z_0)| = |z_0|$, then $\left|\dfrac{f(z_0)}{z_0}\right| = 1$, (by the maximum principle applied to $\dfrac{f(z)}{z}$) hence $f = cz$, $|c| = 1$.

Since $f(z) = f'(o)z + f''(o)z^2 + \cdots$ in a neighbourhood of the origin, and since $\left|\dfrac{f(z)}{z}\right| \leq 1$ in z 1, we get $|f'(o)| \leq 1$. More generally, we have ◻

Theorem 1′: Let f be regular and $|f| \leq M$ in $|z| < R$, and $f(o) = 0$. Then

$$|f(z)| \leq \frac{M|z|}{R}, \quad |z| < R.$$

19 In particular, this holds if f is regular in $|z| < R$, and continuous in $|z| \leq R$, and $|f| \leq M$ on $|z| = R$.

Theorem 2 (Caratheodory's Inequality)**.** *[12, p.112] Suppose that*

(i) f *is regular in* $|z| < R$ f *is non-constant*

(ii) $f(o) = 0$

(iii) Re $f \leq \cup$ *in* $|z| < R$. *(Thus* $\cup > 0$, *since* f *is not a constant).*

Then,

$$|f| \leq \frac{2\cup\rho}{R-\rho}, \ \textit{for } |z| = \rho < R.$$

Proof. Consider the function

$$\omega(z) = \frac{f(z)}{f(z) - 2\cup}.$$

We have Re$\{f - 2\cup\} \leq -\cup < 0$, so that ω is regular in $|z| < R$.
If $f = u + iv$, we get

$$|\omega| = \sqrt{\frac{u^2 + v^2}{(2\cup - u)^2 + v^2}}$$

so that

$$|\omega| \leq 1, \text{ since } 2\cup - u \geq |u|.$$

But $\omega(o) = o$, since $f(o) = o$. Hence by Theorem 1,

$$|\omega(z)| \leq \frac{|z|}{R}, \quad |z| < R.$$

But $f = -\dfrac{2\cup\omega}{1-\omega}$. Now take $|z| = \rho < R$.

20 Then, we have,

$$|f| \leq \frac{2 \cup |\omega|}{1 - |\omega|} \leq \frac{2 \cup \rho}{R - \rho}$$

N.B. If $|f(z_o)| = \dfrac{2 \cup \rho}{R - \rho}$, for $|z_o| < R$, *then*

$$f(z) = \frac{2U \cdot cz}{1 - cz}, \quad |c| = 1.$$

More generally, we have □

Theorem 2′: Let f be regular in $|z| < R$; f non-constant

$$f(o) = a_o = \beta + i\gamma$$

Re $f \leq \cup(|z| < R)$, so that $\cup \geq \beta$

Then

$$|f(z) - a_o| \leq \frac{2(\cup - \beta)\rho}{R - \rho}$$

for $|z| = \rho < R$

Remark. If f is a constant, then Theorem 2 is trivial. We shall now prove Borel's inequality which is sharper than Theorem 2.

Theorem 3 (Borel's Inequality). *[12, p.114] Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$, *be regular in* $|z| < 1$. *Let* Re $f \leq \cup$. $f(o) = a_o = \beta + i\gamma$.

Then

$$|a_n| \leq 2(\cup - \beta) \equiv 2 \quad \cup_1, \ say, \ n > 0.$$

Proof. We shall first prove that $|a_1| < \cup_1$, and then for general a_n.

(i) Let $f_1 \equiv \sum_{n=1}^{\infty} a_n z^n$. Then Re $f_1 \leq \cup_1$. **21**

For $|z| = \rho < 1$, we have, by Theorem 2,

$$|f_1| \leq \frac{2 \cup_1 \rho}{1 - \rho}$$

$$\text{i.e. } \left|\frac{f_1(z)}{z}\right| \leq \frac{2\cup_1}{1 - \rho}$$

Now letting $z \to 0$, we get

$$|a_1| \leq 2\cup_1$$

(ii) Define $\omega = e^{2\pi i/k}$, k a $+ve$ integer.

Then

$$\sum_{\nu=1}^{k} \omega^{\nu m} = o \begin{cases} & \text{if } m \neq 0 \\ \& & \text{if } m \neq \text{ a multiple of } k \end{cases}$$

$$= k \begin{cases} & \text{if } m = 0 \\ \text{or} & \text{if } m = \text{ a multiple of } k. \end{cases}$$

We have

$$\frac{1}{k} \sum_{r=0}^{k-1} f_1(\omega^r z) = \sum_{n=1}^{\infty} a_{nk} z^{nk} = g_1(z^k) \equiv g_1(\zeta), \text{ say.}$$

The series for g_1 is convergent for $|z| < 1$, and so for $|\zeta| < 1$. Hence g_1 is regular for $|\zeta| < 1$. Since $\operatorname{Re} g_1 \leq \cup_1$, we have

22 | coefficient of $\zeta| \leq 2\cup_1$ by the first part of the proof i.e. $|a_k| \leq 2\cup_1$. □

Corollary 1. *Let f be regular in $|z| < R$, and* $\operatorname{Re}\{f(z) - f(o)\} \leq \cup_1$. *Then*

$$|f'(z)| \leq \frac{2 \cup_1 R}{(R-\rho)^2}, \quad |z| = \rho < R$$

Proof. Suppose $R = 1$. Then

$$|f'(z)| \leq \sum n|a_n|\rho^{n-1} \leq 2 \cup_1 \sum n\rho^{n-1}$$

This argument can be extended to the n^{th} derivative. □

Corollary 2. *If f is regular for every finite z, and*

$$e^{f(z)} = O(e^{|z|^k}), \text{ as } |z| \to \infty,$$

then $f(z)$ is a polynomial of degree $\leq k$.

Proof. Let $f(z) = \sum a_n z^n$. For large R, and a fixed ζ, $|\zeta| < 1$, we have

$$\operatorname{Re}\{f(R\zeta)\} < cR^k.$$

By Theorem 3, we have

$$|a_n R^n| \leq 2(2R^k - \operatorname{Re} a_o), \quad n > 0.$$

Letting $R \to \infty$, we get

$$a_n = 0 \text{ if } n > k.$$

Apropos Schwarz's Lemma we give here a formula and an inequality which are useful. □

Theorem 4. *Let f be regular in $|z - z_o| \leq r$* 23

$$f = P(r, \theta) + Q(r, \theta).$$

Then

$$f'(z_o) = \frac{1}{\pi r} \int_o^{2\pi} P(r, \theta) e^{-i\theta} d\theta.$$

Proof. By Cauchy's formula,

$$\text{(i)} \qquad f(z_o) = \frac{1}{2\pi i} \int_{|z-z_o|=r} \frac{f(z)dz}{(z - z_o)^2} = \frac{1}{2\pi r} \int_0^{2\pi} (P + iQ) e^{-i\theta} d\theta$$

By Cauchy's theorem,

$$\text{(ii)} \qquad 0 = \frac{1}{r^2} \cdot \frac{1}{2\pi i} \int_{|z-z_o|=r} f(z)dz = \frac{1}{2\pi r} \int_0^{2\pi} (P + iQ) e^{i\theta} d\theta$$

We may change i to $-i$ in this relation, and add (i) and (ii). Then

$$f'(z_o) = \frac{1}{\pi r} \int_0^{2\pi} P(r, \theta) e^{-i\theta} d\theta$$

□

Corollary. *If f is regular, and* $|\,\mathrm{Re}\, f \leq M$ *in* $|z - z_0| \leq r$, *then*

$$|f'(z_o)| \leq \frac{2M}{r}.$$

Aliter: We have obtained a series of results each of which depended on the preceding. We can reverse this procedure, and state one general result from which the rest follow as consequences.

24 **Theorem 5.** *[11, p.50] Let* $f(z) \equiv \sum\limits_{n=0}^{\infty} c_n(z-z_0)^n$ *be regular in* $|z-z_0| < R$, *and* $\mathrm{Re}\, f < \cup$. *Then*

$$|e_n| \leq \frac{2(\cup - \mathrm{Re}\, c_o)}{R^n}, \quad n = 1, 2, 3, \ldots \tag{5.1}$$

and in $|z - z_o| \leq \rho < R$, *we have*

$$|f(z) - f(z_0)| \leq \frac{2\rho}{R-\rho}\{\cup - \mathrm{Re}\, f(z_o)\} \tag{5.2}$$

$$\left|\frac{f^{(n)}(z)}{n!}\right| \leq \frac{2R}{(R-\rho)^{n+1}}\{\cup - \mathrm{Re}\, f(z_o)\}\, n = 1, 2, 3, \ldots \tag{5.3}$$

Proof. We may suppose $z_o = 0$.

$$\text{Set} \quad \phi(z) = \cup - f(z) = \cup - c_0 - \sum_{1}^{\infty} c_n z^n$$

$$\equiv \sum_{0}^{\infty} b_n z^n, \quad |z| < R.$$

Let γ denote the circle $|z| = \rho < R$. Then

$$b_n = \frac{1}{2\pi i}\int_{\gamma} \frac{\phi(z)dz}{z^{n+1}} = \frac{1}{2\pi\rho^n}\int_{-\pi}^{\pi} (P + iQ)e^{-\mathrm{in}\theta} d\theta, n \geq 0, \tag{5.4}$$

where $\phi(r, \theta) = P(r, \theta) + iQ(r, \theta)$. Now, if $n \geq 1$, then $\phi(z)z^{n-1}$ is regular is γ, so that

$$0 = \frac{\rho^n}{2r}\int_{-\pi}^{\pi} (P + iQ)e^{in\theta} d\theta, n \geq 1 \tag{5.5}$$

Changing i to $-i$ in (5.5) and adding this to (5.4) we get

$$b_n \rho^n = \frac{1}{\pi} \int\limits_{-\pi}^{\pi} P e^{-n\theta i} d\theta, \quad n \geq 1.$$

But $P = \cup - \operatorname{Re} f \geq 0$ in $|z| < R$ and so in γ. Hence, if $n \geq 1$, **25**

$$|b_n| \rho^n \leq \frac{1}{\pi} \int\limits_{-\pi}^{\pi} |P e^{-n\theta i}| d\theta = \frac{1}{\pi} \int\limits_{-\pi}^{\pi} P d\theta = 2 \operatorname{Re} b_o, \tag{5.6}$$

on using (5.4) with $n = 0$. Now letting $\rho \to R$, we get

$$|b_n| R^n \leq 2\beta_o \equiv 2 \operatorname{Re} b_o.$$

Since $b_o = \cup - c_o$, and $b_n = -c_n$, $n \geq 1$, we at once obtain (5.1). We then deduce, for $|z| \leq \rho < R$

$$|\phi(z) - \phi(o)| = |\sum_1^\infty b_n z^n| \leq \sum_1^\infty 2\beta_0 \left(\frac{\rho}{R}\right)^n = \frac{2\beta_o \rho}{R - \rho} \tag{5.7}$$

and if $n \geq 1$,

$$\begin{aligned}
|\phi^{(n)}(z)| &\leq \sum_{r=n}^{\infty} r(r-1)\ldots(r-n+1) \frac{2\beta_0 \rho^{r-n}}{R^r} \\
&= \left(\frac{d}{d\rho}\right)^n \sum_{n=0}^{\infty} 2\beta_0 \left(\frac{\rho}{R}\right)^r \\
&= \left(\frac{d}{d\rho}\right)^n \frac{2\beta_e R}{R - \rho} \\
&= 2 \frac{\beta_o R \cdot n!}{(R - \rho)^{n+1}} \qquad (5.8)
\end{aligned}$$

(5.7) and (5.8) yield the required results on substituting $\phi = \cup - f$ and $\beta_0 = \cup - \operatorname{Re} f(0)$. ◻

Lecture 4

Entire Functions

An entire function is an analytic function (in the complex plane) which has no singularities except at ∞. A polynomial is a simple example. A polynomial $f(z)$ which has zeros at $z_1, \ldots, z_n$ can be factorized as **26**

$$f(z) = f(0)\left(1 - \frac{z}{z_1}\right)\ldots\left(1 - \frac{z}{z_n}\right)$$

The analogy holds for entire functions in general. Before we prove this, we wish to observe that if $G(z)$ is an entire function with no zeros it can be written in the form $e^{g(z)}$ where g is entire. For consider $\dfrac{G'(z)}{G(z)}$; every (finite) point is an 'ordinary point' for this function, and so it is entire and equals $g_1(z)$, say. Then we get

$$\log\left\{\frac{G(z)}{G(z_0)}\right\} = \int_{z_0}^{z_1} g_1(\zeta)d\zeta = g(z) - g(z_0), \text{ say,}$$

so that

$$G(z) = G(z_0)e^{g(z)-g(z_0)} = e^{g(z)-g(z_0)+\log G(z_o)}$$

As a corollary we see that if $G(z)$ is an entire function with n zeros, distinct or not, then

$$G(z) = (z - z_1)\ldots(z - z_n)e^{g(z)}$$

where g is entire. We wish to uphold this in the case when G has an infinity of zeros.

27 **Theorem 1** (Weierstrass)**.** *[15, p.246] Given a sequence of complex numbers a_n such that*

$$0 < |a_1| \leq |a_2| \leq \ldots \leq |a_n| \leq \ldots$$

whose sole limit point is ∞, there exists an entire function with zeros at these points and these points only.

Proof. Consider the function

$$\left(1 - \frac{z}{a_n}\right) e^{Q_\nu(z)},$$

where $Q_\nu(z)$ is a polynomial of degree q. This is an entire function which does not vanish except for $z = a_n$. Rewrite it as

$$\begin{aligned}\left(1 - \frac{z}{a_n}\right) e^{Q_\nu(z)} &= e^{Q_\nu(z) + \log\left(1 - \frac{z}{a_n}\right)} \\ &= e^{-\frac{z}{a_n} - \frac{z^2}{2a^2} \ldots + Q_\nu(z)},\end{aligned}$$

and choose

$$Q_\nu(z) = \frac{z}{a_n} + \ldots + \frac{z^\nu}{\nu a^{\nu_n}}$$

so that

$$\begin{aligned}\left(1 - \frac{z}{a_n}\right) e^{Q_\nu(z)} &= e^{-\left(\frac{z}{a_n}\right)^{\nu+1} \frac{1}{\nu+1} \cdots} \\ &\equiv 1 + \cup_n(z), \text{ say.}\end{aligned}$$

We wish to determine ν in such a way that

$$\prod_1^\infty \left(1 - \frac{z}{a_n}\right) e^{Q_\nu(z)}$$

28 is absolutely and uniformly convergent for $|z| < R$, however large R may

be.

Choose an $R > 1$, and an α such that $0 < \alpha < 1$; then there exists a q (+ ve integer) such that

$$|a_q| \leq \frac{R}{\alpha}, \quad |a_{q+1}| > \frac{R}{\alpha}.$$

Then the partial product

$$\prod_{1}^{q} \left(1 - \frac{z}{a_n}\right) e^{Q_\nu(Z)}$$

is trivially an entire function of z. Consider the remainder

$$\prod_{q+1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{Q_\nu(z)}$$

for $|z| \leq R$.

We have, for $n > q$, $|a_n| > \dfrac{R}{\alpha}$ or $\left|\dfrac{z}{a_n}\right| < \alpha < 1$. Using this fact we shall estimate each of the factors $\{1 + \cup_n(z)\}$ in the product, for $n > q$.

$$\begin{aligned} |\cup_n(z)| &= \left| e^{-\frac{1}{\nu+1}\left(\frac{z}{a_n}\right)^{\nu+1} - \cdots} - 1 \right| \\ &\leq e^{\left|\frac{1}{\nu+1}\left(\frac{z}{a_n}\right)^{\nu+1} \cdots\right|} - 1 \end{aligned}$$

□

Since $|e^m - 1| \leq e^{|m|} - 1$, for $e^m - 1 = m + \dfrac{m^2}{2!} + \ldots$ or $|e^m - 1| \leq$ 29
$|m| + \dfrac{|m^2|}{2} \ldots = e^{|m|} - 1$.

$$\begin{aligned} |U_n(z)| &\leq e^{\left|\frac{z}{a_n}\right|^{\nu+1}\left(1 + \left|\frac{z}{a_n}\right| + \cdots\right)} - 1, \\ &\leq e^{\left|\frac{z}{a_n}\right|^{\nu+1}(1 + \alpha + \alpha^2 + \cdots)} - 1 \\ &= e^{\frac{1}{1-\alpha}\left|\frac{z}{a_n}\right|^{\nu+1}} - 1 \end{aligned}$$

$$\leq \frac{1}{1-\alpha}\left|\frac{z}{a_n}\right|^{\nu+1} \cdot e^{\frac{1}{1-\alpha}\left|\frac{z}{a_n}\right|^{\nu+1}}, \text{ since if } x \text{ is}$$

real, $e^x - 1 \leq xe^x$, for $e^x - 1 = x\left(1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots\right) \leq x(1 + x + \frac{x^2}{2!} + \cdots) = xe^x$

Hence

$$|U_n(z)| \leq \frac{e^{\frac{1}{1-\alpha}}}{1-\alpha}\left|\frac{z}{a_n}\right|^{\nu+1}$$

30 Now arise two cases: (i) either there exists an integer $p > 0$ such that $\sum\limits_{n=1}^{\infty}\frac{1}{|a_n|^p} < \infty$ or (ii) there does not. In case (i) take $\nu = p - 1$, so that

$$|U_n(z)| \leq \frac{R^p}{|a_n|^p} \cdot \frac{e^{\frac{1}{1-a}}}{1-\alpha}, \text{ since } |z| \leq R;$$

and hence $\sum |U_n(z)| < \infty$ for $|z| \leq R$, i.e. $\prod\limits_{q+1}^{\infty}(1 + U_n(z))$ is absolutely and uniformly convergent for $|z| \leq R$.

In case (ii) take $\nu = n - 1$, so that

$$|U_n(z)| < \frac{e^{\frac{1}{1-\alpha}}}{1-\alpha}\left|\frac{z}{a_n}\right|^n, \; n > q$$

$$\left|\frac{z}{a_n}\right| < 1$$

$$|z| \leq R$$

$$|a_n| \to \infty$$

Then by the 'root-test' $\sum |U_n(z)| < \infty$, and the same result follows as before. Hence the product

$$\prod_1^{\infty}\left(1 - \frac{z}{a_n}\right)e^{Q_\nu(z)}$$

31 is analytic in $|z| \leq R$; since R is arbitrary and ν does not depend on R we see that the above product represents an entire function.

Remark. If in addition $z = 0$ is also a zero of $G(z)$ then $\dfrac{G(z)}{z^m G(z)}$ has no zeros and equals $e^{g(z)}$, say, so that

$$G(z) = e^{g(z)} \cdot z^m \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{Q_\nu(z)}$$

In the above expression for $G(z)$, the function $g(z)$ is an *arbitrary* entire function. If G is subject to further restrictions it should be possible to say more about $g(z)$; this we shall now proceed to do. The class of entire functions which we shall consider will be called "entire functions of finite order".

Definition. *An entire function $f(z)$ is of finite order if there exists a constant λ such that $|f(z)| < e^{r^\lambda}$ for $|z| = r > r_0$. For a non-constant f, of finite order, we have $\lambda > 0$. If the above inequality is true for a certain λ, it is also true for $\lambda' > \lambda$. Thus there are an infinity of λ's o satisfying this. The* lower bound *of such λ's is called the "order of f". Let us denote it by ρ. Then, given $\varepsilon > 0$, there exists an r_o such that*

$$|f(z)| < e^{r^{\rho+\varepsilon}}$$

for $|z| = r > r_0$.

This would imply that

$$M(r) = \max_{|z|=r} |f(z)| < e^{r^{\rho+\varepsilon}}, r > r_0,$$

while **32**

$$M(r) > e^{r^{\rho-\varepsilon}} \text{ for an infinity of values of } r \text{ tending}$$

to $+\infty$

Taking logs. twice, we get

$$\frac{\log\log M(r)}{\log r} < \rho + \varepsilon$$

and

$$\frac{\log\log M(r)}{\log r} > \rho - \varepsilon \text{ for an infinity of values of } r \to \infty$$

Hence

$$\rho = \overline{\lim_{r\to\infty}} \frac{\log\log M(r)}{\log R}$$

Theorem 2. *If $f(z)$ is an entire function of order $\rho < \infty$, and has an infinity of zeros, $f(0) \neq 0$, then given $\varepsilon > 0$, there exists an $R_\circ$ such that for $R \geq R_o$, we have*

$$n\left(\frac{R}{3}\right) \leq \frac{1}{\log 2} \cdot \log \frac{e^{R^{\rho+\varepsilon}}}{|f(o)|}$$

Here $n(R)$ denotes the number of zeros of $f(z)$ in $|z| \leq R$.

Proof. We first observe that if $f(z)$ is analytic in $|z| \leq R$, and $a_1 \ldots a_n$ are the zeros of f inside $|z| < R/3$, then for

$$g(z) = \frac{f(z)}{\left(1 - \frac{z}{a_1}\right) \ldots \left(1 - \frac{z}{a_n}\right)}$$

33 we get the inequality

$$|g(z)| \leq \frac{M}{2^n}$$

where M is defined by: $|f(z)| \leq M$ for $|z| = R$. For, if $|z| = R$, then since $|a_p| \leq \dfrac{R}{3}$, $p = 1 \ldots n$, we get $\left|\dfrac{z}{a_p}\right| \geq 3$ or $\left|1 - \dfrac{z}{a_p}\right| \geq 2$

By the maximum modules theorem,

$$|g(o)| \leq \frac{M}{2^n}, \text{ i.e. } |f(o)| \leq \frac{M}{2^n}$$

or

$$n \equiv n\left(\frac{R}{3}\right) \leq \frac{1}{\log 2} \cdot \log\left(\frac{M}{f(o)}\right)$$

If, further, f is of order ρ, then for $r > r_\circ$, we have $M < e^{r^{\rho+\varepsilon}}$ which gives the required results.

N.B. The result is trivial if the number of zeros is finite. ◻

Corollary.

$$n(R) = O(R^{\rho+\varepsilon}).$$

For

$$n\left(\frac{R}{3}\right) \leq \frac{1}{\log 2} \cdot [R^{\rho+\varepsilon} - \log|f(o)|],$$

$$\textit{and} \qquad -\log|f(0)| < R^{\rho+\varepsilon}, \ \textit{say}.$$

Then for $R \geq R'$

$$n\left(\frac{R}{3}\right) < \frac{2}{\log 2} R^{\rho+\varepsilon}$$

and hence the result.

Theorem 3. *If* $f(z)$ *is of order* $\rho < \infty$, *has an infinity of zeros* 34

$$a_1, a_2, \ldots, f(o) \neq 0, \sigma > \rho, \ \textit{then} \ \sum \frac{1}{|a_n|^\sigma} < \infty$$

Proof. Arrange the zeros in a sequence:

$$|a_1| \leq |a_2| \leq \ldots$$

Let

$$\alpha_p = |a_p|$$

Then in the circle $|z| \leq r = \alpha_n$, there are exactly n zeros. Hence

$$n < c\ \alpha_n^{\rho+\varepsilon}, \ \text{for } n \geq N$$

or

$$\frac{1}{n} > \frac{1}{c} \cdot \frac{1}{\alpha_n^{\rho+\varepsilon}}$$

Let $\sigma > \rho + \varepsilon$ (i.e. choose $0 < \varepsilon < \sigma - \rho$). Then

$$\frac{1}{n^{\sigma/\rho+\varepsilon}} > \frac{1}{c^{\sigma/\rho+\varepsilon}} \cdot \frac{1}{\alpha_n^\sigma},$$

or

$$\frac{1}{\alpha_n^\sigma} < c^{\sigma/\rho+\varepsilon} \cdot \frac{1}{n^{\sigma/\rho+\varepsilon}} \ \text{for } n \geq N$$

Hence $\sum \frac{1}{\alpha_n^\sigma} < \infty$. □

Remark. (i) There cannot be too dense a distribution of zeros, since $n < c < \alpha_n^{\rho+\varepsilon}$. Nor can their moduli increase too slowly, since, for instance, $\sum \dfrac{1}{(\log n)^p}$ does not converge.

35 (ii) The result is of course trivial if there are only a finite number of zeros.

Definition. *The* lower bound *of the numbers 'σ' for which* $\sum \dfrac{1}{|a_n|\sigma} < \infty$, *is called the* exponent of convergence *of* $\{a_n\}$. *We shall denote it by* ρ_1.

Then

$$\sum \frac{1}{|a_n|^{\rho_1+\varepsilon}} < \infty, \quad \sum \frac{1}{|a_n|^{\rho_1-\varepsilon}} = \infty, \quad \varepsilon > 0$$

By Theorem 1, we have $\rho_1 \leq \rho$

N.B. If the $a_n's$ are finite in number or nil, then $\rho_1 = 0$. Thus $\rho_1 > 0$ implies that f has an infinity of zeros. Let $f(z)$ be entire, of order $\rho < \infty$; $f(o) \neq 0$, and $f(z_n) = 0$, $n = 1, 2, 3, \ldots$. Then there exists an integer $(p+1)$ such that $\sum \dfrac{1}{|z_n|^{\rho+1}} < \infty$

(By Theorem 1, any integer $> \rho$ will serve for $\rho + 1$). Thus, by Weierstrass's theorem, we have

$$f(z) = e^{Q(z)} \prod_1^\infty \left(1 - \frac{z}{z_n}\right) e^{\frac{z}{z_n} + \frac{z^2}{2z_n 2} + \cdots + \frac{z^\nu}{\nu z_n^\nu}},$$

where $\nu = p$ (cf. proof of Weierstrass's theorem).

Definition. *The smallest* integer p *for which* $\sum \frac{1}{|z_n|^{p+1}} < \infty$ *is the 'genus'*
36 *of the 'canonical product'* $\Pi\left(1 - \dfrac{z}{z_n}\right) e^{\frac{z}{z_n} + \cdots + \frac{z^p}{p z_n^p}}$, *with zeros* z_n. *If* $z_n's$ *are finite in number, we (including nil) define* $p = 0$, *and we define the product as* $\Pi\left(1 - \dfrac{z}{z_n}\right)$

Examples. (i) $z_n = n$, $p = 1$ (ii) $z_n = e^n$, $p = 0$

(iii) $z_1 = \frac{1}{2}\log 2$, $z_n = \log n$, $n \geq 2$, no finite p.

Remark. If $\sigma > \rho_1$, then $\sum \dfrac{1}{|z_n|^\sigma} < \infty$

Also,

$$\sum \frac{1}{|z_n|^{p+1}} < \infty$$

But

$$\sum \frac{1}{|z_n|^p} = \infty$$

Thus, if ρ_1 is not an integer, $p = [\rho_1]$, and if ρ_1 is an integer, then either $\sum \dfrac{1}{|z_n|^{\rho_1}} < \infty$ or $\sum \dfrac{1}{|z_n|^{\rho_1}} = \infty$; in the first case, $p + 1 = \rho_1$, which in the second case $p = \rho_1$. Hence,

$$\phi \leq \rho_1$$

But $\rho_1 \leq \rho$. Thus

$$p \leq \rho_1 \leq \rho$$

Also

$$p \leq \rho_1 \leq p + 1, \text{ since } \sum \frac{1}{|z_n|^{p+1}} < \infty.$$

Lecture 5

Entire Functions (Contd.)

Theorem 1 (Hadamard)**.** *[8] Let f be an entire function with zeros* **37**

$$|z_1| \leq |z_2| \leq \ldots \to \infty \{z_n\} \text{ such that}$$

Let $f(o) \neq 0$. Let f be of order $\rho < \infty$. Let $G(z)$ be the canonical product with the given zeros.

Then

$$f(z) = e^{Q(z)} G(z),$$

where Q is a polynomial of degree $\leq \rho$.

We shall give a proof without using Weierstrass's theorem. The proof will depend on integrating the function $\dfrac{f'(z)}{f(z)} \cdot \dfrac{1}{z^{\mu}(z-x)}$ over a sequence of expending circles.

Given $R > 0$, let N be defined by the property:

$$|z_N| \leq R, \quad |z_{N+1}| > R.$$

We need the following

Lemma. *Let $0 < \varepsilon < 1$. Then*

$$\left| \frac{f'(z)}{f(z)} - \sum_{n+1}^{N} \frac{1}{z - z_n} \right| < cR^{\rho - 1 + \varepsilon}$$

for $|z| = \dfrac{R}{2} > R_0$, and $|z| = \dfrac{R}{2}$ is free from any z_n.

For we can choose an R_0 such that for $|z| = R > R_0$,

38 (i) $|f(z)| < e^{R^{\rho+\varepsilon}}$,

(ii) no z_n lies on $|z| = \dfrac{R}{2}$, and

(iii) $\log \dfrac{1}{|f(0)|} < (2R)^{\rho+\varepsilon}$

For such an $R > R_0$, define

$$g_R(z) = \frac{f(z)}{f(0)} \prod_{n=1}^{N} \left(1 - \frac{z}{z_n}\right)^{-1}$$

Then for $|z| = 2R$, we have

$$|g_R(z)| < \frac{1}{|f(o)|} e^{(2R)^{\rho+\varepsilon}}$$

since $|f| < e^{(2R)^{\rho+\varepsilon}}$, and $|1 - \dfrac{z}{z_n}| \geq 1$ for $1 \leq n \leq N$.

Hence, by the maximum modulus principle,

$$|g_R(z)| < \frac{1}{f(0)} e^{(2R)^{\rho+\varepsilon}} \text{ for } |z| = R.$$

That is

$$\log |g_R(z)| < 2^{\rho+2} R^{\rho+\varepsilon}. \quad (0 < \varepsilon < 1)$$

If $h_R(z) \equiv \log g_R(z)$, and the log vanishes for $z = 0$, then, since $g_R(z)$ is analytic and $g_R(z) \neq 0$ for $|z| \leq R$, we see that $h_R(z)$ is analytic for $|z| \leq R$, and $h_R(0) = 0$. Further, for $|z| = R$,

$$\operatorname{Re} h_R(z) = \log |g_R(z)| < 2^{\rho+2} R^{\rho+\varepsilon}$$

39 Hence, by the Borel-Caratheodory inequality, we have for $|z| = R/2$,

$$|h'_R(z)| < cR^{\rho-1+\varepsilon},$$

which gives the required lemma.

Proof of theorem. Let $\mu = [\rho]$ Then consider

$$I \equiv \int \frac{f'(z)}{f(z)} \cdot \frac{1}{z^{\mu}(z-x)} dz$$

taken along $|z| = R/2$, where $|x| < \dfrac{R}{4}$. The only poles of the integrand are: 0, x and those $z_n's$ which lie inside $|z| = R/2$. Calculating the residues, we get

$$\begin{aligned}\frac{1}{2\pi i} \int\limits_{|z|=R/2} \frac{f'(z)}{f(z)} \frac{1}{z^{\mu}(z-x)} dz = &\sum_{|Z_n|<R/2} \frac{1}{z_n^{\mu} \cdot (z_n - x)} \\ &+ \frac{f'(x)}{f(x)} \cdot \frac{1}{x^{\mu}} \\ &+ \frac{1}{(\mu-1)!} \left(D^{\mu-1}\right)_{z=0} \left(\frac{f'(z)}{f(z)} - \frac{1}{z-x}\right)\end{aligned}$$

We shall prove that the l.h.s tends to o as $R \to \infty$ in such a way that $|z| = R/2$ is free from any z_n. Rewriting I as

$$\begin{aligned} I &= \int \left(\frac{f'}{f} - \sum_{1}^{N} \frac{1}{z - z_n}\right) \frac{dz}{z^{\mu}(z-x)} + \sum_{1}^{N} \int \frac{1}{z - z_n} \cdot \frac{dz}{z^{\mu}(z-x)} \\ &\equiv I_1 + I_2, \end{aligned}$$

we get for $|z| = R/2$ (by the Lemma), **40**

$$|I_1| = O(R^{\rho-\mu-1+\varepsilon}) = O(1).$$

Let k denote the number of $z_n's$ lying inside $|z| = R/2$

Then

$$\begin{aligned} I_2 &= \sum_{n=1}^{k} \int\limits_{|z|=R/2} \ldots + \sum_{n=k+1}^{N} \int \ldots \\ &\equiv I_{2,1} + I_{2,2}, \text{ say.} \end{aligned}$$

The value of $I_{2,1}$ remains unaltered when we integrate along $|z| = 3R/4$, and along the new path, $|z - z_n| \geq R/4$ since $n \leq k$. Since $N = O(R^{\rho+\varepsilon})$ we get

$$I_{2,1} = O(R^{\rho-\mu-1+\varepsilon}) = O(1)$$

Similarly integrating $I_{2,2}$ over $|z| = R/4$, we get

$$I_{2,2} = O(1)$$

Thus $I = o(1)$. Hence

$$\frac{f'(x)}{f(x)}\frac{1}{x^\mu} + \sum_{n=1}^{\infty} \cdot \frac{1}{z_n^\mu (z-x)} \frac{1}{(\mu-1)!} \Big[D\Big]_{z=0}^{\mu-1} \left(\frac{f'}{f}\ \frac{1}{z-x}\right) = 0$$

$$\text{i.e. } \frac{f'(x)}{f(x)} + \sum_{0}^{\mu-1} b_n x^n \sum_{n=1}^{\infty} \left(\frac{x^{-1}}{z_n^\mu (z-x)}\right) = 0, \quad b_n = a_{\mu-n}, a_n = -\frac{\left[\frac{f'}{f}\right]_{z=0}^{(\mu-n)}}{(\mu-n)!}$$

$$\text{i.e. } \frac{f'(x)}{f(x)} = \sum_{0}^{\mu-1} c_n x^n + \sum_{n=1}^{\infty} \left(\frac{x^{\mu-1}}{z_n^\mu} + \frac{x^{\mu-2}}{z_n^{\mu-1}} + \cdots + \frac{1}{x-z_n}\right)$$

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Integrating and raising to the power of e we get the result:

$$f(x) = e^{\sum\limits_0^\mu d_n x^n} \prod_1^\infty \left(1 - \frac{x}{z_n}\right) e^{x/z_p + \cdot + x^\mu/\mu z_n^\mu}, d_n = \frac{c_{n-1}}{n}.$$

If f is an entire functions of order ρ, which is not a constant, and such that $f(o) \neq 0$, and $f(z_n) = 0$, $n = 1, 2, 3, \ldots$ $|z_n| \leq |z_{n+1}|$, then

$$f(z) = e^{Q_1(z)} G_1(z),$$

where $Q_1(z)$ is a polynomial of degree $q \leq \rho$ and

$$G_1(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) e^{\frac{z}{z_n} + \cdots + \frac{1}{p}\left(\frac{z}{z_n}\right)^p},$$

where $p + 1$ is the smallest integer for which $\sum \dfrac{1}{|z_n|^{p+1}} < \infty$, this integer existing, since f is of finite order.

Remark. This follows at once from Weierstrass's theorem but can also be deduced from the infinite-product representation derived in the proof of Hadamard's theorem (without the use of Weierstrass's theorem). One has only to notice that

$$\prod e^{-\frac{1}{p+1}\left(\frac{z}{z_n}\right)^{p+1}-\cdots-\frac{1}{\mu}\left(\frac{z}{z_n}\right)^{\mu}}$$

is convergent, where p is the genus of the entire function in question, so 42
that $f(z) = e^{Q(z)}G(z)$ may be multiplied by this product yielding $f(z) = e^{Q_1(z)}G_1(z)$, where Q_1 is again of degree $\leq \mu$.

We recall the convention that if the $z_n's$ are finite in number (or if there are no $z_n's$), then $p = 0$ and $\rho_1 = 0$.

We have also seen that

$$q \leq \rho, \rho_1 \leq \rho \text{ i.e. } \max(q, \rho_1) \leq \rho$$

We shall now prove the following

Theorem 2. $\rho \leq \max(q, \rho_1)$

Let

$$E(u) \equiv (1-u)e^{u+\dfrac{u^2}{2}+\cdots+\dfrac{u^p}{p}}$$

If $|u| \leq \frac{1}{2}$, *then*

$$\begin{aligned}
|E(u)| &= \left|e^{\log(1-u)+u+\cdots+\frac{u^p}{p}}\right| = \left|e^{\frac{-u^{p+1}}{p+1}-\frac{u^{p+2}}{p+2}-\cdots}\right| \\
&\leq e^{|u|^{p+1}(1+\frac{1}{2}+\frac{1}{4}+\cdots)} \\
&= e^{2|u|^{p+1}} \leq e^{|2u|^{p+1}} \\
&\leq e^{|2u|^{\tau}}, \ \textit{if } \tau \leq p+1
\end{aligned}$$

If $|u| > \frac{1}{2}$, *then*

$$\begin{aligned}
|E(u)| &\leq (1+|u|)e^{|u|+\cdots+\frac{|u|^p}{p}} \\
&\leq (1+|u|)e^{|u|^p\cdot(|u|^{1-p}+\cdots+1)} \\
&\leq (1+|u|)e^{|u|^p(2^{p-1}+\cdots+1)}
\end{aligned}$$

$$< e^{|2u|^p + \log(1+|u|)}$$
$$\leq e^{|2u|^\tau + \log(1+|u|)}, \text{ if } p \leq \tau, \text{ since } |2u| > 1.$$

43 *Hence, for all u, we have*

$$|E(u)| \leq e^{|2u|^\tau + \log(1+|u|)},$$

if $p \leq \tau \leq p+1$.

It is possible to choose τ in such a way that

(i) $p \leq \tau \leq p+1$,

(ii) $\tau > 0$,

(iii) $\sum \dfrac{1}{|z_n|^\tau} < \infty$, *(if there are an infinity of z_n's) and*

(iv) $\rho_1 \leq \tau \leq \rho_1 + \varepsilon$

For, if $\sum \dfrac{1}{|z_n|^{\rho_1}} < \infty$ (if the series is infinite this would imply that $\rho_1 > 0$), we have only to take $\tau = \rho_\tau > 0$. And in all other cases we must have $p \leq \rho_1 < p+1$, [for, if $\rho_1 = p+1$, then $\sum \dfrac{1}{|z_n|^{\rho_1}} < \infty$] and so we choose τ such that $\rho_1 < \tau < p+1$; this implies again that $\sum \dfrac{1}{|z_n|^\tau} < \infty$ and $\tau > 0$. For such a τ we can write the above inequality as

$$|E(u)| \leq e^{c_1|u|^\tau}, c_1 \textit{ is a constant.}$$

44 *Hence*

$$|f(z)| \leq e^{|Q(z)| + c_1} \cdot \sum_{n=1}^{\infty} \left|\frac{z}{z_n}\right|^\tau$$
$$\textit{i.e. } M(r) < e^{c_2 r^q + c_3 r^\tau}, |z| = r > 1, \textit{ and } c_1 \sum |1/z_n|^\tau = c_3.$$

Hence

$$\log M(r) = O(r^q) + O(r^\tau), \textit{ as } r \to \infty,$$

$$= O(r^{\max(q,\tau)}),\ as\ r \to \infty \tag{2.1}$$

so that

$$\rho \leq \max(q, \tau)$$

Since τ is arbitrarily near ρ_1, we get

$$\rho \leq \max(q, \rho_1),$$

and this proves the theorem.

Theorem 3. *If the order of the canonical product $G(z)$ is ρ', then $\rho' = \rho_1$*

Proof. As in the proof of Theorem 2, we have

$$|E(u)| \leq e^{c_1 |u|^\tau} \begin{cases} p \leq \tau \leq p+1; \\ \tau > 0 \\ \sum \dfrac{1}{|z_n|^\tau} < \infty \\ \rho_1 \leq \tau \leq \rho + \varepsilon, \quad \varepsilon > 0 \end{cases}$$

Hence

$$|G(z)| \leq e^{c_1 \sum\limits_{n=1}^{\infty} \left|\frac{z}{z_n}\right|^\tau}$$
$$\leq e^{c_3 r^\tau}, |z| = r.$$

If $M(r) = \max\limits_{|z|=r} |G(z)|$, then **45**

$$M(r) < e^{c_3 r^\tau}$$

Hence

$$\rho' \leq \tau$$

which implies $\rho' \leq \rho_1$; but $\rho_1 \leq \rho'$. Hence $\rho' = \rho_1$ □

Theorem 4. *If, either*

(i) *$\rho_1 < q$ or*

(ii) $\sum \frac{1}{|z_n|^{\rho_1}} < \infty$ *(the series being infinite),*

then

$$\log M(r) = O(r^{\beta}), \ \textit{for}\ \beta = \max(\rho_1, q).$$

Proof. (i) If $\rho_1 < q$, then we take $\tau < q$ and from (2.1) we get

$$\log M(r) = O(r^q) = O(r^{\max(\rho_1, q)})$$

(ii) If $\sum\limits_{n=1}^{\infty} \frac{1}{|z_n|\rho_1} < \infty$, then we may take $\tau = \rho_1 > 0$ and from (2.1) we get

$$\begin{aligned} \log M(r) &= O(r^q) + O(r^{\rho_1}) \\ &= O(r^{\beta}) \end{aligned}$$

□

Theorem 5. (i) *We have*

$$\rho = \max(\rho_1, q)$$

(The existence of either side implies the other)

46 (ii) *If $\rho > 0$, and if $\log M(r) = O(r^{\rho})$ does not hold, then $\rho_1 = \rho$, $f(z)$ has an infinity of zeros z_n, and $\sum |z_n|^{-\tau}$ is divergent.*

Proof. The first part is merely an affirmation of Hadamard's theorem, $\rho_1 \leq \rho$ and Theorem 2.

For the second part, we must have $\rho_1 = \rho$; for if $\rho_1 < \rho$, then $\rho = q$ by the first part, so that $\rho_1 < q$, which implies, by Theorem 4, that $logM(r) = O(r^q) = O(r^{\rho})$ in contradiction with our hypothesis. Since $\rho > 0$, and $\rho_1 = \rho$ we have therefore $\rho_1 > 0$, *which implies that $f(z)$ has an infinity of zeros.* Finally $\sum\limits_{1}^{\infty} |z_n|^{-\rho_1}$ is divergent, for if $\sum |z_n|^{\rho_1} < \infty$, then by Theorem 4, we would have $\log M(r) = O(r^{\rho})$ which contradicts the hypothesis. □

Remarks. (1) ρ_1 is called the 'real order' of $f(z)$, and ρ the 'apparent order Max (q, p) is called the 'genus' of $f(z)$.

(2) If ρ is not an integer, then $\rho = \rho_r$, for q is an integer and $\rho = \max(\rho_1, q)$. In particular, if ρ is non-integral, then f must have an infinity of zeros.

Lecture 6

The Gamma Function

1 Elementary Properties [15, p.148]

47

Define the function Γ by the relation.

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt, \ \ x \text{ real}.$$

(i) This integral converges at the 'upper limit' ∞, for all x, since $t^{x-1}e^{-t} = t^{-2}t^{x+1}e^{-t} = O(t^{-2})$ as $t \to \infty$; it converges at the lower limit o for $x > 0$.

(ii) The integral converges uniformly for $0 < a \leq x \leq b$. For

$$\int_0^\infty t^{x-1}e^{-t}dt = \int_0^1 + \int_1^\infty = O\left(\int_0^1 t^{a-1}dt\right) + O\left(\int_1^\infty t^{b-1}e^{-t}dt\right)$$

$$= O(1), \ \text{independently of } x.$$

Hence the integral represents a *continuous function for* $x > 0$.

(iii) If z is complex, $\int_0^\infty t^{z-1}e^{-t}dt$ is again uniformly convergent over any finite region in which $\operatorname{Re} z \geq a > 0$. For if $z = x + iy$, then

$$|t^{z-1}| = t^{x-1},$$

and we use (ii). Hence $\Gamma(z)$ is an *analytic function for* $\operatorname{Re} z > 0$

(iv) If $x > 1$, we integrate by parts and get

$$\Gamma(x) = \left[-t^{x-1}e^{-t}\right]_0^\infty + (x-1)\int_0^\infty t^{x-2}e^{-t}dt$$
$$= (x-1)\Gamma(x-1).$$

48 However,

$$\Gamma(1) = \int_0^\infty e^{-t}dt = 1; \text{ hence}$$
$$\Gamma(n) = (n-1)!,$$

if n is a $+ve$ integer. Thus $\Gamma(x)$ is a generalization of n!

(v) We have

$$\log\Gamma(n) = (n - \frac{1}{2})\log n - n + c + O(1),$$

where c is a constant.

$$\log\Gamma(n) = \log(n-1)! = \sum_{r=1}^{n-1}\log r.$$

We estimate $\log r$ by an integral: we have

$$\int_{r-\frac{1}{2}}^{r+\frac{1}{2}}\log t\, dt = \int_{-\frac{1}{2}}^{\frac{1}{2}}\log(s+r)ds = \int_0^{\frac{1}{2}} + \int_{-\frac{1}{2}}^{0}$$
$$= \int_0^{1/2}[\log(t+r) + \log(r-t)]\ dt$$

$$= \int_0^{1/2} \left[\log r^2 + \log\left(1 - \frac{t^2}{r^2}\right)\right] dt$$

$$= \log r + c_r, \text{ where } c_r = O\left(\frac{1}{r^2}\right).$$

Hence

$$\log \Gamma(n) = \sum_{r=1}^{n-1} \left(\int_{r-\frac{1}{2}}^{r+\frac{1}{2}} \log t \, dt - c_r \right)$$

$$= \int_{\frac{1}{2}}^{n-\frac{1}{2}} \log t \, dt - \sum_{r=1}^{n-1} c_r$$

$$= \left\{ \left(n - \frac{1}{2}\right) \log \left(n - \frac{1}{2}\right) - \frac{1}{2} \log \frac{1}{2} \right\} - (n - \frac{1}{2}) + \frac{1}{2}$$

$$- \sum_{r=1}^{\infty} c_r + \sum_{n}^{\infty} C_r$$

$$= (n - \frac{1}{2}) \log n - n + c + o(1), \text{ where } c \text{ is a constant.}$$

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Hence, also

$$(n!) = a.e^{-n} n^{n+\frac{1}{2}} e^{o(1)},$$

where 'a' is a constant such that $c = \log a$

(vi) We have

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_O^{\infty} \frac{t^{y-1}}{(1+t)^{x+y}} dt, \; x > 0, \; y > 0.$$

For

$$\Gamma(x)\Gamma(y) = \int_0^{\infty} t^{x-1} e^{-t} dt \cdot \int_0^{\infty} s^{y-1} e^{-s} ds, \; x > o, \; y > 0$$

Put $s = tv$; then

$$\begin{aligned}
\Gamma(x)\Gamma(y) &= \int_0^\infty t^{x-1}e^{-t}\,dt \int_0^\infty t^y v^{y-1}e^{-tv}\,dv \\
&= \int_0^\infty v^{y-1}dv \int_0^\infty t^{x+y-1}e^{-t(1+v)}dt \\
&= \int_0^\infty v^{y-1}dv \int_0^\infty \frac{u^{x+y-1}e^{-u}}{(1+v)^{x+y}}du \\
&= \Gamma(x+y)\int_0^\infty \frac{v^{y-1}}{(1+v)^{x+y}}dv.
\end{aligned}$$

50 The inversion of the repeated integral is justified by the use of the fact that the individual integrals converge uniformly for $x \geq \varepsilon > 0$, $y \geq \varepsilon > 0$.

(vii) Putting $x = y = \dfrac{1}{2}$ in (vii), we get [using the substitution $t = \tan^2\theta$]

$$\{\Gamma(\frac{1}{2})\}^2 = 2\Gamma(1)\int_0^{\pi/2} d\theta = \pi.$$

Since $\Gamma(\frac{1}{2}) > 0$, we get $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

Putting $y = 1 - x$, on the other hand, we get the important relation

$$\Gamma(x)\Gamma(1-x) = \int_0^\infty \frac{u^{-x}}{1+u}du = \frac{\pi}{\sin(1-x)\pi},\ 0 < x < 1,$$

since

$$\int_0^\infty \frac{x^{a-1}}{1+x}dx = \frac{\pi}{\sin a\pi} \quad \text{for} \quad 0 < a < 1, \ \text{by contour}$$

integration. Hence

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin x\pi}, \quad 0 < x < 1.$$

2 Analytic continuation of $\Gamma(z)$ [15, p. 148]

We have seen that the function

$$\Gamma(z) = \int_o^\infty e^{-t}t^{z-1}\,dt$$

is a regular function for $\operatorname{Re} z > 0$. We now seek to extend it analytically into the rest of the complex z-plane. Consider

$$I(z) \equiv \int_C e^{-\zeta}(-\zeta)^{z-1}d\zeta,$$

where C consists of the real axis from ∞ to $\delta > 0$, the circle $|\zeta| = \delta$ in **51**
the 'positive' direction $\curvearrowleft$ and again the real axis from δ to ∞.

We define

$$(-\zeta)^{z-1} = e^{(z-1)\log(-\zeta)}$$

by choosing $\log(-\zeta)$ to be real when $\zeta = -\delta$

The integral $I(z)$ is uniformly convergent in any finite region of the Z-plane, for the only possible difficulty is at $\zeta = \infty$, but this was covered by case (iii) in § 1. *Thus $I(z)$ is regular for all finite values of z.*

We shall evaluate $I(z)$. For this, set

$$\zeta = \rho e^{i\varphi}$$

so that

$$\log(-\zeta) = \log\rho + i(\varphi - \pi) \text{ on the contour}$$

[so as to conform with the requirement that $\log(-\zeta)$ is real when $\zeta = -\delta$]

Now the integrals on the portion of C corresponding to (∞, δ) and (δ, ∞) give [since $\varphi = 0$ in the first case, and $\varphi = 2\pi$ in the second]:

$$\int_\infty^\delta e^{-\rho+(z-1)\cdot(\log\rho - i\pi)}d\rho + \int_\delta^\infty e^{-\rho+(z-1)(\log\rho+i\pi)}d\rho$$

$$= \int_{\delta}^{\infty} \left[-e^{-\rho+(z-1)(\log \rho - i\pi)} + e^{-\rho+(z-1)(\log \rho + i\pi)} \right] d\rho$$

$$= \int_{\delta}^{\infty} e^{-\rho+(z-1)\log \rho} \left[e^{(z-1)i\pi} - e^{-(z-1)i\pi} \right] d\rho$$

$$= -2i \sin z\pi \int_{\delta}^{\infty} e^{-\rho} \cdot \rho^{z-1} d\rho$$

On the circle $|\zeta| = \delta$, we have

$$\begin{aligned} |(-\zeta)^{z-1}| &= |e^{(z-1)} \log(-\zeta)| = |e^{(z-1)[\log \delta + i(\varphi - \pi)]}| \\ &= e^{(x-1)\log \delta - y(\varphi-\pi)} \\ &= O(\delta^{x-1}). \end{aligned}$$

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The integral round the circle $|\zeta| = \delta$ therefore gives

$$O(\delta^x) = O(1), \text{ as } \delta \to 0, \text{ if } x > 0.$$

Hence, letting $\delta \to 0$, we get

$$\begin{aligned} I(z) &= -2i \sin \pi z \int_{0}^{\infty} e^{-\rho} \rho^{z-1} d\rho, \operatorname{Re} z > 0 \\ &= -2i \sin \pi z \Gamma(z), \ \operatorname{Re} z > 0. \end{aligned}$$

We have already noted that $I(z)$ is regular for all finite z; so

$$\frac{1}{2} i I(z) \operatorname{cosec} \pi z$$

is regular for all finite z, except (possibly) for the poles of $\operatorname{cosec} \pi z$ namely, $z = 0, \pm 1, \pm 2, \ldots$; and it equals $\Gamma(z)$ for $\operatorname{Re} z > 0$.

Hence - $\frac{1}{2} i I(z) \operatorname{cosec} z\pi$ *is the analytic continuation of* $\Gamma(z)$ *all over the z-plane.*

We know, however, by (iii) of § 1, that $\Gamma(z)$ is regular for $z = 1, 2, 3, \ldots$. Hence the only possible poles of $\frac{1}{2} i I(z) \operatorname{cosec} \pi x$ are $z = 0, -1, -2, \ldots$. These are actually poles of $\Gamma(z)$, for if z is one of these numbers, say $-n$, then $(-\zeta)^{z-1}$ is single-valued in 0 and $I(z)$ can be evaluated directly by Cauchy's theorem. Actually

$$(-1)^{n+1} \int\limits_C \frac{e^{-\zeta}}{\zeta^{n+1}} d\zeta = \frac{2\pi i}{n!} (-1)^{n+n+1} = \frac{-2\pi i}{n!}$$

$$\text{i.e. } I(-n) = -\frac{-2\pi i}{n!}.$$

So the poles of $\operatorname{cosec} \pi z$, at $z = 0, -n$, are actually poles of $\Gamma(z)$. The 53
residue at $z = -n$ is therefore

$$\lim_{z \to -n} \left(\frac{-2\pi i}{n!} \right) \left(\frac{z+n}{-2i \sin z\pi} \right) = \frac{(-1)^n}{n!}$$

The formula in (viii) of § 1 can therefore be upheld for complex values.

Thus

$$\Gamma(z) \cdot \Gamma(1-z) = \pi \operatorname{cosec} \pi z$$

for all *non-integral values of* z. Hence, also,

$$\frac{1}{\Gamma(z)} \text{ is an entire function.}$$

(Since the poles of $\Gamma(1-z)$ are cancelled by the zeros of $\sin \pi z$)

3 The Product Formula

We shall prove that $\frac{1}{\Gamma(z)}$ is an entire function of order 1.

Since $I(z) = -2i \sin z\pi \, \Gamma(z)$, and

$$\Gamma(z) \cdot \Gamma(1-z) = \frac{\pi}{\sin \pi z},$$

we get

$$I(1-z) = -2i \sin(\pi - z\pi) \Gamma(1-z)$$

$$= -2i \sin z\pi \frac{\pi}{\sin \pi z} \cdot \frac{1}{\Gamma(z)} \qquad = -\frac{2i\pi}{\Gamma(z)}$$

or

$$\frac{1}{\Gamma(z)} = \frac{I(1-z)}{-2i\pi}.$$

54 Now

$$I(z) = O\left[e^{\pi|z|}\left\{\int_0^1 e^{-t}t^{x-1}dt + \int_1^\infty e^{-t}t^{x-1}dt\right\}\right]$$

$$= O\left[e^{\pi|z|}\left\{1 + \int_1^\infty e^{-t}t^{|z|}dt\right\}\right]$$

$$= O\left[e^{\pi|z|}\left\{1 + \int_1^{1+|z|} + \int_{1+|z|}^\infty\right\}\right]$$

$$= O\left[e^{\pi|z|}(|z|+1)^{|z|+1}\right]$$

$$= O\left[e^{|z|^{1+\varepsilon}}\right], \ \varepsilon > 0.$$

Hence $\frac{1}{\Gamma(z)}$ is of order ≤ 1. However, the exponent of convergence of its zeros equals 1. Hence the order is $= 1$, and the genus of the canonical product is also equal to 1.

Thus

$$\frac{1}{\Gamma(z)} = z, e^{az+b}\prod_{n=1}^{\infty}(1 + \frac{z}{n})e^{\frac{-z}{n}}$$

by Hadamard's theorem. However, we know that $\lim\limits_{z\to 0+} \frac{1}{z\Gamma(z)} = 1$ and $\Gamma(1) = 1$. These substitutions give $b = 0$, and

$$1 = e^a \Pi\left(1 + \frac{1}{\pi}\right)e^{-1/n}$$

or,

$$0 = a + \sum_1^\infty \left[\log\left(1 + \frac{1}{n}\right) - \frac{1}{n}\right]$$

$$
\begin{aligned}
&= a - \lim_{m\to\infty} \left\{ \sum_1^m \left[\log\left(1+\frac{1}{n}\right) - \frac{1}{n} \right] \right\} \\
&= a + \lim_{m\to\infty} \left\{ \sum_1^m [\log(n+1) - \log n] - \sum_1^m \frac{1}{n} \right\} \\
&= a + \lim_{m\to\infty} \left[\log(m+1) - \sum_1^m \frac{1}{n} \right] \\
&= a - \gamma, \text{ where } \gamma \text{ is Euler's constant.}
\end{aligned}
$$

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Hence

$$\frac{1}{\Gamma(z)} = e^{\gamma z} z \prod_{n=1}^{\infty} \left\{ \left(1+\frac{z}{n}\right) e^{-z/n} \right\}$$

As a consequence of this we can derive Gauss's expression for $\Gamma(z)$. For

$$
\begin{aligned}
\Gamma(z) &= \lim_{n\to\infty} \left[e^{z(\log n - 1 - \frac{1}{2} - \ldots - \frac{1}{n})} \frac{1}{z} \cdot \frac{e^{z/1}}{1+\frac{z}{1}} \cdots \frac{e^{z/n}}{1+\frac{z}{n}} \right] \\
&= \lim_{n\to\infty} \left[n^z \cdot \frac{1}{z} \cdot \frac{1}{z+1} \cdot \frac{2}{z+2} \cdots \frac{n}{z+n} \right] \\
&= \lim_{n\to\infty} \left[\frac{n^z \cdot n!}{z(z+1)\ldots(z+n)} \right]
\end{aligned}
$$

We shall denote

$$\frac{n^z \cdot n!}{z(z+1)\ldots(z+n)} \equiv \Gamma_n(z),$$

so that **56**

$$\Gamma(z) = \lim_{n\to\infty} \Gamma_n(z).$$

Further, we get by logarithmic-differentiation,

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} - \sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n} \right).$$

Thus

$$\frac{d^2}{dz^2}[\log \Gamma(z)] = \sum_{n=0}^{\infty} \frac{1}{(n+z)^2};$$

Hence

$$\frac{d^2}{dz^2}[\log \Gamma(z)] > 0 \text{ for real, positive } z.$$

Lecture 7

The Gamma Function: Contd

4 The Bohr-Mollerup-Artin Theorem [7, Bd.I, p.276]

We shall prove that the functional equation of $\Gamma(x)$, namely $\Gamma(x+1) =$ 57
$x\Gamma(x)$, together with the fact that $\dfrac{d^2}{dx^2}[\log\Gamma(x)] > 0$ for $x > 0$, determines $\Gamma(x)$ 'essentially' uniquely-essentially, that is, except for a constant of proportionality. If we add the normalization condition $\Gamma(1) = 1$, then these three properties determine Γ uniquely. N.B. The functional equation *alone* does not define Γ uniquely since $g(x) = p(x)\Gamma(x)$, where p is any analytic function of period 1 also satisfies the same functional equation.

We shall briefly recall the definition of 'Convex functions'. A real-valued function $f(x)$, defined for $x > 0$, is *convex*, if the corresponding function

$$\phi(y) = \frac{f(x+y) - f(x)}{y},$$

defined for all $y > -x$, $y \neq 0$, is monotone increasing throughout the range of its definition.

If $0 < x_1 < x < x_2$ are given, then by choosing $y_1 = x_1 - x$ and

$y_2 = x_2 - x$, we can express the condition of convexity as:

$$f(x) \leq \frac{x_2 - x}{x_2 - x_1} f(x_1) + \frac{x - x_1}{x_2 - x_1} f(x_2)$$

58 Letting $x \to x_1$, we get $f(x_1 + 0) \leq f(x_1)$; and letting $x_2 \to x$, we get $f(x) \leq f(x + 0)$ and hence $f(x_1 + 0) = f(x_1)$ for all x_1. Similarly $f(x_1 - 0) = f(x_1)$ for all x_1.

Thus *a convex function is continuous*.

Next, if $f(x)$ is twice (continuously) differentiable, we have

$$\varphi'(y) = \frac{yf'(x+y) - f(x+y) + f(x)}{y^2}$$

and writing $x + y = u$, we get

$$f(x) = f(u - y) = f(u) - yf'(u) + \frac{y^2}{2} f''[u - (1 - \theta)y], 0 \leq \theta \leq 1$$

which we substitute in the formula for $\varphi'(y)$ so as to obtain

$$\varphi'(y) = \frac{1}{2} f''[x + \theta y].$$

Thus if $f''(x) \geq 0$ for all $x > 0$, then $\varphi'(y)$ is monotone increasing and f is convex. [The converse is also true].

Thus $\log \Gamma(x)$ is a convex function; by the last formula of the previous section we may say that $\Gamma(x)$ is *'logarithmically convex'* for $x > 0$. Further $\Gamma(x) > 0$ for $x > 0$.

Theorem. $\Gamma(x)$ *is the positive function uniquely defined for* $x > 0$ *by the conditions:*

(1.) $\Gamma(x + 1) = x\Gamma(x)$

(2.) $\Gamma(x)$ *is logarithmically convex*

(3.) $\Gamma(1) = 1$.

Proof. Let $f(x) > 0$ be any function satisfying the above three conditions satisfied by $\Gamma(x)$.

59 Choose an integer $n > 2$, and $0 < x \leq 1$.

Let

$$n - 1 < n < n + x \leq n + 1.$$

By logarithmic convexity, we get

$$\begin{aligned}\frac{\log f(n-1) - \log f(n)}{(n-1) - n} &< \frac{\log f(n+x) - \log f(n)}{(n+x) - n}\\ &\leq \frac{\log f(n+1) - \log f(n)}{(n+1) - n}\end{aligned}$$

Because of conditions 1 and 3, we get

$$f(n-1) = (n-2)!,\ \ f(n) = (n-1)!, f(n+1) = n!$$

and

$$f(n+x) = x(x+1)\ldots(x+n-1)f(x).$$

Substituting these in the above inequalities we get

$$\log(n-1)^x < \log \frac{f(n+x)}{(n-1)!} \leq \log n^x,$$

which implies, since log is monotone,

$$\text{or} \qquad (n-1)^x < \frac{f(n+x)}{(n-1)!} \leq n^x$$

$$\frac{(n-1)!(n-1)^x}{x(x+1)\ldots(x+n-1)} < f(x) \leq \frac{(n-1)!n^x}{x(x+1)\ldots(x+n-1)}$$

Replacing $(n-1)$ by n in the first inequality, we get

$$\Gamma_n(x) < f(x) \leq \Gamma_n(x) \cdot \frac{x+n}{n}, n = 2, 3, \ldots$$

where Γ_n is defined as in Gauss's expression for Γ.

Letting $n \to \infty$, we get

$$f(x) = \Gamma(x),\ \ 0 < x \leq 1.$$

For values of $x > 1$, we uphold this relation by the functional equation. $\square$

5 Gauss's Multiplication Formula. [7, Bd.I, p.281]

60 Let p be a positive integer, and

$$f(x) = p^x \Gamma\left(\frac{x}{p}\right)\Gamma\left(\frac{x+1}{p}\right)\ldots\Gamma\left(\frac{x+p-1}{p}\right)$$

[If $p = 1$, then $f(x) = \Gamma(x)$]. For $x > 0$, we then have

$$\frac{d^x}{dx^2}[\log f(x)] > 0.$$

Further

$$f(x+1) = xf(x).$$

Hence by the Bohr-Artin theorem, we get

$$f(x) = a_p \Gamma(x).$$

Put $x = 1$. Then we get

$$a_p = p\Gamma\left(\frac{1}{p}\right)\cdot\Gamma\left(\frac{2}{p}\right)\ldots\Gamma\left(\frac{p}{p}\right) \tag{5.1}$$

[$a_1 = 1$, by definition]. Put $k = 1, 2, \ldots p$, in the relation

$$\Gamma_n\left(\frac{k}{p}\right) = \frac{n^{k/p} n! p^{n+1}}{k(k+p)\ldots(k+np)},$$

and multiply out and take the limit as $n \to \infty$. We then get from (5.1),

$$a_p = p. \lim_{n\to\infty} \frac{n^{\frac{p+1}{2}}(n!)^p p^{np+p}}{(np+p)!}$$

However

$$(np+p)! = (np)!(np)^p \left[\left(1+\frac{1}{np}\right)\left(1+\frac{2}{np}\right)\ldots\left(1+\frac{p}{np}\right)\right]$$

Thus, for fixed p, as $n \to \infty$, we get

$$a_p = p \lim_{n\to\infty} \frac{(n!)^p p^{np}}{(np)! n^{(p-1)/2}}$$

61 Now by the asymptotic formula obtained in (v) of § 1, [see p.49]

$$(n!)^p = a^p n^{np+p/2} e^{-np} e^{o(1)}$$
$$(np)! = a n^{np+\frac{1}{2}} p^{np+\frac{1}{2}} e^{-np} e^{o(1)}$$

Hence

$$a_p = \sqrt{p} \cdot a^{p-1}.$$

Putting $p = 2$ and using (5.1) we get

$$a = \frac{1}{\sqrt{2}}, \; a_2 = \frac{1}{\sqrt{2}} 2\Gamma\left(\frac{1}{2}\right)\Gamma(1) = \sqrt{2\pi} \tag{5.2}$$

Hence

$$a_p = \sqrt{p}(2\pi)^{(p-1)/2}.$$

Thus

$$p^x \Gamma\left(\frac{x}{p}\right)\ldots\Gamma\left(\frac{x+p-1}{p}\right) = \sqrt{p}(2\pi)^{(p-1)/2}\Gamma(x)$$

For $p = 2$ and $p = 3$ we get:

$$\Gamma\left(\frac{x}{2}\right)\Gamma\left(\frac{x+1}{2}\right) = \frac{\sqrt{\pi}}{2^{x-1}}\Gamma(x); \quad \Gamma\left(\frac{x}{3}\right)\Gamma\left(\frac{x+1}{3}\right)\Gamma\left(\frac{x+2}{3}\right) = \frac{2\pi}{3^{x-\frac{1}{2}}}\Gamma(x).$$

6 Stirling's Formula [15, p.150]

From (v) of § 1, p.49 and (5.2) we get

$$(n!) = \sqrt{2\pi} \cdot e^{-n} \cdot n^{n+\frac{1}{2}} e^{o(1)},$$

which is the same thing as

$$\log(n-1)! = \left(n - \frac{1}{2}\right)\log n - n + \log \sqrt{2\pi} + o(1) \tag{6.1}$$

Further 62

$$1 + \frac{1}{2} + \cdots + \frac{1}{N-1} - \log N = \gamma + 0(1), \text{ as } N \to \infty \tag{6.2}$$

and

$$\begin{aligned}\log(N+z) &= \log N + \log\left(1+\frac{z}{N}\right)\\ &= \log N + \frac{z}{N} + O\left(\frac{1}{N^2}\right), \text{ as } N\to\infty. \qquad (6.3)\end{aligned}$$

We need to use (6.1)-(6.3) for obtaining the asymptotic formula for $\Gamma(z)$ where z is complex.

From the product-formula we get

$$\log\Gamma(z) = \sum_{n=1}^{\infty}\left\{\frac{z}{n} - \log\left(1+\frac{z}{n}\right)\right\} - \gamma z - \log z, \qquad (6.4)$$

each logarithm having its principal value.

Now

$$\begin{aligned}\int_0^N \frac{[u]-u+\frac{1}{2}}{u+z}du &= \sum_{n=0}^{N-1}\int_n^{n+1}\left(\frac{n+\frac{1}{2}+z}{u+z}-1\right)du\\ &= \sum_{n=0}^{N-1}(n+\frac{1}{2}+z)[\log(n+1+z)-\log(n-z)] - N\\ &= \sum_{n=0}^{N-1} n[\log(n+1+z)-\log(n+z)] + \left(z+\frac{1}{2}\right)\sum_{n=0}^{N-1}\\ &\qquad [\log(n+1+z)-\log(n+z)] - N\\ &= (N-1)\log(N+z) - \sum_{n=1}^{N-1}\log(n+z) + \left(z+\frac{1}{2}\right)\log(N+z)\\ &\qquad - \left(z+\frac{1}{2}\right)\log z - N\\ &= \left(N-\frac{1}{2}+z\right)\log(N+z) - \left(z+\frac{1}{2}\right)\log z\\ &\qquad - N - \sum_{n=1}^{N-1}\left[\log n + \log\left(1+\frac{z}{n}\right)\right]\end{aligned}$$

$$= \left(N - \frac{1}{2} + z\right)\log(N+z) - \left(z + \frac{1}{2}\right)\log z - N - \log(N-1)!$$
$$+ \sum_{n=1}^{N-1}\left\{\frac{z}{n} - \log\left(1 + \frac{z}{n}\right)\right\} - z\sum_{1}^{N-1}\frac{1}{n}$$

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Now substituting for $\log(N+z)$ etc., from (6.1)-(6.3), we get

$$\int_0^N \frac{[u] - u + \frac{1}{2}}{u+z}du = (N - \frac{1}{2} + z)\left\{\log N + \frac{z}{N} + O\left(\frac{1}{N^2}\right)\right\} - \left(z + \frac{1}{2}\right)$$
$$\log z - N - \left(N - \frac{1}{2}\right)\log N + N - c + O(1) +$$
$$\sum_{n=1}^{N-1}\left\{\frac{z}{n} - \log\left(1 + \frac{z}{n}\right)\right\} - z\sum_{1}^{N-1}\frac{1}{n}$$
$$= z\left(\log N - \sum_{1}^{N-1}\frac{1}{n}\right) + z + O(1) - \log\sqrt{2\pi}$$
$$-\left(z - \frac{1}{2}\right)\log z - \log z + \sum_{n=1}^{N-1}\left\{\frac{z}{n} - \log\left(1 + \frac{z}{n}\right)\right\}$$

Now letting $N \to \infty$, and using (6.4), and (6.2), we get **64**

$$\int_0^\infty \frac{[u] - u + \frac{1}{2}}{u+z}du + \frac{1}{2}\log 2\pi - z + \left(z - \frac{1}{2}\right)\log z = \log\Gamma(z) \qquad (6.5)$$

If $\varphi(u) = \int_0^u([v] - v + \frac{1}{2})dv$, then

$$\varphi(u) = O(1), \text{ since } \varphi(n+1) = \varphi(n),$$

if n is an integer.

Hence

$$\int_0^\infty \frac{[u] - u + \frac{1}{2}}{u+z}du = \int_0^\infty \frac{d\varphi(u)}{u+z} = \int_0^\infty \frac{\varphi(u)}{(u+z)^2}$$

$$= O\left[\int_0^\infty \frac{dv}{|u+z|^2}\right]$$

If we write $z = re^{i\theta}$, and $u = ru_1$, then

$$\int_0^\infty \frac{[u] - u + \frac{1}{2}}{u+z} du = O\left[\frac{1}{r}\int_0^\infty \frac{du_1}{|u_1 + \theta^{i\theta}|^2}\right]$$

The last integral is finite if $\theta \neq \pi$. Hence

$$\int_0^\infty \frac{[u] - u + \frac{1}{2}}{u+z} du = O\left(\frac{1}{r}\right), \text{ uniformly for } |arg\ z| \leq \pi - \varepsilon < \pi$$

Thus we get from (6.5):

$$\log\Gamma(z) = \left(z - \frac{1}{2}\right)\log z - z + \frac{1}{2}\log 2\pi + O\left(\frac{1}{|z|}\right)$$

for $|arg\ z| \leq \pi - \varepsilon < \pi$

Corollaries. *(1)* $\log\Gamma(z+\alpha) = (z+\alpha-\frac{1}{2})\log z - z + \frac{1}{2}\log 2\pi + O\left(\frac{1}{|z|}\right)$
65 *as* $|z| \to \infty$ *uniformly for* $|arg\ z| \leq \pi - \varepsilon < \pi$. α *bounded.*

(2) For any fixed x, as $y \to \pm\infty$

$$|\Gamma(x+iy)| \sim e^{-\frac{1}{2}\pi|y|}|y|^{x-\frac{1}{2}}\sqrt{2\pi}$$

(3) $\dfrac{\Gamma'(z)}{\Gamma(z)} = \log z - \dfrac{1}{2z} + O\left(\dfrac{1}{|z|^2}\right)$, $|arg\ z| \leq \pi - \varepsilon < \pi$ *[11, p.57]*

This may be deduced from (1) by using

$$f'(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^2} d\zeta,$$

where $f(z) = \log\Gamma(z) - \left(z - \frac{1}{2}\right)\log z + z - \frac{1}{2}\log 2\pi$ *and C is a circle with centre at* $\zeta = z$ *and radius* $|z|\sin\varepsilon/2$.

Lecture 8

The Zeta Function of Riemann

1 Elementary Properties of $\zeta(s)$ [16, pp.1-9]

We define $\zeta(s)$, for s complex, by the relation **66**

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \;\; s = \sigma + it, \;\; \sigma > 1. \tag{1.1}$$

We define x^s, for $x > 0$, as $e^{s \log x}$, where $\log x$ has its real determination. Then $|n^s| = n^\sigma$, and the series converges for $\sigma > 1$, and uniformly in any finite region in which $\sigma \geq 1 + \delta > 1$. Hence $\zeta(s)$ is regular for $\sigma \geq 1 + \delta > 1$. Its derivatives may be calculated by termwise differentiation of the series.

We could express $\zeta(s)$ also as an infinite product called the *Euler Product*:

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \sigma > 1, \tag{1.2}$$

where p runs through all the primes (known to be infinite in number!). It is the Euler product which connects the properties of ζ with the properties of primes. *The infinite product is absolutely convergent for* $\sigma > 1$, since the corresponding series

$$\sum_p \left|\frac{1}{p^s}\right| = \sum_p \frac{1}{p^\sigma} < \infty,\ \sigma > 1$$

Expanding each factor of the product, we can write it as

$$\prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots\right)$$

67 Multiplying out formally we get the expression (1.1) by the unique factorization theorem. This expansion can be justified as follows:

$$\prod_{p \leq p} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots\right) = 1 + \frac{1}{n_1^s} + \frac{1}{n_2^s} + \cdots,$$

where $n_1, n_2, \ldots$ are those integers none of whose prime factors exceed P. Since all integers $\leq P$ are of this form, we get

$$\left|\sum_{n=1}^{\infty} \frac{1}{n^s} - \prod_{p \leq P} \left(1 - \frac{1}{p^2}\right)^{-1}\right| = \left|\sum_{n=1}^{\infty} \frac{1}{n^s} - 1 - \frac{1}{n_1^s} - \frac{1}{n_2^s} - \cdots\right|$$
$$\leq \frac{1}{(P+1)^\sigma} + \frac{1}{(P+2)^\sigma} + \cdots$$

$\to 0$, as $P \to \infty$, if $\sigma > 1$.

Hence (1.2) has been established for $\sigma > 1$, on the basis of (1.1). As an immediate consequence of (1.2) we observe that $\zeta(s)$ *has no zeros for* $\sigma > 1$, since a convergent infinite product of non-zero factors is non-zero.

We shall now obtain a few formulae involving $\zeta(s)$ which will be of use in the sequel. We first have

$$\log \zeta(s) = -\sum_p \log\left(1 - \frac{1}{p^2}\right),\ \sigma > 1. \tag{1.3}$$

If we write $\pi(x) = \sum\limits_{p \le x} 1$, then

$$\begin{aligned}
\log\zeta(s) &= -\sum_{n=2}^{\infty}\{\pi(n) - \pi(n-1)\}\log\left(1 - \frac{1}{n^s}\right)\\
&= -\sum_{n=2}^{\infty}\pi(n)\left[\log\left(1 - \frac{1}{n^s}\right) - \log\left(1 - \frac{1}{(n+1)^s}\right)\right]\\
&= \sum_{n=2}^{\infty}\pi(n)\int\limits_{n}^{n+1}\frac{s}{x(x^s-1)}\,dx
\end{aligned}$$

Hence 68

$$\log\zeta(s) = s\int\limits_{2}^{\infty}\frac{\pi(x)}{x(x^s-1)}dx, \quad \sigma > 1. \tag{1.4}$$

It should be noted that the rearrangement of the series preceding the above formula is permitted because

$$\pi(n) \le n \text{ and } \log\left(1 - \frac{1}{n^s}\right) = O(n^{-\sigma}).$$

A slightly different version of (1.3) would be

$$\log\zeta(s) = \sum_{p}\sum_{m}\frac{1}{mp^{ms}}, \quad \sigma > 1 \tag{1.3$'$}$$

where p runs through all primes, and m through all positive integers. Differentiating (1.3), we get

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{p}\frac{p^{-s}\log p}{1-p^{-s}} = \sum_{p,m}\sum\frac{\log\rho}{p^{ms}},$$

or

$$\boxed{\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty}\frac{\wedge(n)}{n^s}, \quad \sigma > 1} \quad , \tag{1.5}$$

where $\wedge(n) = \begin{cases} \log p, & \text{if } n \text{ is a + ve power of a prime } p \\ 0, & \text{otherwise.} \end{cases}$ 69

Again

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right)$$

$$= \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}, \quad \sigma > 1 \tag{1.6}$$

(1.6) where $\mu(1) = 1$, $\mu(n) = (-1)^k$ if n is the product of k different primes, $\mu(n) = 0$ if n contains a factor to a power higher than the first.

We also have

$$\zeta^2(s) = \sum_{n=1}^{\infty} \frac{d(n)}{n^s}, \quad \sigma > 1$$

where $d(n)$ denotes the number of divisors of n, including 1 and n. For

$$\zeta^2(s) = \sum_{m=1}^{\infty} \frac{1}{m^s} \cdot \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{\mu=1}^{\infty} \frac{1}{\mu^s} \sum_{mn=\mu} 1$$

More generally

$$\boxed{\zeta^k(s) = \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s}, \quad \sigma > 1} \tag{1.7}$$

70 where $k = 2, 3, 4, \ldots$, and $d_k(n)$ is the number of ways of expressing n as a product of k factors.

Further

$$\zeta(s) \cdot \zeta(s-a) = \sum_{m=1}^{\infty} \frac{1}{m^s} \sum_{n=1}^{\infty} \frac{n^a}{n^s},$$

$$= \sum_{\mu=1}^{\infty} \frac{1}{\mu^s} \cdot \sum_{mn=\mu} n^a,$$

so that

$$\zeta(s) \cdot \zeta(s-a) = \sum_{\mu=1}^{\infty} \frac{\sigma_2(\mu)}{\mu^\delta} \tag{1.8}$$

where $\sigma_a(\mu)$ denotes the sum of the a^{th} powers of the divisors of μ.

If $a \neq 0$, we get, from expanding the respective terms of the Euler products,

$$\begin{aligned}\zeta(a)\zeta(s-a) &= \prod_p \left(1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\cdots\right)\left(1+\frac{p^a}{p^s}+\frac{p^{2a}}{p^{2a}}+\cdots\right)\\ &= \prod_p \left(1+\frac{1+p^a}{p^s}+\frac{1+p^a+p^{2a}}{p^{2s}}+\cdots\right)\\ &= \prod_p \left(1+\frac{1-p^{2a}}{1-p^a}\cdot\frac{1}{p^s}+\cdots\right)\end{aligned}$$

Using (1.8) we thus get

$$\sigma_2(n) = \frac{1-p_1^{(m_1+1)a}}{1-p_1^a}\cdots\frac{1-p_r^{(m_r+1)a}}{1-p_r^a} \tag{1.9}$$

if $n = p_1^{m_1}, p_2^{m_2}\ldots p_r^{m_r}$ by comparison of the coefficients of $\dfrac{1}{n^s}$. **71**

More generally, we have

$$\frac{\zeta(s)\cdot\zeta(s-a)\cdot\zeta(s-b)\cdot\zeta(s-a-b)}{\zeta(2s-a-b)} = \prod_p \frac{1-p^{-2s+a+b}}{(1-p^{-s})(1-p^{-s+a})(1-p^{-s+b})}$$
$$(1-p^{-s+a+b})$$

for $\sigma > \max\{1, \operatorname{Re} a+1, \ \operatorname{Re} b+1, \operatorname{Re}(a+b)+1\}$. Putting $p^{-s} = z$, we get the general term in the right hand side equal to

$$\begin{aligned}&\frac{1-p^{a+b}z^2}{(1-z)\cdot(1-p^az)\cdot(1-p^bz)(1-p^{a+b}z)}\\ &= \frac{1}{(1-p^a)(1-p^b)}\left\{\frac{1}{1-z}-\frac{p^a}{1-p^az}-\frac{p^b}{1-p^bz}+\frac{p^{a+b}}{1-p^{a+b}z}\right\}\\ &= \frac{1}{(1-p^a)(1-p^b)}\sum_{m=0}^{\infty}\left\{1-p^{(m+1)a}-p^{(m+1)b}+p^{(m+1)(a+b)}\right\}z^m\\ &= \frac{1}{(1-p^a)(1-p^b)}\sum_{m=o}^{\infty}\left\{1-p^{(m+1)a}\right\}\left\{1-p^{(m+1)b}\right\}z^m\end{aligned}$$

Hence **72**

$$\frac{\zeta(s)\cdot\zeta(s-a)\zeta(s-b)\cdot\zeta(s-a-h)}{\zeta(2s-a-b)}$$
$$=\prod_p\sum_{m=0}^{\infty}\frac{1-p^{(m+1)a}}{1-p^a}\cdot\frac{1-p^{(m+1)b}}{1-p^b}\cdot\frac{1}{p^{ms}}$$

Now using (1.9) we get

$$\frac{\zeta(s)\cdot\zeta(s-a)\cdot\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)}=\sum_{n=1}^{\infty}\frac{\sigma_a(n)\sigma_b(n)}{n^s} \tag{1.10}$$

$$\sigma>\max\{1,\operatorname{Re}a+1,\operatorname{Re}b+1,\operatorname{Re}(a+b)+1\}$$

If $a=b=0$, then

$$\frac{\{\zeta(s)\}^4}{\zeta(2s)}=\sum_{n=1}^{\infty}\frac{\{d(n)^2\}}{n^s},\quad \sigma>1. \tag{1.11}$$

If α is real, and $\alpha\neq 0$, we write αi for a and $-\alpha i$ for b in (1.10), and get

$$\frac{\zeta^2(s)\zeta(s-\alpha i)\zeta(s+\alpha i)}{\zeta(2s)}=\sum_{n=1}^{\infty}\frac{|\sigma_{\alpha i}(n)|^2}{n^s},\ \sigma>1, \tag{1.12}$$

where $\sigma_{\alpha i}(n)=\sum\limits_{d/n}d^{\alpha i}$.

Lecture 9

The Zeta Function of Riemann (Contd.)

2 Elementary theory of Dirichlet Series [10]

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The Zeta function of Riemann is the sum-function associated with the Dirichlet series $\sum \frac{1}{n^s}$. We shall now study, very briefly, some of the elementary properties of general Dirichlet series of the form

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}, o \leq \lambda_1 < \lambda_2 < \ldots \to \infty$$

Special cases are: $\lambda_n = \log n$; $\lambda_n = n$, $e^{-s} = x$. We shall first prove a lemma applicable to the summation of Dirichlet series.

Lemma. *Let* $A(x) = \sum_{\lambda_n \leq x} a_n$, *where* a_n *may be real or complex. Then, if* $x \geq \lambda_1$, *and* $\phi(x)$ *has a continuous derivative in* $(0, \infty)$, *we have*

$$\sum_{\lambda_n \leq \omega} a_n \phi(\lambda_n) = -\int_{\lambda_1}^{\omega} A(x)\phi'(x)dx + A(\omega)\phi(\omega).$$

If $A(\omega)\phi(\omega) \to 0$ as $\omega \to \infty$, then

$$\sum_{n=1}^{\infty} a_n \phi(\lambda_n) = -\int_{\lambda_1}^{\infty} A(x)\phi'(x)dx,$$

provided that either side is convergent.

Proof.

$$\begin{aligned} A(\omega)\phi(\omega) - \sum_{\lambda_n \leq \omega} a_n \phi(\lambda_n) \\ &= \sum_{\lambda_n \leq \omega} a_n \{\phi(\omega) - \phi(\lambda_n)\} \\ &= \sum_{\lambda_n \leq \omega} \int_{\lambda_n}^{\omega} a_n \phi'(x)dx \\ &= \int_{\lambda_1}^{\omega} A(x)\phi'(x)dx \end{aligned}$$

74 □

Theorem 1. *If $\sum\limits_{n=1}^{\infty} a_n e^{-\lambda_n s}$ converges for $s = s_0 \equiv \sigma_0 + it_o$, then it converges uniformly in the angular region defined by*

$$|am(s - s_0)| \leq \theta < \pi/2, \quad \textit{for} \quad 0 < \theta < \pi/2$$

Proof. We may suppose that $s_o = 0$. For

$$\sum a_n e^{-\lambda_n s} = \sum a_n e^{-\lambda_n s_0} e^{-\lambda_{n(s-s_0)}} \equiv \sum b_n e^{-\lambda_n s'}, \text{ say,}$$

and the new series converges at $s' = 0$. By the above lemma, we get

$$\sum_{\mu+1}^{\nu} a_n e^{-\lambda_n s} = \left(\sum_{1}^{\nu} - \sum_{1}^{\mu}\right) a_n e^{-\lambda n^s}$$

$$= \int_{\lambda_\mu}^{\lambda_\nu} A(x) \cdot e^{-xs} \cdot s dx + A(\lambda_\nu) e^{-\lambda_\nu s} - A(\lambda_\mu) e^{-\lambda_\mu s}$$

$$= s \int_{\lambda_\mu}^{\lambda_\nu} \{A(x) - A(\lambda_\mu)\} e^{-xs} dx + [A(\lambda_\nu) - A(\lambda_\mu)] e^{-\lambda_\nu s}$$

We have assumed that $\sum a_n$ converges, therefore, given $\epsilon > 0$, there exists an n_0 such that for $x > \lambda_\mu \geq n_0$, we have 75

$$|A(x) - A(\lambda_\mu)| < \varepsilon$$

Hence, for such μ, we have

$$\begin{aligned} \left| \sum_{\mu+1}^{\nu} a_n e^{-\lambda_n s} \right| &\leq \varepsilon |s| \int_{\lambda_\mu}^{\lambda_\nu} e^{-x\sigma} dx + \varepsilon e^{-\lambda_\nu \sigma} \\ &\leq 2\varepsilon \frac{|s|}{\sigma} + \varepsilon \\ &< 2\varepsilon\,(\sec\theta + 1), \end{aligned}$$

if $\sigma \neq 0$, since $\dfrac{|s|}{\sigma} < \sec\theta$, and this proves the theorem. As a consequence of Theorem 1, we deduce □

Theorem 2. *If* $\sum a_n e^{-\lambda n^s}$ *converges for* $s = s_0$*, then it converges for* Re $s > \sigma_o$*, and uniformly in any bounded closed domain contained in the half-plane* $\sigma > \sigma_o$*. We also have*

Theorem 3. *If* $\sum a_n e^{-\lambda_n s}$ *converges for* $s = s_0$ *to the sum* $f(s_0)$*, then* $f(s) \to f(s_0)$ *as* $s \to s_0$ *along any path in the region* $|am(s - s_0)| \leq \theta < \pi/2$.

A Dirichlet series may converge for *all* values of s, or *some* values of s, or *no* values of s.

Ex.1. $\sum a_n n^{-s}$, $a_n = \dfrac{1}{n!}$ converges for *all* values of s.

Ex.2. $\sum a_n n^{-s}$, $a_n = n!$ converges for *no* values of s.

Ex.3. $\sum n^{-s}$ converges for Re $s > 1$, and diverges for Re $s \leq 1$.

76 If a Dirichlet series converges for some, but not all, values of s, and if s_1 is a point of convergence while s_2 is a point of divergence, then, on account of the above theorems, we have $\sigma_1 > \sigma_2$. All points on the real axis can then be divided into two classes L and U; $\sigma \in$ U if the series converges at σ, otherwise $\sigma \in$ L. Then any point of U lies to the right of any point of L, and this classification defines a number σ_o such that the series converges for every $\sigma > \sigma_o$ and diverges for $\sigma < \sigma_o$, the cases $\sigma = \sigma_0$ begin undecided. Thus the region of convergence is a half-plane. The line $\sigma = \sigma_0$ is called the *line of convergence*, and the number σ_o is called the *abscissa of convergence*. We have seen that σ_0 may be $\pm\infty$. On the basis of Theorem 2 we can establish

Theorem 4. *A Dirichlet series represents in its half-plane of convergence a regular function of s whose successive derivatives are obtained by termwise differentiation.*

We shall now prove a theorem which implies the '*uniqueness*' of Dirichlet series.

Theorem 5. *If* $\sum a_n e^{-\lambda_n^s}$ *converges for* $s = 0$, *and its sum* $f(s) = 0$ *for an infinity of values of s lying in the region:*

$$\sigma \geq \varepsilon > 0, \quad |am\ s| \leq \theta < \pi/2,$$

then $a_n = 0$ *for all n.*

Proof. $f(s)$ cannot have an infinite number of zeros in the neighbourhood of any *finite* point of the giver region, since it is regular there. Hence there exists an infinity of valves $s_n = \sigma_n + it_n$, say, with $\sigma_{n+1} > \sigma_\eta$, $\lim \sigma_n = \infty$ such that $f(s_n) = 0$.

77 However,

$$g(s) \equiv e^{\lambda_1 s} f(s) = a_1 + \sum_2^\infty a_n e^{-(\lambda_n - \lambda_1)s},$$

(here we are assuming $\lambda_1 > 0$) converges for $s = 0$, and is therefore uniformly convergent in the region given; each term of the series on the right $\to 0$, as $s \to \infty$ and hence the right hand side, as a whole, tends to a_1 as $s \to \infty$. Thus $g(s) \to a_1$ as $s \to \infty$ along any path in the given region. This would contradict the fact that $g(s_n) = 0$ for an infinity of $s_n \to \infty$, unless $a_1 = 0$. Similarly $a_2 = 0$, and so on. □

Remark. In the hypothesis it is essential that $\varepsilon > 0$, for if $\varepsilon = 0$, the origin itself may be a limit point of the zeros of f, and a contradiction does not result in the manner in which it is derived in the above proof.

The above arguments can be applied to $\sum a_n e^{-\lambda_n s}$ so as to yield the existence of an *abscissa of absolute convergence* $\bar{\sigma}$, etc. In general, $\bar{\sigma} \geq \sigma_0$. The strip that separates $\bar{\sigma}$ from σ_o may comprise the whole plane, or may be vacuous or may be a half-plane.

Ex.1. $\sigma_0 = 0, \bar{\sigma} = 1$

$$
\begin{aligned}
(1 - 2^{1-s})\zeta(s) &= \left(\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots\right) - 2\left(\frac{1}{2^s} + \frac{1}{4^s} + \cdots\right) \\
&= \left(\frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \cdots\right)
\end{aligned}
$$

Ex.2. $\sum \dfrac{(-1)^n}{\sqrt{n}} (\log n)^{-s}$ converges for all s but never absolutely. In the 78
simple case $\lambda_n = \log n$, we have

Theorem 6. $\bar{\sigma} - \sigma_0 \leq 1$.

For if $\sum a_n n^{-s} < \infty$, *then* $|a_n| \cdot n^{-\sigma} = O(1)$, *so that* $\sum \dfrac{|a_n|}{n^{s+1+\varepsilon}} < \infty$ *for* $\varepsilon > 0$

While we have observed that the sum-function of a Dirichlet series is regular in the half-plane of convergence, there is no reason to assume that the line of convergence contains at least one singularity of the function (see Ex 1. above!). In the special case where the coefficients are positive, however we can assert the following

Theorem 7. *If* $a_n \geq 0$, *then the point* $s = \sigma_0$ *is a singularity of the function* $f(s)$.

Proof. Since $a_n \geq 0$, we have $\sigma_0 = \bar{\sigma}$, and we may assume, without loss of generality, that $\sigma_0 = 0$. Then, if $s = 0$ is a regular point, the Taylor series for $f(s)$ at $s = 1$ will have a radius of convergence > 1. (since the circle of convergence of a power series must have at least one singularity). Hence we can find a negative value of s for which

$$f(s) = \sum_{\nu=0}^{\infty} \frac{(s-1)^{\nu}}{\nu!} f^{(\nu)}(1) = \sum_{\nu=0}^{\infty} \frac{(1-s)^{\nu}}{\nu 1} \sum_{n=1}^{\infty} a_n \lambda_n^{\nu} e^{-\lambda_n}$$

Here every term is positive, so the summations can be interchanged and
79 we get

$$f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n} \sum_{\nu=0}^{\infty} \frac{(1-s)^{\nu} \lambda_n^{\nu}}{\nu!} = \sum_{n=1}^{\infty} a_n e^{-\lambda_n^s}$$

Hence the series converges for a negative value of s which is a contradiction. ◻

Lecture 10

The Zeta Function of Riemann (Contd)

2 (Contd). Elementary theory of Dirichlet series

We wish to prove a formula for the partial sum of a Dirichlet series. We shall give two proofs, one based on an estimate of the order of the sum-function [10], and another [14, Lec. 4] which is independent of such an estimate. We first need the following **80**

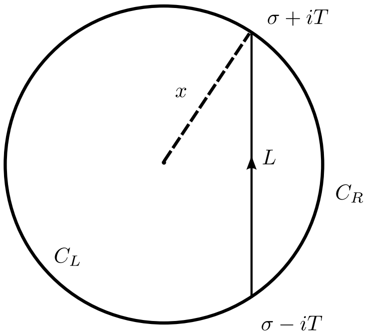
Lemma 1. *If* $\sigma > 0$, *then*

$$\frac{1}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty}\frac{e^{us}}{s}ds = \begin{cases} 1, & \text{if } u > 0 \\ 0, & \text{if } u < 0 \\ \frac{1}{2}, & \text{if } u = 0 \end{cases}$$

where $\int\limits_{\sigma-1\infty}^{\sigma+i\infty} = \lim\limits_{T\to\infty}\int\limits_{\sigma-iT}^{\sigma+iT}$.

Proof. The case $u = 0$ is obvious. We shall therefore study only the cases $u \gtrless 0$. Now

$$\int\limits_{\sigma-iT}^{\sigma+iT}\frac{e^{us}ds}{s} = \frac{e^{us}}{us}\int\limits_{\sigma-iT}^{\sigma+iT} + \int\limits_{\sigma-iT}^{\sigma+iT}\frac{e^{us}}{us^2}ds$$



However

$$|e^{u(\sigma+iT)}| = e^{u\sigma}$$

and $|s| > T$; hence, letting $T \to \infty$, we get

$$\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{e^{us}ds}{s} = \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{e^{us}}{us^2} ds \equiv I, \text{ say.}$$

81 We shall evaluate I on the contour suggested by the diagram. If $u < 0$, we use the contour $L + C_R$; and if $u > 0$, we use $L + C_L$.

If $u < 0$, we have, on C_R,

$$\left|\frac{e^{us}}{s^2}\right| \leq \frac{e^{u\sigma}}{r^2}, \text{ since } \operatorname{Re} s > \sigma.$$

If $u > 0$, we have, on C_L,

$$\left|\frac{e^{us}}{s^2}\right| \leq \frac{e^{u\sigma}}{r^2}, \text{ since } \operatorname{Re} s < \sigma.$$

Hence

$$\left|\int_{C_R, C_L} \frac{e^{us}ds}{us^2}\right| \leq 2\pi r \cdot \frac{e^{u\sigma}}{|u|\gamma^2} \to 0, \text{ as } r \to \infty.$$

By Cauchy's theorem, however, we have

$$\int_L = -\int_{C_R};$$

hence

$$\frac{1}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty}\frac{e^{us}}{us^2}ds=0,\quad \text{if } u<0. \tag{1}$$

If, however, $u > 0$, the contour $L + C_L$ encloses a pole of the second order at the origin, and we get

$$\int\limits_{L} = -\int\limits_{C_L} +1,$$

so that

$$\frac{1}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty}\frac{e^{us}}{us^2}ds=1,\quad \text{if } u>0. \tag{2}$$

□

Remarks. We can rewrite the Lemma as: if $a > 0$, 82

$$\frac{1}{2\pi i}\int\limits_{a-i\infty}^{a+i\infty}\frac{e^{(\lambda-\lambda_n)s}}{s}ds=\begin{cases}0, & \text{if } \lambda<\lambda_n\\ 1, & \text{if } \lambda>\lambda_n\\ \frac{1}{2}, & \text{if } \lambda=\lambda_n\end{cases}$$

Formally, therefore, we should have

$$\begin{aligned}\frac{1}{2\pi i}\int\limits_{a-i\infty}^{a+i\infty}f(s)\frac{e^{\lambda s}}{s}ds &= \frac{1}{2\pi i}\sum_{1}^{\infty}a_n\int\limits_{a-i\infty}^{a+i\infty}\frac{e^{(\lambda-\lambda_n)s}}{s}ds\\ &= {\sum_{\lambda_n\le\lambda}}' a_n\end{aligned}$$

the dash denoting that if $\lambda = \lambda_n$ the last term should be halved. We wish now to establish this on a rigorous basis.

We proceed to prove another lemma first.

Lemma 2. *Let $\sum a_n e^{-\lambda_n s}$ converge for $s = \beta$ real. Then*

$$\sum_1^n a_\nu e^{-\lambda_\nu s} = o(|t|),$$

the estimate holding uniformly both in $\sigma \geq \beta + \varepsilon > \beta$ and in n. In particular, for $n = \infty$, $f(s) = o(|t|)$ uniformly for $\sigma \geq \beta + \varepsilon > \beta$

Proof. Assume that $\beta = 0$, without loss of generality. Then

$$a_\nu = O(1) \text{ and } A(\lambda_\nu) - A(\lambda_\mu) = O(1) \text{ for all } \nu \text{ and } \mu$$

If $1 < N < n$, then

$$\begin{aligned}
\sum_1^n a_\nu e^{-\lambda_\nu s} &= \sum_1^N + \sum_{N+1}^n a_\nu e^{-\lambda_\nu s} \\
&= \sum_1^N a_\nu e^{-\lambda_\nu s} + s \int_{\lambda_N}^{\lambda_n} \{A(x) - A(\lambda_N)\} e^{-xs} dx \\
&\qquad \{A(\lambda_n) - A(\lambda_N)\} e^{-\lambda_n^s} \\
&= O(N) + O(1) + O\left(|s| \int_{\lambda_N}^{\lambda_n} e^{-x\sigma} dx \right) \\
&= O(N) + O\left(\frac{|s|}{\sigma} e^{-\lambda_N \sigma} \right) \\
&= O(N) + O\left(|t| e^{-\lambda_N \varepsilon} \right)
\end{aligned}$$

83 since $|s|^2 = t^2 + \sigma^2$ and $\sigma \geq \varepsilon$.

Hence

$$\sum_1^n a_\nu e^{-\lambda_\nu s} = O(N) + O(|t| e^{-\lambda_N \varepsilon}), \text{ if } 1 < N < n$$

If $N \geq n$, then, trivially,

$$\sum_1^n a_\nu e^{-\lambda_\nu s} = O(N).$$

If we choose N as a function of $|t|$, tending to ∞ more slowly than $|t|$, we get

$$\sum_{1}^{n} a_\nu e^{-\lambda_\nu s} = O(|t|)$$

in either case. $\square$ **84**

Theorem (Perron's Formula). *Let σ_0 be the abscissa of convergence of $\sum a_n e^{-\lambda_n s}$ whose sum-function is $f(s)$. Then for $\sigma > \sigma_0$ and $\sigma > 0$, we have*

$$\sideset{}{'}\sum_{\lambda_\nu \le \omega} a_n = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{f(s)}{s} e^{\omega s} ds,$$

where the dash denotes the fact that if $\omega = \lambda_n$ the last term is to be halved.

Proof. Let $\lambda_m \le \omega < \lambda_{m+1}$, and

$$g(s) = e^{\omega s} \left\{ f(s) - \sum_{1}^{m} a_n e^{-\lambda_n s} \right\}$$

(This has a meaning since $\operatorname{Re} s \equiv \sigma > \sigma_0$). Now

$$\frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{g(s)}{s} ds$$

$$= \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{f(s) e^{\omega s}}{s} ds - \sideset{}{'}\sum_{\lambda_n \le \omega} a_n$$

by the foregoing lemma. We have to show that the last expression is zero.

Applying Cauchy's theorem to the rectangle

$$\sigma - iT_1, \sigma + iT_2, \Omega + iT_2, \Omega - iT_1,$$

where $T_1, T_2 > 0$, and $\Omega > \sigma$, we get

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{g(s)}{s}ds = 0,$$

85 for

$$\frac{1}{2\pi i}\int_{\sigma-iT_1}^{\sigma+iT_2} = \frac{1}{2\pi i}\left[\int_{\sigma-iT_1}^{\Omega-iT_1} + \int_{\Omega-iT_1}^{\Omega+iT_2} + \int_{\Omega+iT_2}^{\sigma+iT_2}\right]$$
$$= I_1 + I_2 + I_3, \text{ say.}$$

Now, if we fix T_1 and T_2 and let $\Omega \to \infty$, then we see that $I_2 \to 0$, for $g(s)$ is the sum-function of $\sum\limits_1^\infty b_n^{-\mu_n s}$, where $b_n = a_{m+n}$, $\mu_n = \lambda_{m+n} - \omega$, with $\mu_1 > 0$, and hence $g(s) = o(1)$ as $s \to \infty$ in the angle $|ams| \leq \dfrac{\pi}{2} - \delta$.

Thus

$$\frac{1}{2\pi i}\int_{\sigma-iT_1}^{\sigma+iT_2}\frac{g(s)}{s}ds = \frac{1}{2\pi i}\left[\int_{\sigma-iT_1}^{\infty-iT_1} - \int_{\sigma+iT_2}^{\infty+iT_2}\right]$$

if the two integrals on the right converge.

Now if

$$h(s) \equiv g(s)e^{(\lambda_{m+1}-\omega)s} = a_{m+1} + a_{m+2}e^{-(\lambda_{m+2}-\lambda_{m+1})s} + \cdots$$

then by lemma 2, we have

$$|h(s)| < \varepsilon T_2$$

for $s = \sigma + iT_2$, $\sigma \geq \sigma_0 + \varepsilon$, $T_2 \geq T_0$. Therefore for the second integral

86 on the right we have

$$\left|\int_{\sigma+iT_2}^{\infty+iT_2}\frac{g(s)}{s}ds\right| < \frac{\varepsilon T_2}{\sqrt{\sigma^2 + T_2^2}}\int_{\sigma}^{\infty} e^{-\mu_1\xi}di < \varepsilon/\mu_1$$

$\mu_1 > 0$. This proves not only that the integral converges for a finite T_2, but also that as $T_2 \to \infty$, the second integral tends to zero. Similarly for the first integral on the right. □

N.B. An estimate for $g(s)$ alone (instead of $h(s)$) will not work!

Remarks. More generally we have for $\sigma > \sigma_0$ and $\sigma > \sigma^*$

$$\sum_{\lambda_n \leq \omega}{}' a_n e^{-\lambda_n s^*} = \frac{1}{2\pi i} \int\limits_{\sigma - i\infty}^{\sigma + i\infty} \frac{f(s)}{s - s^*} e^{\omega(s - s^*)} ds,$$

$$\text{for } \sum a_n e^{-\lambda_n S^*} \cdot e^{\lambda_n (s - s^*)} = \sum a_n e^{-\lambda_n s} = f(s),$$

and writing $s' = s - s^*$ and $b_n = a_n e^{-\lambda_n s^*}$, we have $f(s) \equiv f(s') = \sum b_n e^{-\lambda_n s'}$.

We shall now give an alternative proof of Person's formula *without* using the order of $f(s)$ as $s \to \infty$ [14, Lec. 4]

Aliter. If $f(s) = \sum\limits_1^\infty a_n e^{-\lambda_n s}$ has $\sigma_0 \neq \infty$ as its abscissa of convergence, and if $a > 0$, $a > \sigma_0$, then for $\omega \neq \lambda_n$, we have

$$\sum_{\lambda_n < \omega} a_n = \frac{1}{2\pi i} \int\limits_{a - i\infty}^{a + i\infty} f(s) \frac{e^{\omega s}}{s} ds$$

Proof. Define 87

$$f_m(s) = \sum_1^m a_n e^{-\lambda_n s}$$

and

$$r_m(s) = \sum_{m+1}^\infty a_n . e^{-\lambda_n s} = f(s) - f_m(s)$$

Then

$$(A) \quad \frac{1}{2\pi i} \int\limits_{a - iT}^{a + iT} f(s) \frac{e^{\omega s}}{s} ds = \frac{1}{2\pi i} \int\limits_{a - iT}^{a + iT} f_m(s) \frac{e^{\omega s}}{s} ds + \frac{1}{2\pi i} \int\limits_{a - iT}^{a + iT} r_m(s) \frac{e^{\omega s}}{s} ds$$

Here the integral on the left exists since $f(s)$ is regular on $\sigma = a > \sigma_0$. By Lemma 1, as applied to the first integral on the right hand side, we get

$$(B) \quad \lim_{T\to\infty} \frac{1}{2\pi i} \int\limits_{a-iT}^{a+iT} f(s)\frac{e^{\omega s}}{s} ds - \sum_{\lambda_n<\omega} a_n = \lim_{T\to\infty} \frac{1}{2\pi i} \int\limits_{a-iT}^{a+iT} r_m(s)\frac{e^{\omega s}}{s} ds$$

the limits existing.

Since $\lambda_n \to \infty$, we can choose m_0 such that for all $m \geq m_0$ we have $\lambda_m > \omega$. For all such m the last relation holds.

Now write $b_n = a_n e^{-\lambda_n \sigma'}, \phi(x) = e^{-x(s-\sigma')}$, so that $\sum a_n e^{-\lambda_n s} = \sum b_n \phi(\lambda)$. Write $B(x) = \sum\limits_{n\leq x} b_n$.

Then applying the Lemma of the 9th lecture, we get

$$\begin{aligned}
\sum_{m+1}^{N} a_n e^{-\lambda_n s} &= \sum_1^N - \sum_1^m \\
&= (s-\sigma') \int\limits_{\lambda_m}^{\lambda_N} B(x) e^{-x(s-\sigma')}\, dx \\
&\qquad + B(\lambda_N) e^{-\lambda_N(s-\sigma')} - B(m) e^{-\lambda_m(s-\sigma')} \\
&= \{B(\lambda_N) - B(\lambda_m)\} e^{-\lambda_N(s-\sigma')} \\
&\qquad + (s-\sigma') \int\limits_{\lambda_m}^{\lambda_N} \{B(x) - B(\lambda_m)\} e^{-x(s-\sigma')}\, dx.
\end{aligned}$$

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Now choose $\sigma' > 0$ and $a > \sigma' > \sigma_0$. [If $\sigma_o \geq 0$ the second condition implies the first; if $\sigma_0 < 0$, then since $a > 0$, choose $0 < \sigma' < a$].

Then $B(x)$ tends to a limit as $x \to \infty$, so that $B(x) = o(1)$ *for all* x; while $|e^{-x(s-\sigma')}| = e^{-x(\sigma-\sigma')} \to 0$ as $x \to \infty$ for $\sigma > \sigma'$. Hence, letting $N \to \infty$, we get

$$r_m(s) = (s-\sigma') \int\limits_{\lambda_m}^{\infty} \{B(x) - B(\lambda_m)\} e^{-x(s-\sigma')} dx,$$

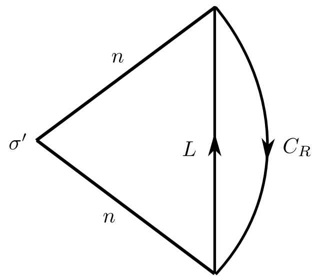
or

$$|r_m(s)| \leq c \frac{|s-\sigma'|}{\sigma-\sigma'} e^{-\lambda_m(\sigma-\sigma')}$$

Now consider

$$\frac{1}{2\pi i} \int\limits_{L+C_R} r_m(s) \frac{e^{\omega s}}{s} ds$$

where $L + C_R$ is the contour indicated in the diagram.



$r_m(s)$ is regular in this contour, and the integral is therefore zero. **89**
Hence

$$\int\limits_L = -\int\limits_{C_R}$$

On C_R, however, $s = \sigma' + re^{i\theta}$ and $\sigma \geq a$, so that

$$|r_m(s)| \leq c \cdot \frac{r}{a-\sigma'} \cdot e^{-\lambda_m r\cos\theta}$$

$$|e^{\omega s}| = e^{\omega(\sigma' + r\cos\theta)}, |s| \geq r.$$

Hence

$$\left| \frac{1}{2\pi i} \int\limits_{C_R} r_m(s) \frac{e^{\omega s}}{s} ds \right| \leq \frac{cr}{2\pi(a-\sigma')} e^{\omega\sigma'} \int\limits_{-\pi/2}^{\pi/2} e^{(\omega-\lambda_m)r\cos\theta} d\theta$$

$$= \frac{cr}{\pi(a-\sigma')} e^{\omega\sigma'} \int\limits_0^{\pi/2} e^{(\omega-\lambda_m)r\sin\theta} d\theta$$

However, $\sin\theta \geq \dfrac{2\theta}{\pi}$ for $0 \leq \theta \leq \pi/2$; hence

$$\left|\frac{1}{2\pi i}\int\limits_{a-iT}^{a+iT} r_m(s)\frac{e^{\omega s}}{s}ds\right| \leq \frac{cre^{\omega\sigma'}}{\pi(a-\sigma')}\int\limits_{0}^{\pi/2} e^{\frac{2}{\pi}(\omega-\lambda_m)r\theta}d\theta$$
$$\leq \frac{cre^{\omega\sigma'}}{2(a-\sigma')(\lambda_m-\omega)r},$$

for all T. Hence

$$(C) \qquad \overline{\lim_{T\to\infty}}\left|\int\limits_{a-iT}^{a+iT} r_m(s)\frac{e^{\omega s}}{s}ds\right| \leq \frac{c_1}{(\lambda_m-\omega)}$$

90 for *all* $m \geq m_o$.

Now reverting to relation (A), we get

$$(D) \quad \overline{\lim_{T\to\infty}}\left|\frac{1}{2\pi i}\int\limits_{a-iT}^{a+iT} f(s)\frac{e^{\omega s}}{s}ds - \frac{1}{2\pi i}\int\limits_{a-iT}^{a+iT} f_m(s)\frac{e^{\omega s}}{s}ds\right|$$
$$= \overline{\lim_{T\to\infty}}\left|\frac{1}{2\pi i}\int\limits_{a-iT}^{a+iT} r_m(s)\frac{e^{\omega s}}{s}ds\right|$$

for every m. However,

$$\lim_{T\to\infty}\frac{1}{2\pi i}\int\limits_{a-iT}^{a+iT} f_m(s)\frac{e^{\omega s}}{s}ds = \sum_{\lambda_n<\omega} a_n$$

independently of m. Hence[1]

---

[1]We use the fact that if

$$\overline{\lim_{T\to\infty}}|f(T)-g(T)| = \alpha$$

and $\lim\limits_{T\to\infty} g(T) = \beta$, then

$$\overline{\lim_{T\to\infty}}|f(T)-\beta| \leq \alpha$$

$$\overline{\lim_{T\to\infty}} \left| \frac{1}{2\pi i} \int\limits_{a-iT}^{a+iT} f(s) \frac{e^{\omega s}}{s} ds - \sum_{\lambda_n < \omega} a_n \right| \leq \text{ left hand side in (D)}$$

$$\leq \frac{c_1}{\lambda_m - \omega} \text{ for } m > m_o.$$

Letting $m \to \infty$ we get the result. □ 91

Remarks. (i) If $\omega = \lambda_n$ the last term in the sum $\sum\limits_{\lambda_n < \omega} a_n$ has to be multiplied by $\frac{1}{2}$.

(ii) If $a < 0$, (and $a > \sigma_0$) then in the contour-integration the residue at the origin has to be taken, and this will contribute a term - $f(0)$. In the proof, the relation between $|s|$ and r has to be modified.

Lecture 11

The Zeta Function of Riemann (Contd)

3 Analytic continuation of $\zeta(s)$. First method [16, p.18]

We have, for $\sigma > 0$, 92

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} dx.$$

Writing nx for x, and summing over n, we get

$$\begin{aligned}
\sum_{n=1}^\infty \frac{\Gamma(s)}{n^s} &= \sum_{n=1}^\infty \int_{n=0}^\infty x^{s-1} e^{-nx} dx \\
&= \int_0^\infty \left(\sum_{n=1}^\infty e^{-nx} \right) x^{s-1} dx, \text{ if } \sigma > 1.
\end{aligned}$$

since

$$\sum_{n=1}^\infty \left| \int^\infty x^{s-1} e^{-nx} dx \right| \leq \sum_{n=1}^\infty \int_0^\infty x^{\sigma-1} e^{-nx} dx = \sum_{n=1}^\infty \frac{\Gamma(\sigma)}{n^\sigma} < \infty,$$

if $\sigma > 1$. Hence

$$\sum_{n=1}^{\infty} \frac{\Gamma(s)}{n^s} = \int\limits_0^{\infty} \frac{e^{-x}}{1-e^{-x}} x^{s-1} dx = \int\limits_0^{\infty} \frac{x^{s-1} dx}{e^x - 1}$$

or

$$\boxed{\Gamma(s)\zeta(s) = \int\limits_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx, \quad \sigma > 1}$$

In order to continue $\zeta(s)$ analytically all over the s-plane, consider the complex integral

$$I(s) = \int\limits_C \frac{z^{s-1}}{e-1} dz$$

where C is a contour consisting of the real axis from $+\infty$ to ρ, $0 < \rho <$
93 2π, the circle $|z| = \rho$; and the real axis from ρ to ∞. $I(s)$, if convergent, is independent of ρ, by *Cauchy's* theorem.

Now, on the circle $|z| = \rho$, we have

$$\begin{aligned} |z^{s-1}| = |e^{(s-1)\log z}| &= |e^{\{(\sigma-1)+it\}\{\log|z|+i\arg z\}}| \\ &= e^{(\sigma-1)\log|z|-t\arg z} \\ &= |z|^{\sigma-1} e^{2\pi|t|}, \end{aligned}$$

while

$$|e^z - 1| > A|z|;$$

Hence, for fixed s,

$$\Big|\int\limits_{|z|=\rho}\Big| \leq \frac{2\pi\rho\cdot\rho^{\sigma-1}}{A\rho}\cdot e^{2\pi|t|} \to 0 \text{ as } \rho \to 0, \text{ if } \sigma > 1.$$

Thus, on letting $\rho \to 0$, we get, if $\sigma > 1$,

$$I(s) = -\int\limits_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx + \int\limits_0^{\infty} \frac{(xe^{2\pi i})^{s-1}}{e^x - 1}\, dx$$

$$= -\Gamma(s)\zeta(s) + e^{2\pi is}\Gamma(s)\zeta(s)$$
$$= \Gamma(s)\zeta(s)(e^{2\pi s} - 1).$$

Using the result
$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}$$
we get **94**
$$I(s) = \frac{\zeta(s)}{\Gamma(1-s)} \cdot 2\pi i. \frac{e^{2\pi is} - 1}{e^{\pi is} - e^{-\pi is}}$$
$$= \frac{\zeta(s)}{\Gamma(1-s)} \cdot 2\pi i \cdot e^{\pi is},$$
or
$$\boxed{\zeta(s) = \frac{e^{-i\pi s}\Gamma(1-s)}{2\pi i} \int\limits_C \frac{z^{s-1}dz}{e^z - 1}, \quad \sigma > 1.}$$
The integral on the right is uniformly convergent in any finite region of the s-plane (by obvious majorization of the integrand), and so defines an entire function. Hence the above formula, proved first for $\sigma > 1$, defines $\zeta(s)$, as a meromorphic function, all over the s-plane. This only possible poles are the poles of $\Gamma(1-s)$, namely $s = 1, 2, 3, \ldots$. We know that $\zeta(s)$ is regular for $s = 2, 3, \ldots$ (As a matter of fact, $I(s)$ vanishes at these points). Hence the only possible pole is at $s = 1$.

Hence
$$I(1) = \int\limits_C \frac{dz}{e^z - 1} = 2\pi i,$$
while
$$\Gamma(1-s) = -\frac{1}{s-1} + \cdots$$
Hence *the residue of* $\zeta(s)$ *at* $s = 1$ *is* 1.

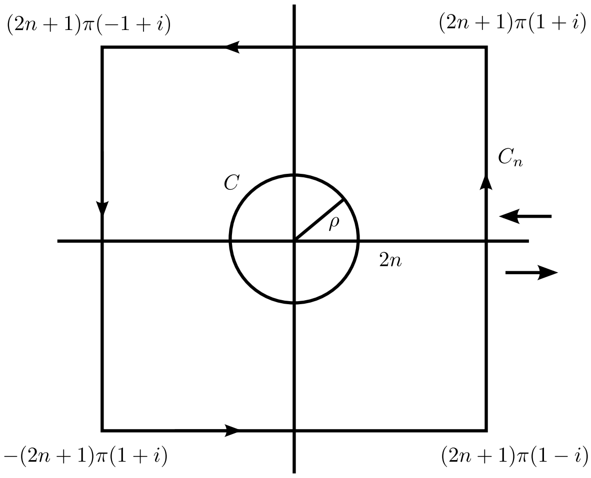
We see in passing, since
$$\frac{1}{e^z - 1} = \frac{1}{z} - \frac{1}{2} + B_1\frac{z}{2!} - B_2\frac{z^3}{4!} + \cdots$$
that **95**
$$\zeta(0) = -\frac{1}{2}, \zeta(-2m) = 0, \zeta(1-2m) = \frac{(-1)^m B_m}{2m}, m = 1, 2, 3, \ldots$$

4 Functional Equation (First method) [16, p.18]

Consider the integral

$$\int \frac{z^{s-1}dz}{e^z - 1}$$

taken along C_n as in the diagram.



Between C and C_n, the integrand has poles at

$$\pm 2i\pi, \ldots, \pm 2ni\pi.$$

The residue at

$$2m\pi i \text{ is } (2m\pi e^{\frac{\pi}{2}i})^{s-1}$$

while the residue at $-2m\pi i$ is $(2m\pi e^{3/2\pi i})^{s-1}$; taken together they amount to

$$\begin{aligned}
&(2m\pi)e^{\pi i(s-1)}\left[e^{\frac{\pi i}{2}}(s-1) + e^{-\frac{\pi i}{2}(s-1)}\right] \\
&= (2m\pi)^{s-1}e^{\pi i(s-1)} 2\cos\frac{\pi}{2}(s-1) \\
&= -2(2m\pi)^{s-1}e^{\pi i s}\sin\frac{\pi}{2}s
\end{aligned}$$

96 Hence

$$I(s) = \int_{C_n} \frac{z^{s-1}dz}{e^z - 1} + 4\pi i \frac{\sin \pi s}{2} e^{\pi i s} \sum_{m=1}^{n} (2m\pi)^{s-1}$$

by the theorem of residues.

Now let $\sigma < 0$, and $n \to \infty$. Then, on C_n,

$$|z^{s-1}| = O(|z|^{\sigma-1}) = O(n^{\sigma-1}),$$

and

$$\frac{1}{e^z - 1} = O(1),$$

for

$$\begin{aligned} |e^z - 1|^2 = |e^{x+iy} - 1|^2 &= |e^x(\cos y + i \sin y) - 1|^2 \\ &= e^{2x} - 2e^x \cos y + 1, \end{aligned}$$

which, on the vertical lines, is $\geq (e^x - 1)^2$ and, on the horizontal lines, $= (e^x + 1)^2$. (since $\cos y = -1$ there).

Also the length of the square-path is $O(n)$. Hence the integral round the square $\to 0$ as $n \to \infty$.

Hence

$$\begin{aligned} I(s) &= 4\pi i e^{\pi i s} \frac{\sin \pi s}{2} \sum_{m=1}^{\infty} (2m\pi)^{s-1} \\ &= 4\pi i e^{\pi i s} \sin \frac{\pi s}{2} \cdot (2\pi)^{s-1} \zeta(1-s), \text{ if } \sigma < 0. \end{aligned}$$

or 97

$$\begin{aligned} \zeta(s)\Gamma(s)(e^{2\pi i s} - 1) &= (4\pi i)(2\pi)^{s-1} e^{\pi i s} \sin \frac{\pi s}{2} \zeta(1-s) \\ &= 2\pi i e^{\pi i s} \frac{\zeta(s)}{\Gamma(1-s)} \end{aligned}$$

Thus

$$\boxed{\zeta(s) = \Gamma(1-s)\zeta(1-s) 2^s \pi^{s-1} \sin \frac{\pi s}{2},}$$

for $\sigma < 0$, and hence, by analytic continuation, for all values of s (each side is regular except for poles!).

This is the functional equation.

Since

$$\Gamma\left(\frac{x}{2}\right)\Gamma\left(\frac{x+1}{2}\right) = \frac{\sqrt{\pi}}{2^{x-1}}\Gamma(x),$$

we get, on writing $x = 1 - s$,

$$2^{-s}\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(1 - \frac{s}{2}\right) = \sqrt{\pi}\Gamma(1-s);$$

also

$$\Gamma\left(\frac{s}{2}\right)\Gamma\left(1 - \frac{s}{2}\right) = \frac{\pi}{\sin\frac{\pi s}{2}}$$

Hence

$$\Gamma(1-s) = 2^{-s}\Gamma\left(\frac{1-s}{2}\right)\left\{\Gamma\left(\frac{s}{2}\right)\right\}^{-1}\sqrt{\pi}\{\sin\frac{\pi s}{2}\}^{-1}$$

Thus

$$\boxed{\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-\frac{(1-s)}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)}$$

98 If

$$\begin{aligned}\xi(s) &\equiv \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)\\ &\equiv \frac{1}{2}s(s-1)\eta(s),\end{aligned}$$

then $\eta(s) = \eta(1-s)$ and $\xi(s) = \xi(1-s)$. If $\equiv(z) = \xi(\frac{1}{2} + iz)$, then $\equiv(z) \equiv (-z)$.

5 Functional Equation (Second Method) [16, p.13]

Consider the lemma given in Lecture 9, and write $\lambda_n = n$, $a_n = 1$, $\phi(x) = x^{-s}$ in it. We then get

$$\sum_{n \leq x} n^{-s} = s\int_1^X \frac{[x]}{x^{s+1}}dx + \frac{[X]}{X^s}, \text{ if } X \geq 1.$$

$$= \frac{s}{s-1} - \frac{s}{(s-1)X^{s-1}} - s\int\limits_1^X \frac{x-[x]}{x^{s+1}}dx + \frac{1}{X^{s-1}} - \frac{X-[X]}{X^s}$$

Since
$$\left|\frac{1}{X^{s-1}}\right| = 1/X^{\sigma-1}, \text{ and } \left|\frac{X-[X]}{X^s} \leq 1/X^{\sigma},\right|$$
we deduce, on making $X \to \infty$,

$$\boxed{\zeta(s) = \frac{s}{s-1} - s\int\limits_1^\infty \frac{x-[x]}{x^{s+1}}dx, \text{ if } \sigma > 1}$$

or

$$\boxed{\zeta(s) = s\int\limits_1^\infty \frac{[x]-x+\frac{1}{2}}{x^{s+1}}dx + \frac{1}{s-1} + \frac{1}{2}, \text{ if } \sigma > 1}$$

5.1 Since $[x] - x + \dfrac{1}{2}$ is bounded, the integral on the right hand side **99**
converges for $\sigma > 0$, and uniformly in any finite region to the right of $\sigma = 0$. Hence it represents an analytic function of *s regular for* $\sigma > 0$, and so provides the continuation of $\zeta(s)$ up to $\sigma = 0$, and $s = 1$ is clearly a simple pole with residue 1.

For $0 < \sigma < 1$, we have, however,

$$\int\limits_0^1 \frac{[x]-x}{x^{s+1}}dx = -\int\limits_0^1 x^{-s}dx = \frac{1}{s-1},$$

and
$$\frac{s}{2} = \int\limits_1^\infty \frac{dx}{x^{s+1}} = \frac{1}{2}$$

Hence

5.2

$$\zeta(s) = s\int_0^\infty \frac{[x]-x}{x^{s+1}}dx, \quad 0 < \sigma < 1$$

We have seen that (5.1) gives the analytic continuation up to $\sigma = 0$. By refining the argument dealing with the integral on the right-hand side of (5.1) we can get the continuation all over the s-plane. For, if

$$f(x) \equiv [x] - x + \frac{1}{2}, \quad f_1(x) \equiv \int_1^x f(y)dy,$$

then $f_1(x)$ is *also* bounded, since $\int_n^{n+1} f(y)dy = 0$ for any integer n.

100 Hence

$$\int_{x_1}^{x_2} \frac{f(x)}{x^{s+1}}dx = \left.\frac{f_1(x)}{x^{s+1}}\right]_{x_1}^{x_2} + (s+1)\int_{x_1}^{x_2} \frac{f_1(x)}{x^{s+2}}dx$$

$$\to 0, \text{ as } x_1 \to \infty, x_2 \to \infty,$$

$$\boxed{\text{if } \sigma > -1.}$$

Hence the integral in (5.1) converges for $\sigma > -1$.

Further

$$s\int_0^1 \frac{[x]-x+\frac{1}{2}}{x^{s+1}}dx = \frac{1}{s-1} + \frac{1}{2}, \text{ for } \sigma < 0;$$

$$\boxed{\zeta(s) = s\int_0^\infty \frac{[x]-x+\frac{1}{2}}{x^{s+1}}dx, \quad -1 < \sigma < 0} \tag{5.3}$$

Now the function $[x] - x + \frac{1}{2}$ has the Fourier expansion

$$\sum_{n=1}^\infty \frac{\sin 2n\pi x}{n\pi}$$

if x is *not* an integer. The series is boundedly convergent. If we substitute it in (5.3), we get

$$\begin{aligned}
\zeta(s) &= \frac{s}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\int\limits_{0}^{\infty}\frac{\sin 2n\pi x}{x^{s+1}}dx\\
&= \frac{s}{\pi}\sum_{n=1}^{\infty}\frac{(2n\pi)^{s}}{n}\int\limits_{0}^{\infty}\frac{\sin y}{y^{s+1}}dy\\
\zeta(s) &= \frac{s}{\pi}(2\pi)^{s}\{-\Gamma(1-s)\}\sin\frac{s\pi}{2}\zeta(1-s),
\end{aligned}$$

5.4 If termwise integration is permitted, *for* $-1 < \sigma < 0$. The right 101
hand side is, however, analytic for all values of s such that $\sigma < 0$. Hence (5.4) provides the analytic continuation (not only for $-1 < \sigma < 0$) all over the s-plane.

The term-wise integration preceding (5.4) is certainly justified over any *finite* range since the concerned series is boundedly convergent. We have therefore only to prove that

$$\lim_{X\to\infty}\sum_{n=1}^{\infty}\frac{1}{n}\int\limits_{X}^{\infty}\frac{\sin 2n\pi x}{x^{s+1}}dx = 0, -1 < \sigma < 0$$

Now

$$\begin{aligned}
\int\limits_{X}^{\infty}\frac{\sin 2n\pi x}{x^{s+1}}dx &= \left[-\frac{\cos 2n\pi x}{2n\pi x^{s+1}}\right]_{X}^{\infty} - \frac{s+1}{2n\pi}\int\limits_{X}^{\infty}\frac{2n\pi x}{x^{s+2}}dx\\
&= O\left(\frac{1}{nX^{\sigma+1}}\right) + O\left(\frac{1}{n}\int\limits_{X}^{\infty}\frac{dx}{x^{\sigma+2}}\right)\\
&= O\left(\frac{1}{nX^{\sigma+1}}\right)
\end{aligned}$$

and hence

$$\lim_{X\to\infty}\sum_{n=1}^{\infty}\frac{1}{n}\int\limits_{X}^{\infty}\frac{\sin 2n\pi x}{x^{s+1}}dx = 0, \text{ if } -1<\sigma<0$$

This completes the analytic continuation and the proof of the Functional equation by the second method.

As a consequence of (5.1), we get

$$\begin{aligned}
\lim_{s\to 1}\left\{\zeta(s)-\frac{1}{s-1}\right\} &= \int\limits_{1}^{\infty}\frac{[x]-x+\frac{1}{2}}{x^2}dx+\frac{1}{2}\\
&= \lim_{n\to\infty}\int\limits_{1}^{n}\frac{[x]-x}{x^2}dx+1\\
&= \lim_{n\to\infty}\left\{\sum_{m=1}^{n-1} m\int\limits_{m}^{m+1}\frac{dx}{x^2}-\log n+1\right\}\\
&= \lim_{n\to\infty}\left\{\sum_{m=1}^{n-1}\frac{1}{m+1}+1-\log n\right\}\\
&= \gamma
\end{aligned}$$

102

Hence, near $s=1$, we have

$$\zeta(s)=\frac{1}{s-1}+\gamma+O(|s-1|)$$

Lecture 12

The Zeta Function of Riemann (Contd)

6 Some estimates for $\zeta(s)$ [11, p.27]

In any fixed half-plane $\sigma \geq 1 + \varepsilon > 1$, we know that $\zeta(s)$ is bounded, since 103

$$|\zeta(s)| \leq \zeta(\sigma) \leq \zeta(1+\varepsilon).$$

We shall now use the formulae of §5 to estimate $|\zeta(s)|$ for large t, where $s = \sigma + it$.

Theorem.

$$|\zeta(s)| < A \log t, \quad \sigma \geq 1, \quad t \geq 2.$$
$$|\zeta(s)| < A(\delta) t^{1-\delta}, \quad \sigma \geq \delta, \quad t \geq 1, \; if\, 0 < \delta < 1.$$

Proof. From (5.1) we get

$$\zeta(s) = \frac{s}{s-1} - s \int_1^\infty \frac{x-[x]}{x^{s+1}} dx, \quad \sigma > 0$$

Also from §5 of the 11th Lecture,

$$\sum_{n\le X}\frac{1}{n^s}=s\int\limits_1^X\frac{[x]}{x^{s+1}}dx+\frac{[X]}{X^s},\ \text{if } X\ge 1, s\ne 1$$

If $\sigma>0$, $t\ge 1$, $X\ge 1$, we get

$$\zeta(s)-\sum_{n\le X}\frac{1}{n^s}=-s\int\limits_X^\infty\frac{x-[x]}{x^{s+1}}dx+\frac{1}{(s-1)X^{s-1}}+\frac{X-[X]}{X^s}$$

Hence

$$|\zeta(s)|<\sum_{n\le X}\frac{1}{n^\sigma}+\frac{1}{tX^{\sigma-1}}+\frac{1}{X^\sigma}+|s|\int\limits_X^\infty\frac{dx}{x^{\sigma+1}}$$
$$<\sum_{n\le X}\frac{1}{n^\sigma}+\frac{1}{tx^{\sigma-1}}+\frac{1}{X^\sigma}+\left(1+\frac{t}{\sigma}\right)\frac{1}{X^\sigma},\ \text{since } |s|<\sigma+t.$$

104 If $\sigma\ge 1$,

$$|\zeta(s)|<\sum_{n\le X}\frac{1}{n}+\frac{1}{t}+\frac{1}{X}+\frac{1+t}{X}$$
$$\le(\log X+1)+3+\frac{t}{X},\ \text{since } t\ge 1,\quad X\ge 1.$$

Taking $X=t$, we get the first result.

If $\sigma\ge\eta$, where $0<\eta<1$,

$$|\zeta(s)|<\sum_{n\le X}\frac{1}{n^\eta}+\frac{1}{tX^{\eta-1}}+\left(2+\frac{1}{\eta}\right)\frac{1}{x^\eta}$$
$$<\int\limits_0^{[X]}\frac{dx}{x^\eta}+\frac{X^{1-\eta}}{t}+\frac{3t}{\eta X^\eta}$$
$$\le\frac{X^{1-\eta}}{1-\eta}+X^{1-\eta}+\frac{3t}{\eta X^\eta}$$

Taking $X = t$, we deduce

$$|\zeta(s)| < t^{1-\eta}\left(\frac{1}{1-\eta} + 1 + \frac{3}{\eta}\right), \quad \sigma \geq \eta, \quad t \geq 1.$$

□

7 Functional Equation (Third Method) [16, p.21]

The third method is based on the 'theta-relation'

$$\vartheta(x) = \frac{1}{\sqrt{x}}\vartheta\left(\frac{1}{x}\right),$$

where

$$\vartheta(x) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 x}, \mathrm{Re}\, x > 0$$

This relation can be proved directly, or obtained as a special case of 105
'*Poisson's formula*'. We shall here give a proof of that formula [2, p.37]

Let $f(x)$ be continuous in $(-\infty, \infty)$. Let

$$\phi(\alpha) = \int_{-\infty}^{\infty} f(x)e^{2\pi i\alpha x}dx \equiv \lim_{T\to\infty}\int_{-T}^{T} f(x)e^{2\pi i\alpha x}dx,$$

if the integral exists. Then, for α *and pro-positive integers*, we have

$$\int_{-p-\frac{1}{2}}^{p+\frac{1}{2}} f(x)e^{2\pi i\alpha x}dx = \int_{-\frac{1}{2}}^{\frac{1}{2}}\left\{\sum_{-p}^{p} f(x+k)\right\}e^{2\pi i\alpha x}dx.$$

If we assume that

$$\lim_{p\to\infty}\sum_{-p}^{p} f(x+k) = g(x) \tag{7.1}$$

uniformly in $-\frac{1}{2} \leq x \leq \frac{1}{2}$, then $g(x)$ is continuous, and on letting $p \to \infty$ in (7.1), we get

$$\phi(\alpha) = \int\limits_{-\frac{1}{2}}^{\frac{1}{2}} g(x) e^{2\pi i\alpha x} dx.$$

Thus $\phi(\alpha)$ is the Fourier coefficient of the continuous periodic function $g(x)$. Hence, by Fejer's theorem,

$$g(o) = \sum_{(C,1)} \phi(\alpha)$$

If we assume, however, that $\sum\limits_{-\infty}^{\infty} \phi(\alpha) < \infty$, then $\sum\limits_{(C,1)} \phi(\alpha) = \sum\limits_{-\infty}^{\infty} \phi(\alpha)$ Hence

$$\lim_{p\to\infty} \sum_{-p}^{p} f(k) = \lim_{p\to\infty} \sum_{-p}^{p} \phi(\alpha)$$

106 or

$$\sum_{-\infty}^{\infty} f(k) = \sum_{-\infty}^{\infty} \phi(\alpha) \tag{7.2}$$

This is Poisson's formula. We may summarize our result as follows. *Poisson's Formula*. If

(i) $f(x)$ is continuous in $(-\infty, \infty)$,

(ii) $\sum\limits_{-\infty}^{\infty} f(x+k)$ converges uniformly in $-\frac{1}{2} \leq x < \frac{1}{2}$ and

(iii) $\sum\limits_{-\infty}^{\infty} \phi(\alpha)$ converges,

then

$$\sum_{-\infty}^{\infty} f(k) = \sum_{-\infty}^{\infty} \phi(\alpha)$$

Remarks. This result can be improved by a relaxation of the hypotheses.

The Theta Relation. If we take $f(x) = e^{-\pi x^2 t}$, $t > 0$, in the above discussion, we obtain the required relation. However, we have to show that for this function the conditions of Poisson's formula are satisfied. This is easily done. $f(x)$ is obviously continuous in $(-\infty, \infty)$. Secondly

$$e^{-\pi(x+k)^2 t} \leq e^{|k|t} \text{ for } |k| \geq k_0(t)$$

This proves the uniform convergence of

$$\sum_{k=-\infty}^{\infty} e^{-\pi(x+k)^2 t}, \quad t > 0. -\frac{1}{2} \leq x < \frac{1}{2}$$

Thirdly we show that, for $t > 0$

$$\phi(\alpha) \equiv \int_{-\infty}^{\infty} e^{-y^2 \pi t + 2\pi\alpha y i} dy = \frac{1}{t^{1/2}} e^{-\pi\alpha^2/t}$$

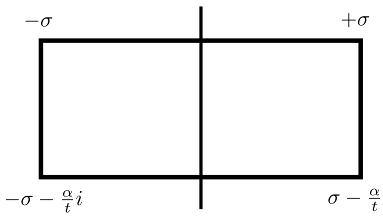
For 107

$$\phi(\alpha) = e^{-\frac{\pi\alpha^2}{t}} \int_{-\infty}^{\infty} e^{-\pi t} \left(y - \frac{\alpha}{t} i\right)^2 dy, \text{ and}$$

If $\alpha \gtrless 0$, we consider the contour integral

$$\int_C e^{-\pi t s^2} ds, \quad s = \sigma + i\tau$$

taken over C as indicated. Then on the vertical lines, we have



$$e^{-\pi t\,\mathrm{Re}(\pm\sigma+\tau i)^2} = e^{-\pi t(\sigma^2-\tau^2)} \leq e^{-\pi t(\sigma^2-\frac{\alpha^2}{t^2})}$$

Letting $\sigma \to \infty$, we thus see that the integrals along the vertical lines vanish. Hence we have, for all real α,

$$\int_{-\infty}^{\infty} e^{-\pi t\sigma^2} d\sigma = \int_{-\infty}^{\infty} e^{-\pi t(\sigma-\frac{\alpha}{t}i)^2} d\sigma$$

Thus

$$\begin{aligned}
\phi(\alpha) &= e^{-\pi\alpha^2/t} \int_{-\infty}^{\infty} e^{-\pi t\sigma^2} d\sigma \\
&= e^{-\pi\alpha^2/t} \frac{1}{\sqrt{t}} \int_{-\infty}^{\infty} e^{-\pi u^2} du \\
&= \frac{e^{-\pi\alpha^2/t}}{\sqrt{t}} \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} e^{-v^2} dv = \frac{e^{-\pi\alpha^2/t}}{\sqrt{t}} \frac{1}{\pi} \int_{0}^{\infty} e^{-z} z^{-\frac{1}{2}} dz \\
&= \frac{e^{-\pi\alpha^2/t}}{\sqrt{t}} \frac{1}{\sqrt{\pi}} \Gamma(\frac{1}{2}) = \frac{e^{-\pi\alpha^2/t}}{\sqrt{t}}
\end{aligned}$$

108 Hence we have the desired relation:

$$\boxed{\sum_{-\infty}^{\infty} e^{-\pi k^2 t} = \frac{1}{\sqrt{t}} \sum_{-\infty}^{\infty} e^{-\pi k^2/t}, \text{ for } t > 0,} \tag{7.3}$$

or, writing

$$\psi(t) = \sum_{1}^{\infty} e^{-\pi k^2 t}, \text{ we get}$$

$$2\psi(t) + 1 = \frac{1}{\sqrt{t}} \left\{ 2\psi\left(\frac{1}{t}\right) + 1 \right\} \tag{7.4}$$

Remark. (7.3) holds for t complex, with $\mathrm{Re}(t) > 0$.

Functional Equation (Third method).

We have, for $\sigma > 0$,

$$\Gamma\left(\frac{s}{2}\right) = \int_0^\infty e^{-x} x^{s/2-1} dx.$$

If $\sigma > 0$, $n > 0$, we then have on writing $\pi n^2 x$ for x,

$$\Gamma\left(\frac{s}{2}\right) = \int_0^\infty e^{-\pi n^2 x} (\pi n^2 x)^{\frac{s}{2}-1} \pi n^2 dx$$

$$= \pi^{s/2} n^s \int_0^\infty e^{-\pi n^2 dx} x^{s/2-1} dx$$

$$\text{or} \quad \frac{1}{n^s} = \frac{\pi^{s/2}}{\Gamma(s/2)} \int_0^\infty e^{-\pi n^2 x} x^{s/2-1}\ dx.$$

Now *if* $\sigma > 1$, we have **109**

$$\zeta(s) = \sum_1^\infty \frac{1}{n^s} = \frac{\pi^{s/2}}{\Gamma(s/2)} \sum_{n=1}^\infty \int_0^\infty e^{-\pi n^2 x} x^{s/2-1} dx$$

$$= \frac{\pi^{s/2}}{\Gamma(s/2)} \int_0^\infty \left(\sum_1^\infty e^{-\pi n^2 x}\right) x^{s/2-1} dx$$

the inversion being justified by absolute convergence. Hence, for $\sigma > 1$, we have

$$\zeta(s) = \frac{\pi^{s/2}}{\Gamma(s/2)} \int_0^\infty \psi(x) x^{s/2-1} dx.$$

Using (7.4) we rewrite this as

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \int_0^1 x^{s/2-1}\psi(x) dx + \int_1^\infty \psi(x) x^{s/2-1} dx$$

$$= \int_0^1 x^{s/2-1}\left\{\frac{1}{\sqrt{x}}\psi\left(\frac{1}{x}\right) + \frac{1}{2\sqrt{x}} - \frac{1}{2}\right\}dx + \int_1^\infty \psi(x)x^{s/2-1}dx$$

$$= \frac{1}{s-1} - \frac{1}{s} + \int_0^1 x^{s/2-3/2}\psi\left(\frac{1}{x}\right)dx + \int_1^\infty$$

$$= \frac{1}{s(s-1)} + \int_1^\infty \left(x^{-s/2-\frac{1}{2}} + x^{s/2-1}\right)\psi(x)dx, \tag{7.5}$$

for $\sigma > 1$.

110 Now the integral on the right converges uniformly in $a \leq s \leq b$, for if $x \geq 1$, we have

$$\left|x^{-s/2-1/2} + x^{s/2-1}\right| \leq x^{b/2-1} + x^{-a/2-1/2}$$

while

$$\psi(x) < \sum_1^\infty e^{-\pi n x} = \frac{1}{e^{\pi x}-1},$$

and hence the integral on the right hand side of (7.5) is an entire function. Hence (7.5) provides the analytic continuation to the left of $\sigma = 1$. It also yields the functional equation directly. We have also deduced that

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) - \frac{1}{s(s-1)}$$

is an entire function; so is $\pi^{s/2}\{\Gamma(s/2)\}^{-1}$. Hence

$$\zeta(s) - \frac{1}{s(s-1)} \cdot \frac{\pi^{s/2}}{\Gamma(s/2)} = \zeta(s) - \frac{1}{s-1} \cdot \frac{\pi^{s/2}}{2\Gamma(s/2+1)}$$

is an entire function. But $\dfrac{\pi^{1/2}}{2\Gamma(1/2+1)} = 1$. Hence $\zeta(s) - \dfrac{1}{s-1}$ is an entire function.

Lecture 13

The Zeta Function of Riemann (Contd)

8 The zeros of $\zeta(s)$

111

In the 11th Lecture, (p.98) we defined

$$\xi(s) \equiv \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s) \equiv \frac{1}{2}s(s-1)\eta(s).$$

The following theorem about $\xi(s)$ is a consequence of the results which we have already proved.

Theorem 1. *[11, p.45]*

(i) $\xi(s)$ *is an entire function, and*

$$\xi(s) = \xi(1-s);$$

(ii) $\xi(s)$ *is real on* $t = 0$ *and* $\sigma = \frac{1}{2}$*;*

(iii) $\xi(0) = \xi(1) = \frac{1}{2}$.

Proof. That $\xi(s) = \xi(1-s)$ is immediate from the functional equation of $\zeta(s)$ (cf. p.97, Lecture 11) Clearly $\xi(s)$ is regular for $\sigma > 0$, and since $\xi(s) = \xi(1-s)$ it is also regular for $\sigma < 1$, and hence it is entire.

This proves (i)

$\xi(s)$ is obviously real for real s. Hence by a general theorem it takes conjugate values at conjugate points. Thus $\xi(\frac{1}{2}+ti)$ and $\xi(\frac{1}{2}-ti)$ are conjugate; they are equal by the functional equation. So they are real. This proves (ii).

To prove (iii) we observe that

$$\xi(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}+1\right)\cdot(s-1)\zeta(s),$$

so that

$$\xi(1) = \pi^{-1/2}\Gamma(3/2)\cdot 1 = \frac{1}{2},$$

and by the functional equation, $\xi(0) = \frac{1}{2}$. □

112 We shall next examine the location of the zeros of $\xi(s)$ and of $\zeta(s)$.

Theorem 2. *[11, p.48] (i) The zeros of $\xi(s)$ (if any !) are all situated in the strip $0 \leq \sigma \leq 1$, and lie symmetrical about the lines $t = 0$ and $\sigma = \frac{1}{2}$*

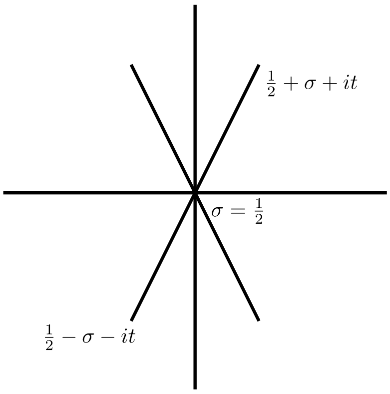
(ii) The zeros of $\zeta(s)$ are identical (in position and order of multiplicity) with those of $\xi(s)$, except that $\zeta(s)$ has a simple zero at each of the points $s = -2, -4. -6, \ldots$

(iii) $\xi(s)$ has no zeros on the real axis.

Proof. (i) $\xi(s) = (s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}+1\right)\zeta(s)$

$$\equiv h(s)\zeta(s), \text{ say.}$$

Now $\zeta(s) \neq 0$; for $\sigma > 1$; (Euler Product!) also $h(s) \neq 0$, for $\sigma > 1$; hence $\xi(s) \neq 0$, for $\sigma > 1$; hence $\xi(s) \neq 0$, for $\sigma < 0$; since $\xi(s) = \xi(1-s)$.



The zeros, if any, are symmetrical about the real axis since $\xi(\sigma \pm ti)$ are conjugates, and about *the point* $s = \frac{1}{2}$, since $\xi(s) = \xi(1-s)$; they are therefore also symmetrical about the line $\sigma = \frac{1}{2}$.

(ii) The zeros of $\zeta(s)$ differ from those of $\xi(s)$ only in so far as $h(s)$ has zeros or poles. The only zero of $h(s)$ is at $s = 1$. But this is not a zero of $\xi(s)$ since $\xi(1) = \frac{1}{2}$, nor of $\zeta(s)$ for it is a pole of the latter.

The poles of $h(s)$ are simple ones at $s = -2, -4, -6, \ldots$. Since these are points at which $\xi(s)$ is regular *and not zero*, they must be *simple zeros* **113**
of $\zeta(s)$. We have already seen this by a different argument on *p*.95.

(iii) Since $\xi(s) \neq 0$ for $\sigma < 0$ and $\sigma > 1$, it is enough, in view of (ii), to show that

$$\zeta(\sigma) \neq 0, \text{ for } 0 < \sigma < 1.$$

Now

$$(1-2^{1-s})\zeta(s) = (1-2^{-s}) + (3^{-s} - 4^{-s}) + \cdots, \text{ for } \sigma > 0.$$

To prove this we first notice that the relation is obvious for $\sigma > 1$, and secondly each side is regular for $\sigma > 0$; the left side is obviously regular for $\sigma > 0$ (in fact it is entire); and, as for the right side, we have

$$\left|\frac{1}{(2n-1)^s} - \frac{1}{(2n)^s}\right| = \left|s \int\limits_{2n-1}^{2n} \frac{dx}{x^{s+1}}\right|$$

$$\leq \frac{|s|}{(2n-1)^{\sigma+1}} < \frac{\Delta}{(2n-1)^{\delta+1}}$$

if $\sigma > \delta$ and $|s| < \Delta$ for any fixed δ and Δ

But, if $0 < \sigma < 1$, the above relation gives

$$(1 - 2^{1-\sigma})\zeta(\sigma) > 0, \text{ or } \zeta(\sigma) < 0.$$

This establishes (iii). □

Remarks. The strip $0 \leq \sigma \leq 1$ is called the '*critical strip*'; the line $\sigma = \frac{1}{2}$ the 'critical line'; the zeros at $-2, -4, -6, \ldots$ are the '*trivial zeros*' of $\zeta(s)$. We still have to show that $\xi(s)$ actually has zeros, i.e. $\zeta(s)$ has
114 'non-trivial' zeros. This is done by an appeal to the theory of entire functions gives in the first few lectures.

Theorem 3. *[11, p.56] If* $M(r) \equiv \max_{|s|=r} |\xi(s)|$, *then*

$$\log M(r) \sim \frac{1}{2} r \log r, \text{ as } r \to \infty$$

Proof. For $\sigma \geq 2$, we have

$$|\zeta(s)| \leq \zeta(2),$$

and for $\sigma \geq \frac{1}{2}$, $|t| \geq 1$, we have

$$|\zeta(s)| < c_1 |t|^{1/2},$$

so that

$$|\zeta(s)| < c_2 |s|^{1/2}, \text{ for } \sigma \geq \frac{1}{2}, |s| > 3.$$

Applying Stirlingl's formula to $\Gamma(s/2)$, we get, for

$$\sigma \geq \frac{1}{2}, \quad |s| = r > 3,$$

$$|\xi(s)| < e^{|\frac{1}{2} s \log \frac{1}{2} s| + c_3 |s|}$$
$$< e^{\frac{1}{2} r \log r + c_4 r},$$

since $\log \dfrac{s}{2} = \log|s| - \log 2 + i \arg s$, $|\arg s| < \frac{\pi}{2}$. From the equation $\xi(s) = \xi(1-s)$, we infer that

$$\begin{aligned} |\xi(s)| &< e^{\frac{1}{2}|1-s|\log|1-s| + c_4|1-s|} \\ &< e^{\frac{1}{2}r\log r + c_5 r} \end{aligned}$$

for $\sigma \leq \frac{1}{2}$, $|s| = r > 4$. Combining the results we get

$$M(r) < e^{\frac{1}{2}r\log r + c_6 r}, r > 4.$$

On the other hand, if $r > 2$, **115**

$$M(r) \geq \xi(r) > 1.\pi^{-\frac{1}{2}r}\Gamma\left(\frac{1}{2}r\right) e^{\frac{1}{2}r\log r - c_7 r}$$

Thus we have, for $r > 4$,

$$\frac{1}{2}r\log r - C_7 r < \log M(r) < \frac{1}{2}r\log r + c_6 r,$$

and hence the result. □

Theorem 4. *[11, p.57]*

(i) *$\xi(s)$ has an infinity of zeros;*

(ii) *If they are denoted by 'ρ', then $\sum|\rho|^{-\alpha}$ converges for $\alpha > 1$, diverges for $\alpha \leq 1$;*

(iii)

$$\xi(s) = e^{b_0 + b_1 s}\pi_\rho \left\{\left(1 - \frac{s}{\rho}\right) e^{s/\rho}\right\},$$

where b_0, b_1 are constants;

(iv) $$\frac{\xi'(s)}{\xi(s)} = b_1 + \sum\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right)$$

(v) $$\frac{\zeta'(s)}{\zeta(s)} = b - \frac{1}{s-1} - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2} + 1\right) + \sum_\rho\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right),$$

where $b = b_1 + \frac{1}{2}\log 2\pi$.

Proof. $\xi(0) = \frac{1}{2} \neq 0$, and we apply the theory of entire functions as developed in the earlier lectures. By Theorem 3, $\xi(s)$ is of order 1, and the relation $\log M(r) = O(r)$ does not hold. Hence $\rho_1 = 1$, $\xi(s)$ has an infinity of zeros and $\sum \dfrac{1}{|\rho|}$ diverges (by a previous theorem). This proves (i), (ii) and (iii). (iv) follows by logarithmic differentiation. (v) is obtained from $\xi(s) = (s-1)\pi^{-s/2}\Gamma(\frac{s}{2}+1)\zeta(s)$.

We shall now show that $\zeta(s)$ has no zeros on the line $\sigma = 1$. □

Theorem 5. *[11, p.28]* $\zeta(1+it) \neq 0$.

116 **First Proof.** $2(1+\cos\theta t)^2 = 3 + 4\cos\theta + \cos 2\theta \geq 0$ for real θ. Since

$$\log\zeta(s) = \sum_m \sum_p \frac{1}{mp^{ms}}, \text{ for } \sigma > 1,$$

we have

$$\begin{aligned}\log|\zeta(s)| &= \operatorname{Re}\sum_{n=2}^{\infty} c_n n^{-\sigma-ti}\\ &= \sum_{n=2}^{\infty} c_n n^{-\sigma}\cos(t\log n),\end{aligned}$$

where $c_n = \begin{cases} 1/m & \text{if } n \text{ is the } m^{\text{th}} \text{ power of a prime}\\ 0 & \text{otherwise.}\end{cases}$

Hence

$$\begin{aligned}&\log\left|\zeta^3(\sigma)\cdot\zeta(\sigma+ti)\zeta(\sigma+2ti)\right|\\ &\quad= \sum c_n n^{-\sigma}\{3 + 4\cos(t\log n) + \cos(2t\log n)\}\\ &\quad\geq 0,\end{aligned}$$

because $c_n \geq 0$ and we have the trigonometric identity above.

Thus

$$\{(\sigma-1)\zeta(\sigma)\}^3\left|\frac{\zeta(\sigma+ti)}{\sigma-1}\right|^4|\zeta(\sigma+2ti)| \geq \frac{1}{\sigma-1},$$

if $\sigma > 1$.

Now suppose $1 + ti (t \gtrless 0)$ were a zero of $\zeta(s)$; then, on letting $\sigma \to 1 + 0$, we see that the left hand side in the above inequality would tend to a finite limit, viz. $|\zeta'(1+ti)|^4 |\zeta(1+2ti)|$, while the right hand side 117
tends to ∞. Hence $\zeta(1 + it) \neq 0$.

Second Proof. [11, p.89] We know from the 8th Lecture (1.12) that if α is real, and $\alpha \neq 0$, then

$$f(s) \equiv \frac{\zeta^2(s)\zeta(s-\alpha i)\zeta(s+\alpha i)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{|\sigma_{\alpha i}(n)|^2}{n^s}, \sigma > 1 \tag{8.1}$$

Let σ_0 be the abscissa of convergence of the series on the right hand side. Then $\sigma_0 \leq 1$. The sum-function of the series is then regular in the half-plane $\sigma > \sigma_0$, and hence, by analytic continuation, (8.1) can be upheld for $\sigma > \sigma_0$. And since the coefficients of the series are positive, the point $s = \sigma_0$ is a singularity of $f(s)$.

Now, if $1 + \alpha i$ is a zero of $\zeta(s)$, then so is $(1 - \alpha i)$, and these two zeros cancel the double pole of $\xi^2(s)$ at $s = 1$. Hence $f(s)$ is regular on the real axis, as far to the left as $s = -1$, where $\zeta(2s) = 0$. Hence $\sigma_0 = -1$. This is, however, impossible since (8.1) gives

$$f\left(\frac{1}{2}\right) \geq |\sigma_{\alpha i}(1)|^2 = 1,$$

while $f(\frac{1}{2}) = 0$. Thus $\zeta(1 + \alpha i) \neq 0$.

Remarks. (i) We shall show later on that this fact ($\zeta(1 + it) \neq 0$) is equivalent to the Prime Number Theorem.

(ii) If $\rho = \beta + \gamma i$ is a zero of $\xi(s)$, then we have seen that $0 \leq \beta \leq 1$, and since $\zeta(1 + it) \neq 0$, we actually have $0 \leq \beta < 1$, and by symmetry about the line $\sigma = \frac{1}{2}$, we have $0 < \beta < 1$.

Lecture 14

The Zeta Function of Riemann (Contd)

9 Riemann-Von Magoldt Formula [11, p.68]

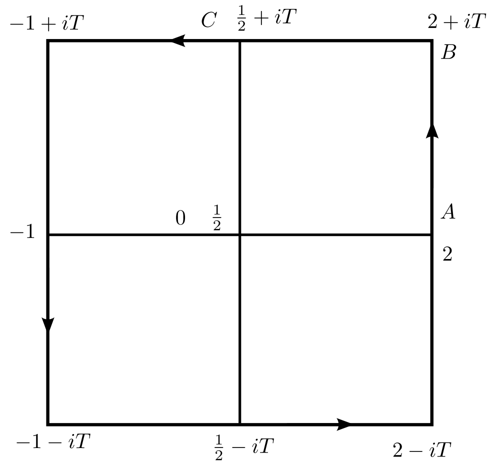
Let $N(T)$ denote the number of zeros (necessarily finite) of $\zeta(s)$ in $0 \leq$ **118**
$\sigma \leq 1$, $0 \leq t \leq T$. Then, by what we have proved, $N(T)$ is the number of zeros $\rho = \beta + i\gamma$ of $\zeta(s)$, or of $\xi(s)$, for which $0 < \gamma \leq T$

Theorem 1. $N(T) = \dfrac{T}{2n} \log \dfrac{T}{2\pi} - \dfrac{T}{2\pi} + O(\log T)$ *as* $T \to \infty$.

Proof. Suppose first that T is not equal to any γ. Also let $T > 3$. Consider the rectangle R as indicated. $\xi(s)$ has $2N(T)$ zeros *inside* it, and none on the boundary. Hence, by the principle of the argument,

$$2N(T) = \frac{1}{2\pi}[\arg \xi(s)]_R,$$

where $[\arg \xi(s)]_R$ denotes the increase in $\arg \xi(s)$, as s describes the boundary of R.



Now

$$[\arg \xi(s)]_R = [\arg \frac{1}{2} s(s-1)]_R + [\arg \eta(s)]_R,$$

where $\eta(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$. Here

$$[\arg \frac{1}{2} s(s-1)]_R = 4\pi,$$

119 and $\eta(s)$ has the properties: $\eta(s) = \eta(1-s)$, and $\eta(\sigma \pm ti)$ are conjugates. Hence

$$[\arg \eta(s)]_R = 4[\arg \eta(s)]_{ABC}$$

Hence

$$\begin{aligned} \pi N(T) = \pi + \arg[\pi^{-s/2}] + [\arg \Gamma(s/2)]_{ABC} \\ + [\arg \zeta(s)]_{ABC} \end{aligned} \tag{9.1}$$

Now

$$[\arg \pi^{-s/2}]_{ABC} = \left[-\frac{t}{2} \log \pi\right]_{ABC} = -\frac{T}{2} \log \pi. \tag{9.2}$$

Next, since $\Gamma(z+\alpha) = (z+\alpha-\frac{1}{2}) \log z - z + \frac{1}{2} \log 2\pi + O(|z|^{-1})$ as $|z| \to \infty$, in any angle $|\arg z| \leq \pi - \varepsilon < \pi$, we have

$$\left[\arg\Gamma\left(\frac{1}{2}\right)\right]_{ABC} = [\text{im}\log\Gamma(s/2)]_{ABC}$$
$$= \text{im}\log\Gamma\left(\frac{1}{4}+\frac{1}{2}iT\right) - \text{im}\log\Gamma(1)$$
$$= \text{im}\left\{\left(-\frac{1}{4}+\frac{1}{2}iT\right)\log\left(\frac{1}{2}iT\right) - \frac{1}{2}iT + \frac{1}{2}\log 2\pi + O(T^{-1})\right\}$$
$$= \frac{1}{2}T\log\left(\frac{1}{2}T\right) - \frac{1}{8}\pi - \frac{T}{2} + O\left(T^{-1}\right), \text{ as } T\to\infty \tag{9.3}$$

Using (9.2) and (9.3) in (9.1), we get

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + \frac{1}{\pi}[\arg\zeta(s)]_{ABC} + 0\left(\frac{1}{T}\right) \tag{9.4}$$

Consider the broken line ABC exclusive of the end-points A and C. If m is the number of distinct points s' on ABC at which $\text{Re}\,\zeta(s) = 0$, then

$$[\arg\zeta(s)]_{ABC} \leq (m+1)\pi, \tag{9.5}$$

since $\arg\zeta(s)$ cannot vary by more than π (since $\text{Re}\,\zeta$ does not change sign here) on any one of the $m+1$ segments into which ABC is divided by the point s' **120**

Now no point s' can be on AB, for

$$\text{Re}\,\zeta(2+iT) \geq 1 - \sum_{2}^{\infty}\frac{1}{n^2} > 1 - \frac{1}{2^2} - \int_2^\infty \frac{du}{u^2} = \frac{1}{4} \tag{9.6}$$

Thus m is the number of distinct points σ of $\frac{1}{2} < \sigma < 2$ at which $\text{Re}\,\zeta(\sigma+iT) = 0$, i.e. the number of distinct zeros of

$$g(s) = \frac{1}{2}\{\zeta(s+iT) + \zeta(s-iT)\}$$

on the real axis for which $\frac{1}{2} < \sigma < 2$, because $g(\sigma) = \text{Re}\,\zeta(\sigma+iT)$ for real σ, since $\zeta(\sigma \pm iT)$ are conjugate.

Now since $g(s)$ is regular except for $s = 1 \pm iT$, m must be finite, and we can get an upper bound for m by using a theorem proved earlier.

Consider the two circles: $|s-2| \leq \dfrac{7}{4}$, $|s-2| \leq \dfrac{3}{2}$. Since $T > 3$, $g(s)$ is regular in the larger circle. At all points s of this circle, we have

$$\sigma \geq \frac{1}{4}, \quad 1 < |t \pm T| < 2 + T.$$

Hence, by an appeal to the other of $\zeta(s)$ obtained in the 12th Lecture (with $\delta = \frac{1}{4}$), we get

$$\begin{aligned} |g(s)| &< \frac{1}{2} c_1 (|t+T|^{3/4} + |t-T|^{3/4}) \\ &< c_1 (T+2)^{3/4}, \end{aligned}$$

on the larger circle. At the centre, we have

$$g(2) = \operatorname{Re} \zeta(2 + iT) > \frac{1}{4} \text{ by (9.6)}$$

121 Hence, by using the theorem (given in the Appendix), we get

$$\left(\frac{7}{6}\right)^m < \frac{c_1 (T+2)^{3/4}}{\frac{1}{4}} < T, \text{ for } T > T_0 \geq 3.$$

Thus

$$m < c_2 \log T, \text{ for } T > T_0.$$

Substituting this into (9.5) and (9.4) we get

$$\left| N(T) - \frac{T}{2\pi} \log \frac{T}{2\pi} + \frac{T}{2\pi} \right| \leq c_3 \log T, \quad T > T_0,$$

provided that $T \neq \gamma$ for any 'γ'. The last restriction may be removed by first taking T' larger than T and distinct from the $\gamma' s$ and then letting $T' \to T + 0$. $\square$

Remarks. The above theorem was stated by Riemann but proved by Von Mangoldt in 1894. The proof given here is due to Backlund (1914).

Corollary 1. *If $h > 0$ (fixed), then*

$$N(T+h) - N(T) = O(\log T), \text{ as } T \to \infty$$

Proof. If we write
$$f(t) = \frac{t}{2\pi}\log\frac{t}{2\pi} - \frac{t}{2\pi},$$
we get
$$f(t+h) - f(t) = hf'(t+\theta h), 0 < \theta < 1,$$
where $f'(t) = \dfrac{1}{2\pi}\log\dfrac{t}{2\pi}$. Now use the Theorem. □

Corollary 2.

$$S \equiv \sum_{0\le\gamma\le T}\frac{1}{\gamma} = O(\log^2 T); S' \exists \sum_{\gamma>T}\frac{1}{\gamma^2} = O\left(\frac{\log T}{T}\right)$$

(summed over all zeros ρ whose imaginary parts γ satisfy the given conditions).

Let **122**

$$s_m = \sum\frac{1}{\gamma}, \quad m < \gamma \le m+1, \text{ and}$$
$$s'_m = \sum\frac{1}{\gamma^2}, \quad m < \gamma \le m+1.$$

Then
$$S \le \sum_{m=0}^{[T]} s_m, \quad S' \le \sum_{m=[T]}^{\infty} s'_m$$

If $m \ge 1$, the number of terms in s_m (or s'_m) is

$$N(m+1) - N(m) = \nu_m, \text{ say}.$$

By Corollary 1, $\nu_m = O(\log m)$ as $m \to \infty$; and

$$s_m \le \frac{\nu_m}{m}, \quad s_m \le \frac{\nu_m}{m^2}$$

Hence, as $T \to \infty$,

$$S = O(1) + O\left(\sum_{2}^{[T]} \frac{\log m}{m}\right) = O(\log^2 T)$$

$$S' = O\left(\sum_{[T]}^{\infty} \frac{\log m}{m^2}\right) = O\left(\frac{\log T}{T}\right)$$

Corollary 3. *If the zeros 'ρ' for which $\gamma > 0$ are arranged as a sequence $\rho_n = \beta_n + i\gamma_n$, so that $\gamma_{n+1} \geq \gamma_n$, then*

$$|\rho_n| \sim \gamma_n \sim \frac{2\pi n}{\log n}, \text{ as } n \to \infty$$

Proof. Since $N(\gamma_n - 1) < n \leq N(\gamma_n + 1)$, and

$$\begin{aligned} 2\pi N(\gamma_n \pm 1) &\sim (\gamma_n \pm 1)\log(\gamma_n \pm 1) \\ &\sim \gamma_n \log \gamma_n, \end{aligned}$$

123 we have first

$$2\pi n \sim \gamma_n \log \gamma_n$$

whence

$$\log n \sim \log \gamma_n,$$

and $\quad \gamma_n \sim \dfrac{2\pi n}{\log \gamma_n} \sim \dfrac{2\pi n}{\log n}$

And $\quad \gamma_n \leq |\rho_n| < \gamma_n + 1.$ □

Remarks. (i) The main theorem proves incidentally the existence of an infinity of complex zeros of $\zeta(s)$, without the use of the theory of entire functions.

(ii) We see from corollary 3 that

$$\sum \frac{1}{|\rho|(\log|\rho|)^{\alpha}}$$

converges for $\alpha > 2$ and diverges for $\alpha \leq 2$.

(iii) Corollary 1 may be obtained directly by applying the theorem (on the number of zeros) to $\zeta(s)$ and to two circles with centre at $c+iT$ and passing through $\frac{1}{2}+(T+2h)i$, $\frac{1}{2}+(T+h)i$ respectively, where $c=c(h)$ is sufficiently large, and using the symmetry about $\sigma=1/2$

(iv) If $N_0(T)$ denotes the number of complex zeros of $\zeta(s)$ with real part $\frac{1}{2}$ and imaginary part between 0 and T, then Hardy and Littlewood proved that $N_0(T) > cT$ in comparison with $N(T) \sim \dfrac{1}{2\pi}T \log$ A. Selberg has shown, however, that $N_0(T) > cT \log T$.

Appendix

Theorem. *Let $f(z)$ be regular in $|z-z_0| \leq R$ and have n zeros (at least) in $|z-z_0| \leq r < R$. Then, if $f(z_0) \neq 0$.* 124

$$\left(\frac{R}{r}\right)^n \leq \frac{M}{|f(z_0)|},$$

where $M = \max\limits_{|z-z_0|=R} |f(z)|$

Proof. Multiple zeros are counted according to multiplicity. Suppose $z_0 = 0$ (for otherwise put $z = z_0 + z'$). Let $a_1, \ldots a_n$ be the zeros of $f(z)$ in $|z| \leq r$.

Then

$$f(z) = \varphi(z) \prod_{\nu=1}^{n} \frac{R(z-a_\nu)}{R^2 - \overline{\lambda_\nu} z}$$

where a_ν, $\overline{a_\nu}$ are conjugates and φ is regular in $|z| \leq R$. On $|z| = R$, each factor has modulus 1; hence

$$|\varphi(z)| = |f(z)| \leq M \text{ on } |z| = R$$

Since φ is regular in $|z| \leq R$, we have

$$|\varphi(0)| \leq M$$

Hence $|f(0)| = |\varphi(0)| \prod\limits_{\nu=1}^{n} \dfrac{|a_\nu|}{R} \leq M\left(\frac{r}{R}\right)^n$ and $f(0) \neq 0$, hence the result. $\square$

Theorem. *If f is analytic inside and on C and N is the number of zero inside C, then*

$$\frac{1}{2\pi i}\int\limits_{C}\frac{f'(z)}{f(z)}dz = N$$
$$= \Delta_c \log\{f(z)\}$$

so that

$$N = \frac{1}{2\pi}\Delta_c \arg f(z).$$

Lecture 15

The Zeta Function of Riemann (Contd)

10 Hardy's Theorem [16, p.214]

We have proved that $\zeta(s)$ has an infinity of complex zeros in $o < \sigma < 1$. **125**
Riemann conjectured that all these zeros lie on the line $\sigma = \frac{1}{2}$. Though this conjecture is yet to be proved, Hardy showed in 1914 that an infinity of these zeros do lie on the 'critical' line. We shall prove this result.

In obtaining the functional equation by the third method, we have used the formula:

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \frac{1}{s(s-1)} + \int\limits_{1}^{\infty} \psi(x)(x^{\frac{s}{2}-1} + x^{-s/2-1/2})dx,$$

where $\psi(x) = \sum\limits_{n=1}^{\infty} e^{-n^2\pi x}$, $x > 0$. Multiplying by $\dfrac{1}{2}s(s-1)$, and writing $x = \frac{1}{2} + it$, we get

$$\zeta\left(\frac{1}{2} + it\right) = \equiv (t) = \frac{1}{2} - \left(t^2 + \frac{1}{4}\right)\int\limits_{1}^{\infty} \psi(x)x^{-3/4} \cos\left(\frac{1}{2} t \log x\right) dx$$

It is a question of showing that $\equiv (t)$, which is an even, entire function of t, real for real t, has an infinity of real zeros.

Writing $x = e^{4u}$, we get

$$\equiv (t) = \frac{1}{2} - 4\left(t^2 + \frac{1}{4}\right)\int\limits_0^\infty \psi(0^{4u})e^u \cos(2ut)du.$$

If we write

$$\psi(u) = \psi(e^{4u})e^u,$$

126 we obtain

$$\equiv \left(\frac{t}{2}\right) = \frac{1}{2} - (t^2 + 1)\int\limits_0^\infty \phi(u)\cos ut\ du$$

We wish now to get rid of the factor $(t^2 + 1)$ and the additive constant $\frac{1}{2}$ so as to obtain $\equiv (t/2)$ as a cosine transform of ϕ. This we can achieve by two integrations (by parts). However, to get rid of the extra term arising out of partial integration we need to know $\phi'(0)$. We shall show that $\phi'(0) = -\frac{1}{2}$. For, by the functional equation of the theta-function,

$$1 + 2\psi(x) = \frac{1}{\sqrt{x}}\left[1 + 2\psi\left(\frac{1}{x}\right)\right]$$

or

$$1 + 2\psi(e^{4u}) = e^{-2u}[1 + 2\psi(e^{-4u})]$$

or

$$e^u + 2e^u\psi(e^{4u}) = e^{-u} + 2e^{-u}\psi(e^{-4u})$$

or

$$e^u + 2\phi(u) = e^{-u} + 2\phi(-u)$$

so that $\quad \dfrac{d}{du}\left[e^u + 2\phi(u)\right]_{u=0} = 0$

$$\text{i.e.} \quad \phi'(0) = -\frac{1}{2}.$$

We also know that $\psi(x) = O(e^{-\pi x})$ as $x \to \infty$. Hence

$$t^2\int\limits_0^\infty \phi(u)\cos ut\ du = \frac{1}{2} - \int\limits_0^\infty \phi''(u)\cos ut\ du$$

127 Hence

$$\equiv (t/2) = \int_0^\infty [\phi''(u) - \phi(u)] \cos ut \, du \tag{10.1}$$

By actual computation it may be verified that

$$\phi''(t) - \phi(t) = 4[6e^{5t}\psi'(e^{4t}) + 4e^{9t}\psi''(e^{4t})]$$

Again if we write

$$\begin{aligned}\phi(u) &= e^u\psi(e^{4u}) + \frac{1}{2}e^u = \phi(u) + \frac{1}{2}e^u\\ &= \frac{1}{2}e^u\vartheta(e^{4u}),\end{aligned}$$

then by the theta relation, we have

$$\Phi(u) = \Phi(-u), \text{ and } \Phi''(u) - \Phi(u) = \phi''(u) - \phi(u)$$

Let us now write (10.1) in the form

$$\equiv (t) = 2\int_0^\infty f(u) \cos ut \, du$$

We wish to deduce that

$$f(u) = \frac{1}{\pi}\int_0^\infty \equiv (t) \cos ut \, dt, \tag{10.2}$$

by Fourier's integral theorem. We need to establish this result.

Fourier Inversion Formula.[5, p.10] If $f(x)$ is absolutely integrable in $(-\infty, \infty)$, then

$$\phi(\alpha) \equiv \int_{-\infty}^{\infty} f(x)e^{i\alpha x} dx$$

exists for every real α, $-\infty < \alpha < \infty$, is continuous and bounded in $(-\infty, \infty)$. We wish to find out when

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(\alpha) e^{-i\alpha x} d\alpha$$

for a given x. The conditions will bear on the behaviour of f in a neigh-
128 bourhood of x.

If

$$\begin{aligned} S_R(x) &= \frac{1}{2\pi i} \int_{-R}^{R} e^{-i\alpha x} \phi(\alpha) d\alpha \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin Rt}{t} f(x+t)\, dt \\ &= \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin Rt}{t} \frac{f(x+t) + f(x-t)}{2}\, dt, \end{aligned}$$

and

$$g_x(t) = \frac{1}{2} [f(x+t) + f(x-t)] - f(x),$$

then

$$\begin{aligned} S_R(x) - f(x) &= \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin Rt}{t} g_x(t)\, dt \\ &= \frac{2}{\pi} \left[\int_{0}^{\varepsilon} + \int_{\epsilon}^{\infty} \right] = I_1 + I_2, \text{ say,} \end{aligned}$$

for a fixed $\epsilon > 0$.

$I_2 \to 0$ as $R \to \infty$, by the Riemann-Lebesgue Lemma. So, if $\dfrac{g_x(t)}{t}$ is absolutely integrable in $(0, \varepsilon)$, then $I_1 \to 0$ as $\varepsilon \to 0$. Hence we have the

Theorem . *[5, p.10] If* $\dfrac{g_x(t)}{t}$ *is absolutely integrable in* $(0, \varepsilon)$*, then* $\lim\limits_{R\to\infty} S_R(x) = f(x)$.

If, in particular, f is absolutely integrable in $(-\infty, \infty)$ and differentiable in $(-\infty, \infty)$ then $\lim\limits_{R\to\infty} S_R(x) = f(x)$ for every x in $(-\infty, \infty)$. It should be noted that sufficient conditions for the validity of the theorem are: (i) $f(x)$ is of bounded variation in a neighbourhood of x, and (ii) $f(x)$ is absolutely integrable in $(-\infty, \infty)$. Then $\lim\limits_{R\to\infty} S_R(x) = \frac{1}{2}[f(x+0) + f(x-0)]$. **129**

We now return to (10.2) which is an immediate deduction from the formula for $\equiv(t)$, on using Fourier's inversion formula.

Now we know that $\equiv(t) = \xi(\frac{1}{2} + it) = O(t^{\varepsilon_1} e^{-\frac{\pi}{4}t})$; and $\equiv(t)$ is an entire function of t. Hence we are justified in obtaining, by differentiation of (10.2),

$$f^{(2n)}(u) = \frac{(-1)^n}{\pi} \int\limits_0^\infty \equiv(t) t^{2n} \cos ut \, dt.$$

Since $\psi(x)$ is regular for $\operatorname{Re} x > 0$, $f(u)$ is regular for $-\pi/4 < \operatorname{Im} u < \dfrac{\pi}{4}$ (Note that $x = e^{2u}$). Let

$$f(iu) = c_0 + c_1 u^2 + c_2 u^4 + \cdots$$

Then

$$c_n = \frac{(-1)^n f^{(2n)}(0)}{(2n)!} = \frac{1}{(2n)!\pi} \int\limits_0^\infty \equiv(t) t^{2n} dt$$

If $\equiv(t)$ had only a finite number of zeros, then $\equiv(t)$ would be of constant sign for $t > T$. Let $\equiv(t) > 0$ for $t > T$. Then $c_n > 0$ for $n > 2n_0$, since

$$\begin{aligned}\int\limits_0^\infty \equiv(t) t^{2n}\, dt &> \int\limits_{T+1}^{T+2} \equiv(t) t^{2n}\, dt - \int\limits_0^T |\equiv(t)| t^{2n}\, dt \\ &> (T+1)^{2n} \int\limits_{T+1}^{T+2} \equiv(t) dt - T^{2n} \int\limits_0^T |\equiv(t)|\, dt\end{aligned}$$

$$> 0 \text{ for } n > 2n_0, \text{ say.}$$

130

Hence $f^{(n)}(iu)$ increases steadily with u for $n > 2n_0$. However we shall se that $f(u)$, and all its derivatives, tend to zero as $u \to \dfrac{\pi i}{4}$ along the imaginary axis.

We know that

$$\psi(x) = x^{-1/2}\psi\left(\frac{1}{x}\right) + \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}$$

Further

$$\begin{aligned}
\psi(i+\delta) &= \sum_{n=1}^{\infty} e^{-n^2(i+\delta)\pi} = \sum_{n=1}^{\infty}(-1)^n e^{-n^2\pi\delta} \\
&= 2\sum_{1}^{\infty} e^{-(2n)^2\pi\delta} - \sum_{1}^{\infty} e^{-n^2\pi\delta} \\
&= 2\psi(4\delta) - \psi(\delta) \\
&= \frac{1}{\sqrt{\delta}}\psi\left(\frac{1}{4\delta}\right) - \frac{1}{\delta}\psi\left(\frac{1}{\delta}\right) - \frac{1}{2}
\end{aligned}$$

i.e. $$\frac{1}{2} + \psi(i+\delta) = \frac{1}{\sqrt{\delta}}\psi\left(\frac{1}{4\delta}\right) - \frac{1}{\sqrt{\delta}}\psi\left(\frac{1}{\delta}\right)$$

Thus $\frac{1}{2} + \psi(x) \to 0$ as $\delta \to 0$, and so do its derivatives (by a similar argument). Hence $\frac{1}{2} + \psi(u) \to 0$ as $u \to \dfrac{\pi i}{4}$ along the imaginary axis, and so do its derivatives. Thus $f(u)$ and all its derivatives $\to 0$ as $u - \dfrac{\pi i}{4}$.

Lecture 16

The Zeta Function of Riemann (Contd)

Hamburger's Theorem.[13] 131

We have seen that the functional equation of the Gamma function, together with its logarithmic convexity, determine the Gamma function 'essentially' uniquely (i.e. up to a constant). A corresponding result can be proved for a the zeta function.

Theorem 1. *Let* $f(s) = \dfrac{G(s)}{P(s)}$, *where* $G(s)$ *is an entire function of finite order and* $P(s)$ *is a polynomial, and let*

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \tag{1.1}$$

the series on the right converging absolutely for $\sigma > 1$.

Let

$$f(s)\Gamma\left(\frac{s}{2}\right)\pi^{-s/2} = g(1-s)\Gamma\left(\frac{1}{2} - \frac{s}{2}\right)\pi^{-\frac{(1-s)}{2}} \tag{1.2}$$

where

$$g(1-s) = \sum_{n=1}^{\infty} \frac{b_n}{n^{1-s}}, \tag{1.3}$$

The series on the right converging absolutely for $\sigma < -\alpha < 0$. Then $f(s) = a_1\zeta(s)(= g(s))$.

For the proof we need to evaluate two integrals:

$$e^{-y} = \frac{1}{2\pi i}\int\limits_{1-\infty i}^{1+\infty i} y^{-s}\Gamma(s)ds, \quad y > 0 \tag{1}$$

$$\int\limits_0^\infty e^{-a^2x-\frac{b^2}{x}}\frac{dx}{\sqrt{x}} = \frac{\sqrt{\pi}}{a}e^{-2ab}, \quad a > 0, \quad b \geq 0. \tag{2}$$

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The first formula is a classic example of *Mellin's inversion formula* which can be obtained as an instance of Fourier's inversion formula discussed in the last lecture. We have seen that if $f(x) \in L_1(-\infty, \infty)$, then its Fourier transform $\phi(\alpha)$ is defined in $-\infty < \alpha < \infty$, and if in a neighbourhood of x_0, $f(x)$ is differentiable, then

$$f(x_0) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{-ix_0\alpha}d\alpha\int\limits_{-\infty}^{\infty} f(y)e^{i\alpha y}dy.$$

If we define

$$\mathscr{F}(s) = \int\limits_0^\infty f(x)x^{s-1}dx, \tag{3}$$

for $s = c + it$, $c > 0$, where $f(x)x^{c-1} \in L_1(0, \infty)$, so that the integral on the right exists absolutely, we can look upon this as a Fourier transform by a change of variable: $x \to e^y$, for

$$\mathscr{F}(c + it) = \int\limits_{-\infty}^{\infty} f(e^y)e^{yc}e^{it\,y}dy$$

Hence, provided that $f(e^y)e^{yc}$ satisfies a suitable 'local' condition, such as differentiability in the neighbourhood of a point, we can invert this relation and obtain

$$f(e^y)\cdot e^{yc} = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \mathscr{F}(c+it)e^{-iyt}dt$$

or 133

$$f(e^y) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \mathscr{F}(c+it)e^{-(c+it)y}dt$$

$$= \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} \mathscr{F}(s)e^{-ys}ds$$

or $$f(x) = \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} \mathscr{F}(s)x^{-s}ds,\ c>0. \tag{4}$$

Relations (3) and (4) give the Mellin inversion formulae. Since $\Gamma(s) = \int\limits_0^\infty e^{-x}x^{s-1}dx$, Re $s>0$, we get (1) as in immediate application.

We know that, for $k>0$

$$\int\limits_0^A e^{-(k-i\alpha)x}dx = \frac{1}{k-i\alpha}\left(1-e^{-kA}e^{i\alpha A}\right)$$

so that

$$2\int\limits_0^\infty e^{-kx}\cos\alpha x dx = \frac{2.k}{k^2+\alpha^2}$$

and by Fourier's inversion formula,

$$\int\limits_0^\infty \frac{\cos\alpha x}{k^2+\alpha^2}d\alpha = \frac{\pi e^{-kx}}{2k},\quad k>0 \tag{5}$$

On the other hand, we have for $k\geq 0$

$$\frac{\Gamma(1)}{x^2+k^2} = \int\limits_0^\infty e^{-(x^2+k^2)y}dy,$$

134 so that

$$\int_0^\infty \frac{\cos \alpha x}{x^2+k^2}dx = \frac{\sqrt{\pi}}{2}\int_0^\infty y^{-\frac{1}{2}}e^{-(k^2y+\frac{\alpha^2}{4y})}dy$$
$$= \frac{\pi}{2k}e^{-|\alpha|k}, \text{ from (5).}$$

Setting $k^2 = a^2$, $\alpha^2 = 4b^2$, we get

$$\int_0^\infty e^{-a^2x-\frac{b^2}{x}}\frac{dx}{\sqrt{x}} = \frac{\sqrt{\pi}}{a}e^{-2ab}, a > 0\ b \geq 0$$

which proves formula (2).

Proof of Theorem 1. For any $x > 0$, we have by (1.1) and (1.2), (Here $f(s) = O(1)$)

$$S_1 \equiv \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} f(s)\Gamma\ \frac{(s)}{2}\ \pi^{(-s/2)} \cdot x^{-s/2}ds$$
$$= \sum_{n=1}^\infty \frac{a_n}{2\pi i}\int_{2-i\infty}^{2+i\infty}\Gamma(s/2)(\pi n^2 x)^{-s/2}ds$$
$$= 2\sum_{n=1}^\infty a_n e^{-\pi n^2 x}$$

We have, however, by (1.2), $S_1 = S_2$, where

$$S_2 = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} g(1-s)\Gamma\left(\frac{1-s}{2}\right)\pi^{\frac{-(1-s)}{2}}x^{-s/2}ds.$$

135 Now we wish to move the line of integration here from $\sigma = 2$ to $\sigma = -1-\alpha$.

By the hypothesis on the 'order' of $f(s)$ and (1.2), it follows that there exist two numbers $T > 0$, $\gamma > 0$, such that for $|t| \geq T$ and $-1-\alpha \leq$

$\sigma \leq 2$, the function $g(1-s)$ is regular and $O(e^{|t|^\gamma})$. By (1.3), we see that $g(1-s) = O(1)$ on $\sigma = -1-\alpha$; and, since $f(s) = O(1)$ on $\sigma = 2$, while

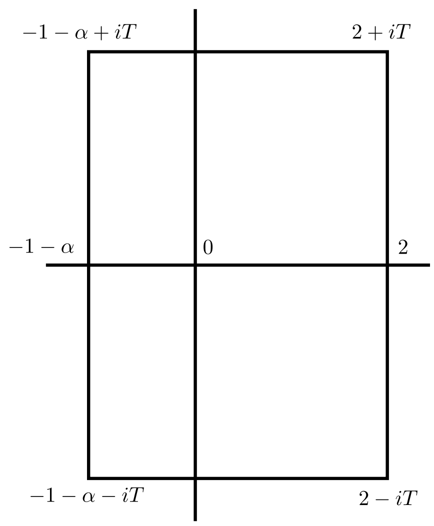
$$\frac{\Gamma(s/2)}{\Gamma(\frac{1-s}{2})} = O(|t|^{3/2}), \text{ on } \sigma = 2,$$

it follows that $g(1-s) = O(|t|^{3/2})$ on $\sigma = 2$. Hence, by the principle of Phragmen-Lindelöf, we observe that

$$g(1-s) = O(|t|^{3/2})$$

for $|t| \geq T$, $-\alpha - 1 \leq \sigma \leq 2$. *Taking a suitable rectangle, and applying Cauchy's theorem*, we get

$$S_2 = \frac{1}{2\pi i} \int\limits_{-\alpha-1-i\infty}^{-\alpha-1+i\infty} g(1-s)\Gamma\left(\frac{1-s}{2}\right)\pi^{-\frac{(1-s)}{2}} x^{-s/2} ds + \sum_{\nu=1}^{m} R_\nu$$



where R_ν is the residue of the integrand at the pole s_ν lying in the region $-\alpha-1 < \sigma < 2$. The integrand, however, equals **136**

$$f(s)\Gamma\left(\frac{s}{2}\right)\pi^{-s/2}x^{-s/2}$$

The residue of this at a pole s_ν of order q_ν is

$$x^{\frac{-s_\nu}{2}}\left(A^{(\nu)}_{q_\nu-1}\log^{q_\nu-1}x+\cdots+A^{(\nu)}_1\log x+A^{(\nu)}_0\right)$$

Hence

$$\sum_{\nu=1}^{m}R_\nu=\sum_{\nu=1}^{m}x^{-s_\nu/2}Q_\nu(\log x)=Q(x),\text{ say, where}$$

$$Q_\nu \text{ is a polynomial}$$

Here $\operatorname{Re}s_\nu\leq 2-\theta$, $\theta>0$. (i.e. we are using the absolute convergence of $\sum a_n n^{-s}$ only for $\sigma>2-\theta$).

Hence

$$\begin{aligned}
S_2&=\frac{1}{\sqrt{x}}\sum_{n=1}^{\infty}\frac{b_n}{2\pi i}\int\limits_{-\alpha-1-i\infty}^{-\alpha-1+i\infty}\frac{\Gamma(1-s)}{2}\left(\frac{\pi n^2}{x}\right)^{-\frac{(1-s)}{2}}ds+Q(x)\\
&=\frac{2}{\sqrt{x}}\sum_{n=1}^{\infty}\frac{b_n}{2\pi i}\int\limits_{\frac{\alpha}{2}+1-i\infty}^{\frac{\alpha}{2}+1+i\infty}\Gamma(s)\left(\frac{\pi n^2}{x}\right)^{-s}ds+Q(x)\\
&=\frac{2}{\sqrt{x}}\sum_{n=1}^{\infty}b_n e^{-\pi n^2/x}+Q(x)
\end{aligned}$$

137 Hence (since $S_1=S_2$) we have

$$2\sum_{1}^{\infty}a_n e^{-\pi n^2 x}=\frac{2}{\sqrt{x}}\sum_{n=1}^{\infty}b_n e^{-\pi n^2/x}+Q(x)$$

Multiplying this by $e^{-\pi t^2 x}$, $t>0$, and integrating over $(0,\infty)$ w.r.t. x, we get

$$2\sum_{n=1}^{\infty}\frac{a_n}{\pi(t^2+n^2)}=2\sum_{n=1}^{\infty}\frac{b_n}{t}e^{-2\pi nt}+\int\limits_{0}^{\infty}Q(x)e^{-\pi t^2 x}dx$$

by (2).

The last integral is convergent. Since $\operatorname{Re} s_\nu \leq 2 - \theta$, $\nu = 1, 2, \ldots m$, each term of $Q(x)$ is $O(x^{-1+\frac{\theta}{4}})$ as $x \to 0$.

Now

$$\int\limits_0^\infty Q(x) e^{-\pi t^2 x} dx = \frac{1}{t^2} \int\limits_0^\infty Q\left(\frac{x}{t^2}\right) e^{-\pi x} dx = \sum_1^m t^{s_\nu - 2} H_\nu(\log t),$$

where H_ν is a polynomial, $= H(t)$, say.

Hence

$$\sum_{n=1}^\infty a_n \left(\frac{1}{t+ni} + \frac{1}{t-ni}\right) - \pi t H(t) = 2\pi \sum_{n=1}^\infty b_n e^{-2\pi n t}$$

The series on the left hand side is uniformly convergent in any finite region of the t-plane which excludes $t = \pm ki$, $k = 1, 2, \ldots$; It is a meromorphic function with poles of the first order at $t = \pm ki$.

$H(t)$ is regular and single-valued in the t-plane with the negative real **138**
axis deleted and $t \neq 0$.

The right hand side is a periodic function of t with period i for $\operatorname{Re} t > 0$. Hence, by analytical continuation, so is the function on the left. Hence the residues at ki, $(k+1)i$ are equal. So $a_k = a_{k+1}$

Lecture 17

The Zeta Function of Riemann (Contd)

12 The Prime Number Theorem [11, pp.1-18]

The number of primes is infinite. For consider any finite set of primes; 139
let P denote their product, and let $Q = P + 1$. Then $(P, Q) = 1$, since any common prime factor would divide $Q - P$, which would be impossible if $(P, Q) \neq 1$. But $Q > 1$, and so divisible by some prime. Hence there exists at least one prime distinct from those occurring in P. If there were only a finite number of primes, by taking P to be their product we would arrive at a contradiction.

Actually we get a little more by this argument: if p_n is the n^{th} prime, the integer $Q_n = p_1 \ldots p_n + 1$ is divisible by some p_m with $m > n$, so that

$$p_{n+1} \leq p_m \leq Q_n$$

from which we get, by induction, $p_n < 2^{2^n}$. For if this inequality was true for $n = 1, 2, \ldots N$, then

$$p_{N+1} \leq p_1 \ldots p_N + 1 < 2^{2+4+\cdots+2^N} + 1 < 2^{2^{N+1}},$$

and the inequality is known to be true for $n = 1$.

The fact that the number of primes is infinite is equivalent to the statement that $\pi(x) = \sum\limits_{p \le x} 1 \to \infty$ as $x \to \infty$. Actually far more is true; we shall show that $\pi(x) \sim \dfrac{x}{\log x}$. Again the relation $p_n < 2^{2^n}$ is far
140 weaker than the asymptotic formula: $p_n \sim n \log n$, which is know to be equivalent to the formula: $\pi(x) \sim \dfrac{x}{\log x}$ (The Prime Number Theorem!). We shall prove the Prime Number Theorem, and show that it is equivalent to the non-vanishing of the zeta function on the line $\sigma = 1$. We shall state a few preliminary results on primes.

Theorem 1. *The series* $\sum \dfrac{1}{p}$ *and the product* $\pi\left(1 - \dfrac{1}{p}\right)^{-1}$ *are divergent.*

Proof. Let $S(x) = \sum\limits_{p \le x} \dfrac{1}{p}$, $\quad P(x) = \prod\limits_{p \le x} \left(1 - \dfrac{1}{p}\right)^{-1}$, $\quad x > 2$.

Then, for $0 < u < 1$,

$$\frac{1}{1-u} > \frac{1-u^{m+1}}{1-u} = 1 + u + \cdots + u^m$$

Hence

$$P(x) \ge \prod_{p \le x} (1 + p^{-1} + \cdots + p^{-m}),$$

where m is any positive integer. The product on the right is $\sum \dfrac{1}{n}$ summed over a certain set of positive integers n, and if m is so chosen that $2^{m+1} > x$, this set will certainly include all integers from 1 to $[x]$. Hence

$$P(x) > \sum_{1}^{[x]} \frac{1}{n} > \int\limits_{1}^{[x]+1} \frac{du}{u} > \log x. \tag{1}$$

141 Since

$$-\log(1-u) - u < \frac{\frac{1}{2}u^2}{1-u}, \text{ for } 0 < u < 1,$$

we have

$$\log P(x) - S(x) < \sum_{p \le x} - \frac{p^{-2}}{2(1-p^{-1})}$$

$$< \sum_{n=2}^{\infty} \frac{1}{2n(n-1)} = \frac{1}{2}$$

Hence, by (1),

$$S(x) > \log\log x - \frac{1}{2}. \tag{2}$$

(1) and (2) prove the theorem.

The Chebyschev Functions. Let

$$\vartheta(x) = \sum_{p \le x} \log p, \quad \psi(x) = \sum_{p^m \le x} \log p, \quad x > 0.$$

Grouping together terms of $\psi(x)$ for which m has the same value, we get

$$\psi(x) = \vartheta(x) + \vartheta(x^{1/2}) + \vartheta(x^{1/3}) + \cdots, \tag{3}$$

this series having only a finite number of non-zero terms, since $\vartheta(x) = 0$ if $x < 2$.

Grouping together terms for which 'p' has the same value ($\le x$), we obtain

$$\psi(x) = \sum_{p \le x} \left[\frac{\log x}{\log p}\right] \log p, \tag{4}$$

since the number of values of m associated with a given p is equal to the number of positive integers m satisfying $m \log p \le x$, and this is 142
$\left[\frac{\log x}{\log p}\right]$ □

Theorem 2. *The functions*

$$\frac{\pi(x)}{x/\log x}, \quad \frac{\vartheta(x)}{x}, \quad \frac{\psi(x)}{x}$$

have the same limits of indetermination as $x \to \infty$.

Proof. Let the upper limits (may be ∞) be L_1, L_2, L_3 and the lower limits be l_1, l_2, l_3, respectively. Then by (3) and (4),

$$\vartheta(x) \le \psi(x) \le \sum_{p \le x} \frac{\log x}{\log p} \log p = \pi(x) \log x.$$

Hence
$$L_2 \leq L_3 \leq L_1. \tag{5}$$
On the other hand, if $0 < \alpha < 1$, $x > 1$, then
$$\vartheta(x) \geq \sum_{x^\alpha < p \leq x} \log p \geq \{\pi(x) - \pi(x^\alpha)\} \log x^\alpha$$
and $\pi(x^\alpha) < x^\alpha$, so that
$$\frac{\vartheta(x)}{x} \geq \alpha\left(\frac{\pi(x)\log x}{x} - \frac{\log x}{x^{1-\alpha}}\right)$$

143 Keep α fixed, and let $x \to \infty$; then $\dfrac{\log x}{x^{1-\alpha}} \to 0$

Hence
$$L_2 \geq \alpha L_1,$$
i.e.
$$L_2 \geq L_1, \text{ as } \alpha \to 1 - 0.$$
This combined with (5) gives $L_2 = L_3 = L_1$, and similarly for the l's. □

Corollary. *If* $\dfrac{\pi(x)}{x/\log x}$ *or* $\dfrac{\vartheta(x)}{x}$ *or* $\dfrac{\psi(x)}{x}$ *tends to a limit, then so do all, and the limits are equal.*

Remarks. If $\wedge(n) = \begin{cases} \log p, & \text{if } n \text{ is a +ve power of a prime } p \\ 0, & \text{otherwise,} \end{cases}$
then
$$\psi(x) = \sum_{p^m \leq x} \log p = \sum_{n \leq x} \wedge(n), \text{ and}$$
$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_1^\infty \frac{\wedge(n)}{n^s}, \quad \sigma > 1 \text{ [p.69]}$$

Also
$$-\frac{\zeta'(s)}{\zeta(s)} = s\int_1^\infty \frac{\psi(x)}{x^{s+1}} dx, \quad \sigma > 1.$$

The Wiener-Ikehara theorem. [3, 4] Let $A(u)$ be a non-negative non-decreasing function defined for $0 \leq u < \infty$. Let

$$f(s) \equiv \int_0^\infty A(u) e^{-us} du, \quad s = \sigma + i\tau,$$

converge for $\sigma > 1$, and be analytic for $\sigma \geq 1$ except at $s = 1$, where it 144
has a simple pole with residue 1. Then $e^{-u} A(u) \to 1$ as $u \to \infty$

Proof. If $\sigma > 1$,

$$\frac{1}{s-1} = \int_0^\infty e^{-(s-1)u} du$$

Therefore

$$g(s) \equiv f(s) - \frac{1}{s-1} = \int_0^\infty \{A(u) - e^u\} e^{-us} du$$

$$\equiv \int_0^\infty \{B(u) - 1\} e^{-(s-1)u} du,$$

where

$$B(u) \equiv A(u) e^{-u}$$

Take

$$s = 1 + \varepsilon + i\tau.$$

Then

$$g(1 + \varepsilon + i\tau) \equiv g_\varepsilon(\tau) = \int_0^\infty \{B(u) - 1\} e^{-\varepsilon u} \cdot e^{-i\tau u} du$$

Now $g(s)$ is analytic for $\sigma \geq 1$; therefore

$$g_\varepsilon(t) \to g(1 + it) \text{ as } \varepsilon \to 0.$$

uniformly in any finite interval $-2\lambda \leq t \leq 2\lambda$.

We should like to form the Fourier transform of $g_\varepsilon(\tau)$, but since we do not know that it is bounded on the whole line, we shall introduce a smoothing kernel. Thus

$$\frac{1}{2}\int_{-2\lambda}^{2\lambda} g_\varepsilon(\tau)\left(1-\frac{|\tau|}{2\lambda}\right)e^{iy\tau}d\tau$$
$$= \frac{1}{2}\int_{-2\lambda}^{2\lambda} e^{iy\tau}\left(1-\frac{|\tau|}{2\lambda}\right)d\tau\int_0^\infty \{B(u)-1\}e^{-\varepsilon u-i\tau u}du.$$

145 For a fixed y, a finite τ-interval, and an infinite u-interval we wish to change the order of integration. This is permitted if

$$\int_0^\infty \{B(u)-1\}e^{-\varepsilon u}\cdot e^{-i\tau u}du$$

converges uniformly in $-2\lambda \leq \tau \leq 2\lambda$. This is so, because it is equal to

$$\int_0^\infty B(u)e^{-\varepsilon u}e^{-i\tau u}du - \int_0^\infty e^{-\varepsilon u}\ e^{-\varepsilon\tau u}du.$$

For a fixed $\varepsilon > 0$, the second converges absolutely and uniformly in τ. In the first we have

$$B(u)\cdot e^{-(\varepsilon/2)u} = A(u)\cdot e^{-(1+\varepsilon/2)u} \to 0 \text{ as } u \to \infty$$

Hence $B(u) = O(e^{(\varepsilon/2)u})$, which implies the first integral converges uniformly and absolutely.

Thus

$$\frac{1}{2}\int_{-2\lambda}^{2\lambda} e^{iy\tau}\left(1-\frac{|\tau|}{2\lambda}\right)g_\varepsilon(\tau)d\tau$$
$$= \left[\int_0^\infty \{B(u)-1\}e^{-\varepsilon u}du\int_{-2\lambda}^{2\lambda}\frac{1}{2}e^{i(y-u)\tau}\left(1-\frac{|\tau|}{2\lambda}\right)d\tau\right]$$

$$= \int_0^\infty [B(u) - 1] e^{-\varepsilon u} \frac{\sin^2 \lambda(y-u)}{\lambda(y-u)^2} du$$

Now consider the limit as $\varepsilon \to 0$. The left hand side tends to **146**

$$\frac{1}{2} \int_{-2\lambda}^{2\lambda} e^{iy\tau} \left(1 - \frac{|\tau|}{2\lambda}\right) g(1 + i\tau) d\tau$$

since g is analytic, or rather, since $g_\varepsilon(\tau) \to g(1 + i\tau)$ uniformly. On the right side, we have

$$\int_0^\infty e^{-\varepsilon u} \frac{\sin^2 \lambda(y-u)}{\lambda(y-u)^2} du \to \int_0^\infty \frac{\sin^2 \lambda(y-u)}{\lambda(y-u)^2} du.$$

Hence

$$\lim_{\varepsilon \to 0} \int_0^\infty e^{-\varepsilon u} B(u) \frac{\sin^2 \lambda(y-u)}{\lambda(y-u)^2} du =$$

$$\int_0^\infty \frac{\sin^2 \lambda(y-u)}{\lambda(y-u)^2} du + \int_{-2\lambda}^{2\lambda} e^{iyt} \left(1 - \frac{|t|}{2\lambda}\right) g(1 + it)\, dt$$

The integrand of the left hand side increases monotonically as $\varepsilon \to 0$; it is positive. Hence by the monotone-convergence theorem,

$$\int_0^\infty B(u) \frac{\sin^2 \lambda(y-u)}{\lambda(y-u)^2} du = \int_0^\infty \frac{\sin^2 \lambda(y-u)}{\lambda(y-u)^2} du$$

$$+ \int_{-2\lambda}^{2\lambda} e^{iyt} \left(1 - \frac{|t|}{2\lambda}\right) g(1 + it)\, dt$$

Now let $y \to \infty$; then the second term on the right-hand side tends to zero by the Riemann-Lebesgue lemma, while the first term is

$$\lim_{y \to \infty} \int_{-\infty}^{\lambda y} \frac{\sin^2 v}{v^2} dv = \int_{-\infty}^{\infty} \frac{\sin^2 v}{v^2} dv = \pi$$

147 Hence

$$\lim_{y\to\infty}\int\limits_{-\infty}^{\lambda y} B\left(y-\frac{v}{\lambda}\right)\frac{\sin^2 v}{v^2}dv = \pi \text{ for every } \lambda.$$

To prove that $\lim\limits_{u\to\infty} B(u) = 1.$ □

Second Part.

(i) $\overline{\lim} B(u) \leq 1.$

For a fixed 'a' such that $0 < a < \lambda y$, we have

$$\overline{\lim_{y\to\infty}}\int\limits_{-a}^{a} B\left(y-\frac{v}{\lambda}\right)\frac{\sin^2 v}{v^2}dv \leq \pi$$

By definition $B(u) = A(u)e^{-u}$, where $A(u)\uparrow$, so that

$$e^{y-\frac{d}{\lambda}}B\left(y-\frac{a}{\lambda}\right) \leq e^{y-\frac{v}{\lambda}}B\left(y-\frac{v}{\lambda}\right)$$

or

$$B\left(y-\frac{v}{\lambda}\right) \geq e^{(v-a)/\lambda}B\left(y-\frac{a}{\lambda}\right).$$

Hence

$$\overline{\lim_{y\to\infty}}\int\limits_{-a}^{a} B\left(y-\frac{a}{\lambda}\right)e^{(v-a)/\lambda}\frac{\sin^2 v}{v^2}dv \leq \pi$$

or $$\int\limits_{-a}^{a} e^{\frac{-2a}{\lambda}}\frac{\sin^2 v}{v^2}\ \overline{\lim} B\left(y-\frac{a}{\lambda}\right)dv \leq \pi$$

or $$e^{-2a/\lambda}\int\limits_{-a}^{a}\frac{\sin^2 v}{v^2}\ \overline{\lim_{y\to\infty}} B(y)\cdot dv \leq \pi$$

148 Now let $a\to\infty$, $\lambda\to\infty$ in such a way that $\dfrac{a}{\lambda}\to 0$.

Then

$$\pi\,\overline{\lim_{y\to\infty}} B(y) \leq \pi$$

(ii) $\underline{\lim}_{u\to\infty} B(u) \geq 1$.

$$\int_{-\infty}^{\lambda y} B\left(y - \frac{v}{\lambda}\right)\frac{\sin^2 v}{v^2}dv = \int_{-\infty}^{-a} + \int_{-a}^{a} + \int_{a}^{\lambda y} \tag{$\circledast$}$$

We know that $|B(u)| \leq c$, so that

$$\int_{-\infty}^{\lambda y} \leq c\left[\int_{-\infty}^{-a}\frac{\sin^2 v}{v^2}dv + \int_{a}^{\infty}\frac{\sin^2 v}{v^2}dv\right] + \int_{-a}^{a}$$

We know that $|B(u)| \leq c$, so that

$$\int_{-\infty}^{\lambda y} \leq c\left[\int_{-\infty}^{-a}\frac{\sin^2 v}{v^2}dv + \int_{a}^{\infty}\frac{\sin^2 v}{v^2}dv\right] + \int_{-a}^{a}$$

As before we get

$$e^{y+a/\lambda}B\left(y + \frac{a}{\lambda}\right) \geq e^{y-v/\lambda}B\left(y - \frac{v}{\lambda}\right)$$

or

$$B\left(y - \frac{v}{\lambda}\right) \leq B\left(y + \frac{a}{\lambda}\right)e^{\frac{a+v}{\lambda}} \leq B\left(y + \frac{a}{\lambda}\right)e^{2a/\lambda}$$

Therefore

$$\int_{-a}^{a} B\left(y - \frac{v}{\lambda}\right)\frac{\sin^2 v}{v^2}dv \leq \int_{-a}^{a} B\left(y + \frac{a}{\lambda}\right)e^{2a/\lambda}\frac{\sin^2 v}{v^2}dv$$

$$\leq e^{2a/\lambda}B\left(y + \frac{a}{\lambda}\right)\int_{-a}^{a}\frac{\sin^2 v}{v^2}dv$$

Hence taking the $\underline{\lim}$ in ($\circledast$) we get, since **149**

$$\lim_{y\to\infty}\int_{-\infty}^{\lambda y} = \lim_{y\to\infty} = \pi$$

and

$$\underline{\lim}(c + \psi(y)) \leq c + \underline{\lim}\psi(y),$$

that

$$\pi \leq c\left(\int_{-\infty}^{-a} + \int_{a}^{\infty}\right)\frac{\sin^2 v}{v^2}dv + \underline{\lim}\int_{-a}^{a}$$

$$\leq \{\cdots\} + \underline{\lim}B\left(y + \frac{a}{\lambda}\right)e^{2a/\lambda}\int_{-a}^{a}\frac{\sin^2 v}{v^2}dv$$

$$\leq \{\cdots\} + e^{2a/\lambda}\int_{-a}^{a}\underline{\lim}B(y)\frac{\sin^2 v}{v^2}dv$$

Let $a \to \infty$, $\lambda \to \infty$ in such a way that $\frac{a}{\lambda} \to 0$.
Then

$$\pi \leq \pi\underline{\lim}B(y)$$

Thus

$$\lim_{u\to\infty} B(u) = \lim_{u\to\infty} A(u)e^{-u} = 1.$$

150 **Proof of the Prime Number Theorem.**

We have seen [p.143] that

$$\frac{-\zeta'(s)}{s\zeta(s)} = \int_{1}^{\infty}\frac{\psi(x)}{x^{s+1}}dx, \quad \sigma > 1$$

$$= \int_{0}^{\infty} e^{-st}\psi(e^t)dt, \quad \sigma > 1.$$

Since $\zeta(s)$ is analytic for $\sigma \geq 1$ except for $s = 1$, where it has a simple pole with residue 1, and has no zeros for $\sigma \geq 1$, and $\psi(e^t) \geq 0$ and non-decreasing, we can appeal to the Wiener-Ikehara theorem with $f(s) = -\frac{\zeta'(s)}{s\zeta(s)}$, and obtain

$$\psi(e^t) \sim e^t \text{ as } t \to \infty$$

or $\qquad \psi(x) \sim x.$

13 Prime Number Theorem and the zeros of $\zeta(s)$

We have seen that the Prime Number Theorem follows from the Wiener-Ikehara Theorem if we assume that $\zeta(1+it) \neq 0$. On the other hand, if we assume the Prime Number Theorem, it is easy to deduce that $\zeta(1+it) \neq 0$. If 151

$$\psi(x) = \sum_{p^m \leq x} \log p = \sum_{n \leq x} \wedge(n),$$

then, for $\sigma > 1$, we have

$$\int_1^\infty \frac{\psi(x) - x}{x^{s+1}} dx = -\frac{\zeta'(s)}{s\zeta(s)} - \frac{1}{1-s} \equiv \phi(s), \text{ say.}$$

Now $\phi(s)$ is regular for $\sigma > 0$ except (possibly) for simple poles at the zeros of $\zeta(s)$. Now, if $\psi(x) = x + O(x)$, [which is a consequence of the p.n. theorem], then, given $\varepsilon > 0$,

$$|\psi(x) - x| < \varepsilon x, \text{ for } x \geq x_0(\varepsilon) > 1.$$

Hence, for $\sigma > 1$,

$$|\phi(s)| < \int_1^{x_0} \frac{|\psi(x) - x|}{x^2} dx + \int_{x_0}^\infty \frac{\varepsilon}{x^\sigma} dx < K + \frac{\varepsilon}{\sigma - 1}$$

where $K = K(x_0) = K(\varepsilon)$. Thus

$$|(\sigma - 1)\phi(\sigma + it)| < K(\sigma - 1) + \varepsilon < 2\varepsilon,$$

for $1 < \sigma < \sigma_0(\varepsilon, K) = \sigma_0(\varepsilon)$. Hence, for any fixed t,

$$(\sigma - 1)\phi(\sigma + it) \to 0 \text{ as } \sigma \to 1 + 0.$$

This shows that $1 + it$ cannot be a zero of $\zeta(s)$, for in that case

$$(\sigma - 1)\phi(\sigma + it)$$

would tend to a limit different from zero, namely the residue of $\phi(s)$ at the simple pole $1 + it$.

14 Prime Number Theorem and the magnitude of p_n

152 It is easy to see that the p.n. theorem is equivalent to the result: $p_n \sim n \log n$. For if

$$\frac{\tau(x) \log x}{x} \to 1 \tag{1}$$

then

$$\log \pi(x) + \log \log x - \log x \to 0,$$

hence

$$\frac{\log \pi(x)}{\log x} \to 1 \tag{2}$$

Now (1) and (2) give

$$\frac{\pi(x) \cdot \log \pi(x)}{x} \to 1$$

Taking $x = p_n$, we get $p_n \sim n \log n$, since $\pi(p_n) = n$.

Conversely, if n is defined by : $p_n \leq x < p_{n+1}$, then $p_n \sim n \log n$ implies

$$p_{n+1} \sim (n+1) \log(n+1) \sim n \log n,$$

as $x \to \infty$. Hence

$$x \sim n \log n$$

or

$$x \sim y \log y, \quad y = \pi(x) = n.$$

Hence

$$\log x \sim \log y, \text{ as above, so that}$$
$$y \sim x / \log x$$

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