LECTURE NOTES OF
ENGINEERING MATHEMATICS–III (Sub Code: MAT31)

COURSE CONTENT

1) Numerical Analysis
2) Fourier Series
3) Fourier Transforms & Z-transforms
4) Partial Differential Equations
5) Linear Algebra
6) Calculus of Variations

Text Book:
Higher Engineering Mathematics by
Dr. B.S.Grewal (36th Edition – 2002)
Khanna Publishers,New Delhi

Reference Book:
Advanced Engineering Mathematics by
E. Kreyszig (8th Edition – 2001)
John Wiley & Sons, INC. New York

FOURIER SERIES

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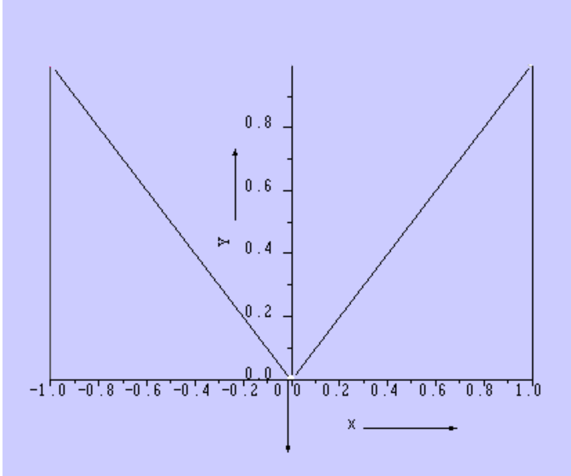
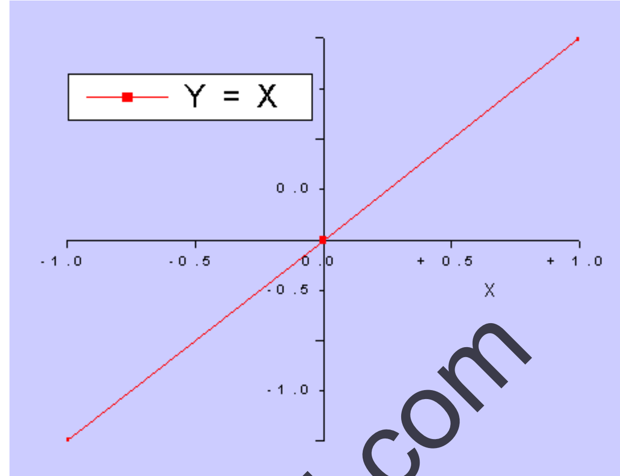
DEFINITIONS :

A function y = f(x) is said to be even, if f(-x) = f(x). The graph of the even function is always symmetrical about the y-axis.

A function y=f(x) is said to be odd, if f(-x) = - f(x). The graph of the odd function is always symmetrical about the origin.

For example, the function $f(x) = |x|$ in [-1,1] is even as $f(-x) = |-x| = |x| = f(x)$ and the function f(x) = x in [-1,1] is odd as f(-x) = -x = -f(x). The graphs of these functions are shown below :





Graph of f(x) = $|x|$ Graph of f(x) = x

Note that the graph of f(x) = $|x|$ is symmetrical about the y-axis and the graph of f(x) = x is symmetrical about the origin.

1. If f(x) is even and g(x) is odd, then

    - h(x) = f(x) x g(x) is odd
    - h(x) = f(x) x f(x) is even
    - h(x) = g(x) x g(x) is even

    For example,
    1. h(x) = x^2 cosx is even, since both x^2 and cosx are even functions
    2. h(x) = xsinx is even, since x and sinx are odd functions
    3. h(x) = x^2 sinx is odd, since x^2 is even and sinx is odd.

2. If f(x) is even, then $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$

3. If f(x) is odd, then $\int_{-a}^{a} f(x)dx = 0$

    For example,

    $$\int_{-a}^{a} \cos x dx = 2\int_{0}^{a} \cos x dx, \quad \text{as cosx is even}$$

    and $\int_{-a}^{a} \sin x dx = 0,$ as sinx is odd

PERIODIC FUNCTIONS :-

A periodic function has a basic shape which is repeated over and over again. The fundamental range is the time (or sometimes distance) over which the basic shape is defined. The length of the fundamental range is called the period.

A general periodic function f(x) of period T satisfies the condition

$$f(x+T) = f(x)$$

Here f(x) is a real-valued function and T is a positive real number.

As a consequence, it follows that

$$f(x) = f(x+T) = f(x+2T) = f(x+3T) = \ldots.. = f(x+nT)$$

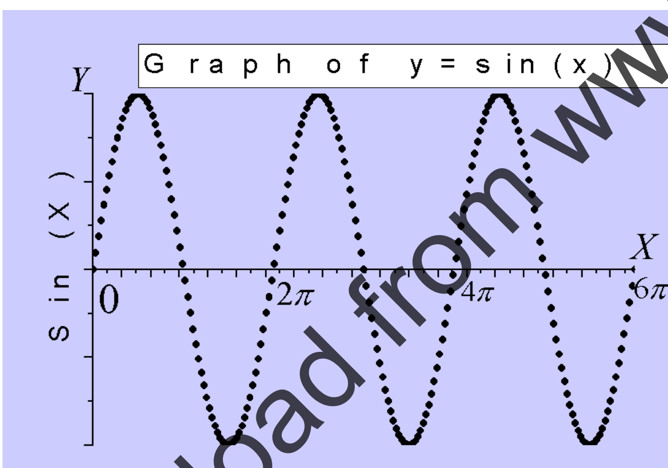
Thus,

$$f(x) = f(x+nT),\ n=1,2,3,\ldots..$$

The function f(x) = sinx is periodic of period 2π since

$$\text{Sin}(x+2n\pi) = \sin x,\quad n=1,2,3,\ldots\ldots..$$

The graph of the function is shown below :

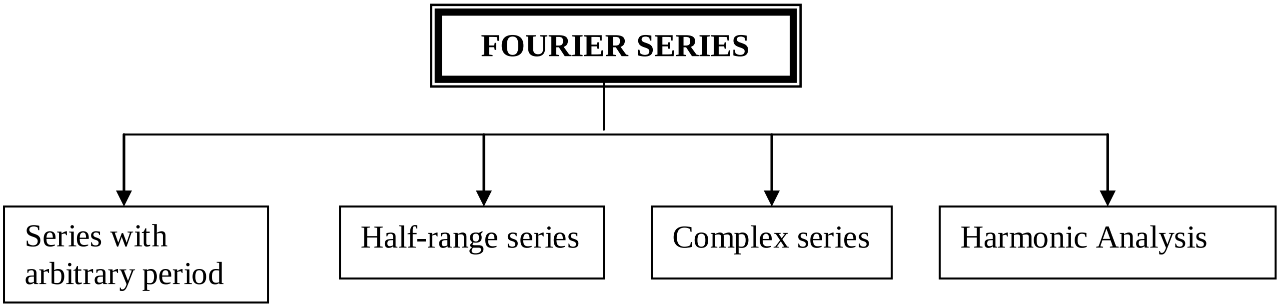


Note that the graph of the function between 0 and 2π is the same as that between 2π and 4π and so on. It may be verified that a linear combination of periodic functions is also periodic.

FOURIER SERIES

A Fourier series of a periodic function consists of a sum of sine and cosine terms. Sines and cosines are the most fundamental periodic functions.

The Fourier series is named after the French Mathematician and Physicist Jacques Fourier (1768 – 1830). Fourier series has its application in problems pertaining to Heat conduction, acoustics, etc. The subject matter may be divided into the following sub topics.



FORMULA FOR FOURIER SERIES

Consider a real-valued function f(x) which obeys the following conditions called Dirichlet's conditions :

1. f(x) is defined in an interval $(a,a+2l)$, and $f(x+2l) = f(x)$ so that f(x) is a periodic function of period $2l$.
2. f(x) is continuous or has only a finite number of discontinuities in the interval $(a,a+2l)$.
3. f(x) has no or only a finite number of maxima or minima in the interval $(a,a+2l)$.

Also, let

$$a_0 = \frac{1}{l}\int_a^{a+2l} f(x)dx \qquad (1)$$

$$a_n = \frac{1}{l}\int_a^{a+2l} f(x)\cos\left(\frac{n\pi}{l}\right)xdx, \qquad n=1,2,3,\ldots.. \quad (2)$$

$$b_n = \frac{1}{l}\int_a^{a+2l} f(x)\sin\left(\frac{n\pi}{l}\right)xdx, \qquad n=1,2,3,\ldots... \quad (3)$$

Then, the infinite series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi}{l}\right)x + b_n\sin\left(\frac{n\pi}{l}\right)x \qquad (4)$$

is called the Fourier series of f(x) in the interval $(a,a+2l)$. Also, the real numbers a_0, a_1, a_2, ….a_n, and b_1, b_2 , ….b_n are called the Fourier coefficients of f(x). The formulae (1), (2) and (3) are called Euler's formulae.

It can be proved that the sum of the series (4) is f(x) if f(x) is continuous at x. Thus we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi}{l}\right)x + b_n\sin\left(\frac{n\pi}{l}\right)x \ldots\ldots. \qquad (5)$$

Suppose f(x) is discontinuous at x, then the sum of the series (4) would be

$$\frac{1}{2}\left[f(x^+) + f(x^-)\right]$$

where $f(x^+)$ and $f(x^-)$ are the values of f(x) immediately to the right and to the left of f(x) respectively.

Particular Cases

Case (i)

Suppose a=0. Then f(x) is defined over the interval (0,2*l*). Formulae (1), (2), (3) reduce to

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)dx$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi}{l}\right)xdx, \qquad n = 1,2,......\infty \qquad (6)$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\pi}{l}\right)xdx,$$

Then the right-hand side of (5) is the Fourier expansion of f(x) over the interval (0,2*l*).

If we set *l*=π, then f(x) is defined over the interval (0,2π). Formulae (6) reduce to

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nxdx, \qquad n=1,2,\ldots..\infty \qquad (7)$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nxdx \qquad n=1,2,\ldots..\infty$$

Also, in this case, (5) becomes

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + b_n\sin nx \qquad (8)$$

Case (ii)

Suppose a=-*l*. Then f(x) is defined over the interval (-*l* , *l*). Formulae (1), (2) (3) reduce to

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi}{l}\right)xdx \qquad n=1,2,\ldots\ldots\infty \qquad (9)$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi}{l}\right)xdx, \qquad n=1,2,\ldots\ldots\infty$$

Then the right-hand side of (5) is the Fourier expansion of f(x) over the interval (-*l* , *l*).

If we set *l* = π, then f(x) is defined over the interval (-π, π). Formulae (9) reduce to

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx, \qquad n=1,2,\ldots..\infty \qquad (10)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx \qquad n=1,2,\ldots..\infty$$

Putting $l = \pi$ in (5), we get

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

PARTIAL SUMS

The Fourier series gives the exact value of the function. It uses an infinite number of terms which is impossible to calculate. However, we can find the sum through the partial sum S_N defined as follows :

$$S_N(x) = a_0 + \sum_{n=1}^{n=N}\left[a_n \cos\left(\frac{n\pi}{l}\right)x + b_n \sin\left(\frac{n\pi}{l}\right)x \right]$$ where N takes positive integral values.

In particular, the partial sums for N=1,2 are

$$S_1(x) = a_0 + a_1 \cos\left(\frac{\pi x}{l}\right) + b_1 \sin\left(\frac{\pi x}{l}\right)$$

$$S_2(x) = a_0 + a_1 \cos\left(\frac{\pi x}{l}\right) + b_1 \sin\left(\frac{\pi x}{l}\right) + a_2 \cos\left(\frac{2\pi x}{l}\right) + b_2 \sin\left(\frac{2\pi x}{l}\right)$$

If we draw the graphs of partial sums and compare these with the graph of the original function f(x), it may be verified that $S_N(x)$ approximates f(x) for some large N.

Some useful results :

1. The following rule called Bernoulli's generalized rule of integration by parts is useful in evaluating the Fourier coefficients.

$$\int uv dx = uv_1 - u' v_2 + u'' v_3 + \ldots\ldots.$$

Here $u', u'', \ldots..$ are the successive derivatives of u and

$$v_1 = \int v dx, v_2 = \int v_1 dx, \ldots\ldots$$

We illustrate the rule, through the following examples :

$$\int x^2 \sin nx dx = x^2\left(\frac{-\cos nx}{n}\right) - 2x\left(\frac{-\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)$$

$$\int x^3 e^{2x} dx = x^3\left(\frac{e^{2x}}{2}\right) - 3x^2\left(\frac{e^{2x}}{4}\right) + 6x\left(\frac{e^{2x}}{8}\right) - 6\left(\frac{e^{2x}}{16}\right)$$

2. The following integrals are also useful :

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]$$

3. If ‘n’ is integer, then

$\sin n\pi = 0$, $\cos n\pi = (-1)^n$, $\sin 2n\pi = 0$, $\cos 2n\pi = 1$

Examples

1. Obtain the Fourier expansion of

$$f(x) = \frac{1}{2}(\pi - x) \text{ in } -\pi < x < \pi$$

We have,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}(\pi - x)dx$$

$$= \frac{1}{2\pi}\left[\pi x - \frac{x^2}{2}\right]_{-\pi}^{\pi} = \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}(\pi - x)\cos nx\,dx$$

Here we use integration by parts, so that

$$a_n = \frac{1}{2\pi}\left[(\pi - x)\frac{\sin nx}{n} - (-1)\left(\frac{-\cos nx}{n^2}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}[0] = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}(\pi - x)\sin nx\,dx$$

$$= \frac{1}{2\pi}\left[(\pi - x)\frac{-\cos nx}{n} - (-1)\left(\frac{-\sin nx}{n^2}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{(-1)^n}{n}$$

Using the values of a_0 , a_n and b_n in the Fourier expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$$

we get,

$$f(x)=\frac{\pi}{2}+\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$$

This is the required Fourier expansion of the given function.

2. Obtain the Fourier expansion of $f(x)=e^{-ax}$ in the interval $(-\pi, \pi)$. Deduce that

$$\operatorname{cosech}\pi=\frac{2}{\pi}\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}$$

Here,

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}e^{-ax}dx=\frac{1}{\pi}\left[\frac{e^{-ax}}{-a}\right]_{-\pi}^{\pi}$$

$$=\frac{e^{a\pi}-e^{-a\pi}}{a\pi}=\frac{2\sinh a\pi}{a\pi}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}e^{-ax}\cos nx\,dx$$

$$a_n=\frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+n^2}\{-a\cos nx+n\sin nx\}\right]_{-\pi}^{\pi}$$

$$=\frac{2a}{\pi}\left[\frac{(-1)^n\sinh a\pi}{a^2+n^2}\right]$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}e^{-ax}\sin nx\,dx$$

$$=\frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+n^2}\{-a\sin nx-n\cos nx\}\right]_{-\pi}^{\pi}$$

$$=\frac{2n}{\pi}\left[\frac{(-1)^n\sinh a\pi}{a^2+n^2}\right]$$

Thus,

$$f(x)=\frac{\sinh a\pi}{a\pi}+\frac{2a\sinh a\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2}\cos nx+\frac{2}{\pi}\sinh a\pi\sum_{n=1}^{\infty}\frac{n(-1)^n}{a^2+n^2}\sin nx$$

For x=0, a=1, the series reduces to

$$f(0)=1=\frac{\sinh\pi}{\pi}+\frac{2\sinh\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}$$

or

$$1=\frac{\sinh\pi}{\pi}+\frac{2\sinh\pi}{\pi}\left[-\frac{1}{2}+\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}\right]$$

or $$1 = \frac{2\sinh\pi}{\pi}\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}$$

Thus,

$$\pi\,\mathrm{cosech}\,\pi = 2\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}$$

This is the desired deduction.

3. Obtain the Fourier expansion of f(x) = x^2 over the interval (-π, π). Deduce that

$$\frac{\pi^2}{6} = 1+\frac{1}{2^2}+\frac{1}{3^2}+......+\infty$$

The function f(x) is even. Hence

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

or $$a_0 = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx, \quad \text{since f(x)cosnx is even}$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,dx$$

Integrating by parts, we get

$$a_n = \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{4(-1)^n}{n^2}$$

Also, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0$ since f(x)sinnx is odd.

Thus

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n \cos nx}{n^2}$$

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\sum_{1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

Hence, $$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + ...$$

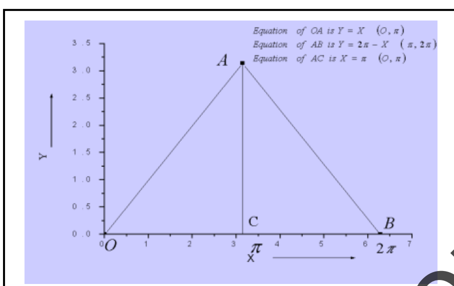
4. Obtain the Fourier expansion of

$$f(x) = \begin{cases} x, 0 \le x \le \pi \\ 2\pi - x, \pi \le x \le 2\pi \end{cases}$$

Deduce that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + ...$$

The graph of f(x) is shown below.



Here OA represents the line f(x)=x, AB represents the line f(x)=(2π-x) and AC represents the line x=π. Note that the graph is symmetrical about the line AC, which in turn is parallel to y-axis. Hence the function f(x) is an even function.

Here,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} xdx = \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nxdx$$

since f(x)cosnx is even.

$$= \frac{2}{\pi}\int_{0}^{\pi} x\cos nxdx$$

$$= \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - 1\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

Also,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = 0, \text{ since f(x)sinnx is odd}$$

Thus the Fourier series of f(x) is

$$f(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[(-1)^n - 1\right]\cos nx$$

For x=π , we get

$$f(\pi) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[(-1)^n - 1\right]\cos n\pi$$

or $$\pi = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{-2\cos(2n-1)\pi}{(2n-1)^2}$$

Thus,

$$\frac{\pi^2}{8} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

or $$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + ...$$

This is the series as required.

5. Obtain the Fourier expansion of

$$f(x) = \begin{cases} -\pi, -\pi < x < 0 \\ x, 0 < x < \pi \end{cases}$$

Deduce that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + ...$$

Here,

$$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi dx + \int_{0}^{\pi} x dx\right] = -\frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi \cos nx dx + \int_{0}^{\pi} x\cos nx dx\right]$$

$$= \frac{1}{n^2\pi}\left[(-1)^n - 1\right]$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi \sin nx dx + \int_{0}^{\pi} x \sin nx dx\right]$$

$$= \frac{1}{n}\left[1 - 2(-1)^n\right]$$

Fourier series is

$$f(x) = \frac{-\pi}{4} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[(-1)^n - 1\right]\cos nx + \sum_{n=1}^{\infty}\frac{\left[1 - 2(-1)^n\right]}{n}\sin nx$$

Note that the point x=0 is a point of discontinuity of f(x). Here $f(x^+) = 0$, $f(x^-) = -\pi$ at x=0.

Hence $$\frac{1}{2}[f(x^+) + f(x^-)] = \frac{1}{2}(0 - \pi) = \frac{-\pi}{2}$$

The Fourier expansion of f(x) at x=0 becomes

$$\frac{-\pi}{2} = \frac{-\pi}{4} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}[(-1)^n - 1]$$

$$or \frac{\pi^2}{4} = \sum_{n=1}^{\infty}\frac{1}{n^2}[(-1)^n - 1]$$

Simplifying we get,

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots$$

6. Obtain the Fourier series of $f(x) = 1-x^2$ over the interval (-1,1).

The given function is even, as f(-x) = f(x). Also period of f(x) is 1-(-1)=2

Here

$$a_0 = \frac{1}{1}\int_{-1}^{1} f(x)dx = 2\int_{0}^{1} f(x)dx$$

$$= 2\int_{0}^{1}(1 - x^2)dx = 2\left[x - \frac{x^3}{3}\right]_0^1$$

$$= \frac{4}{3}$$

$$a_n = \frac{1}{1}\int_{-1}^{1} f(x)\cos(n\pi x)dx$$

$$= 2\int_{0}^{1} f(x)\cos(n\pi x)dx$$ as $f(x)\cos(n\pi x)$ is even

$$= 2\int_{0}^{1}(1 - x^2)\ \cos(n\pi x)dx$$

Integrating by parts, we get

$$a_n = 2\left[(1 - x^2)\left(\frac{\sin n\pi x}{n\pi}\right) - (-2x)\left(\frac{-\cos n\pi x}{(n\pi)^2}\right) + (-2)\left(\frac{-\sin n\pi x}{(n\pi)^3}\right)\right]_0^1$$

$$= \frac{4(-1)^{n+1}}{n^2\pi^2}$$

$$b_n = \frac{1}{1}\int_{-1}^{1} f(x)\sin(n\pi x)dx \quad =0, \text{ since } f(x)\sin(n\pi x) \text{ is odd.}$$

The Fourier series of f(x) is

$$f(x) = \frac{2}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos(n\pi x)$$

7. Obtain the Fourier expansion of

$$f(x) = \begin{cases} 1+\frac{4x}{3} \; in -\frac{3}{2} < x \le 0 \\ 1-\frac{4x}{3} \; in 0 \le x < \frac{3}{2} \end{cases}$$

Deduce that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots$$

The period of f(x) is $\frac{3}{2} - \left(\frac{-3}{2}\right) = 3$

Also f(-x) = f(x). Hence f(x) is even

$$a_0 = \frac{1}{3/2}\int_{-3/2}^{3/2} f(x)dx = \frac{2}{3/2}\int_0^{3/2} f(x)dx$$

$$= \frac{4}{3}\int_0^{3/2}\left(1-\frac{4x}{3}\right)dx = 0$$

$$a_n = \frac{1}{3/2}\int_{-3/2}^{3/2} f(x)\cos\left(\frac{n\pi x}{3/2}\right)dx$$

$$= \frac{2}{3/2}\int_0^{3/2} f(x)\cos\left(\frac{2n\pi x}{3}\right)dx$$

$$= \frac{4}{3}\left[\left(1-\frac{4x}{3}\right)\left(\frac{\sin\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)}\right) - \left(\frac{-4}{3}\right)\left(\frac{-\cos\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^2}\right)\right]_0^{3/2}$$

$$= \frac{4}{n^2\pi^2}\left[1-(-1)^n\right]$$

Also,

$$b_n = \frac{1}{3}\int_{-3/2}^{3/2} f(x)\sin\left(\frac{n\pi x}{3/2}\right)dx = 0$$

Thus

$$\text{f(x)} = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[1-(-1)^n\right]\cos\left(\frac{2n\pi x}{3}\right)$$

putting x=0, we get

$$\text{f(0)} = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[1-(-1)^n\right]$$

or $$1 = \frac{8}{\pi^2}\left[1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\ldots\right]$$

Thus, $$\frac{\pi^2}{8} = 1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\ldots$$

NOTE

Here verify the validity of Fourier expansion through partial sums by considering an example. We recall that the Fourier expansion of f(x) = x^2 over ($-\pi$, π) is

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n\cos nx}{n^2}$$

Let us define

$$S_N(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{n=N}\frac{(-1)^n\cos nx}{n^2}$$

The partial sums corresponding to N = 1,2,…..6

are

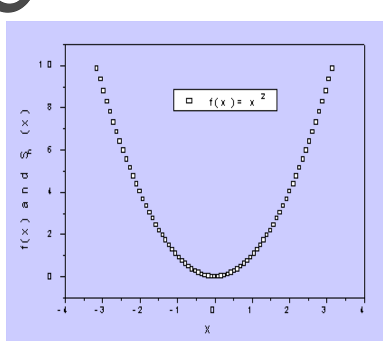
$$S_1(x) = \frac{\pi^2}{3} - 4\cos x$$

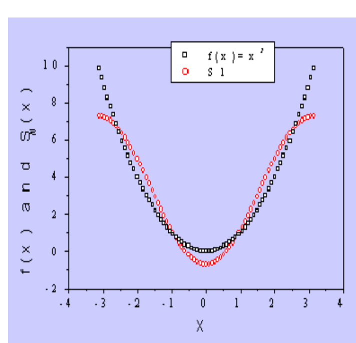
$$S_2(x) = \frac{\pi^2}{3} - 4\cos x + \cos 2x$$

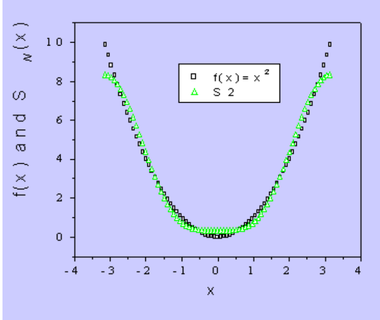
…

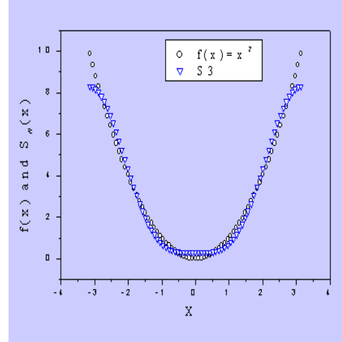
$$S_6(x) = \frac{\pi^2}{3} - 4\cos x + \cos 2x - \frac{4}{9}\cos 3x + \frac{1}{4}\cos 4x - \frac{4}{25}\cos 5x + \frac{1}{9}\cos 5x$$

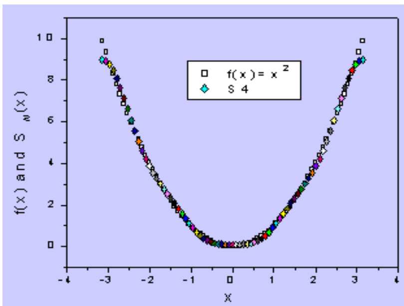
The graphs of S_1 , S_2 , …S_6 against the graph of f(x) = x^2 are plotted individually and shown below :

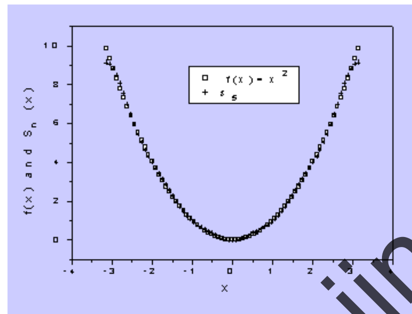












On comparison, we find that the graph of $f(x) = x^2$ coincides with that of S_6 (x). This verifies the validity of Fourier expansion for the function considered.

Exercise

Check for the validity of Fourier expansion through partial sums along with relevant graphs for other examples also.

HALF-RANGE FOURIER SERIES

The Fourier expansion of the periodic function f(x) of period $2l$ may contain both sine and cosine terms. Many a time it is required to obtain the Fourier expansion of f(x) in the interval $(0,l)$ which is regarded as half interval. The definition can be extended to the other half in such a manner that the function becomes even or odd. This will result in cosine series or sine series only.

Sine series :

Suppose $f(x) = \varphi(x)$ is given in the interval $(0,l)$. Then we define $f(x) = -\varphi(-x)$ in $(-l,0)$. Hence f(x) becomes an odd function in $(-l , l)$. The Fourier series then is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \qquad (11)$$

where $$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

The series (11) is called half-range sine series over $(0,l)$.

Putting $l=\pi$ in (11), we obtain the half-range sine series of f(x) over $(0,\pi)$ given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

Cosine series :

Let us define

$$f(x) = \begin{cases} \phi(x) & \text{in } (0,l) \quad .....given \\ \phi(-x) & \text{in } (-l,0) \quad \text{.....in order to make the function even.} \end{cases}$$

Then the Fourier series of f(x) is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \qquad (12)$$

where,

$$a_0 = \frac{2}{l}\int_0^{l} f(x)\,dx$$

$$a_n = \frac{2}{l}\int_0^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

The series (12) is called half-range cosine series over $(0,l)$

Putting l = π in (12), we get

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx \quad n = 1,2,3,.....$$

Examples :

1. Expand f(x) = x(π-x) as half-range sine series over the interval (0,π).

We have,

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2)\sin nx\,dx$$

Integrating by parts, we get

$$b_n = \frac{2}{\pi}\left[(\pi x - x^2)\left(\frac{-\cos nx}{n}\right) - (\pi - 2x)\left(\frac{-\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{4}{n^3\pi}\left[1 - (-1)^n\right]$$

The sine series of f(x) is

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^3}\left[1 - (-1)^n\right]\sin nx$$

2. Obtain the cosine series of

$$f(x) = \begin{cases} x, 0 < x < \frac{\pi}{2} \\ \pi - x, \frac{\pi}{2} < x < \pi \end{cases} \qquad over(0, \pi)$$

Here

$$a_0 = \frac{2}{\pi}\left[\int_0^{\pi/2} x\,dx + \int_{\pi/2}^{\pi}(\pi - x)\,dx\right] = \frac{\pi}{2}$$

$$a_n = \frac{2}{\pi}\left[\int_0^{\pi/2} x\cos nx\,dx + \int_{\pi/2}^{\pi}(\pi - x)\cos nx\,dx\right]$$

Performing integration by parts and simplifying, we get

$$a_n = -\frac{2}{n^2\pi}\left[1 + (-1)^n - 2\cos\left(\frac{n\pi}{2}\right)\right]$$

$$= -\frac{8}{n^2\pi}, n = 2, 6, 10, ...$$

Thus, the Fourier cosine series is

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + ...\infty\right]$$

3. Obtain the half-range cosine series of f(x) = c-x in 0<x<c

Here

$$a_0 = \frac{2}{c}\int_0^c (c - x)\,dx = c$$

$$a_n = \frac{2}{c}\int_0^c (c - x)\cos\left(\frac{n\pi x}{c}\right)dx$$

Integrating by parts and simplifying we get,

$$a_n = \frac{2c}{n^2\pi^2}\left[1-(-1)^n\right]$$

The cosine series is given by

$$f(x) = \frac{c}{2} + \frac{2c}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[1-(-1)^n\right]\cos\left(\frac{n\pi x}{c}\right)$$

Exercices:

Obtain the Fourier series of the following functions over the specified intervals :

1. f(x) = $x + \frac{x^2}{4}$ over (-π, π)
2. f(x) = 2x + 3x^2 over (-π, π)
3. f(x) = $\left(\frac{\pi - x}{2}\right)^2$ over (0, 2π)
4. f(x) = x over (-π, π) ; Deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - ....\infty$
5. f(x) = $|x|$ over (-π, π) ; Deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + ...\infty$
6. f(x) = $\begin{cases} \pi + x, -\pi \le x < 0 \\ \pi - x, 0 \le x < \pi \end{cases}$ over (-π, π)

   Deduce that

   $$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + ...\infty$$
7. f(x) = $\begin{cases} -1, -\pi < x < 0 \\ 0, x = 0 \\ 1, 0 < x < \pi \end{cases}$ over (-π, π)

   Deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - ....\infty$
8. f(x) = x sinx over 0 ≤ x ≤ 2π ; Deduce that

   $$\sum_{n=2}^{\infty}\frac{1}{n^2 - 1} = \frac{3}{4}$$
9. f(x) = $\begin{cases} 0, -2 \le x \le 0 \\ a, 0 < x \le 2 \end{cases}$ over (-2 , 2)
10. f(x) = x(2-x) over (0,3)
11. f(x) = x^2 over (-1,1)
12. f(x) = $\begin{cases} \pi x, 0 \le x \le 1 \\ \pi(2 - x), 1 \le x \le 2 \end{cases}$

Obtain the half-range sine series of the following functions over the specified intervals :

13. $f(x) = \cos x$ over $(0,\pi)$
14. $f(x) = \sin^3 x$ over $(0,\pi)$
15. $f(x) = lx - x^2$ over $(0, l)$

Obtain the half-range cosine series of the following functions over the specified intervals :

16. $f(x) = x^2$ over $(0,\pi)$
17. $f(x) = x\sin x$ over $(0,\pi)$
18. $f(x) = (x-1)^2$ over $(0,1)$
19. $f(x) = \begin{cases} kx, 0 \le x \le \dfrac{l}{2} \\ k(l-x), \dfrac{l}{2} \le x \le l \end{cases}$

HARMONIC ANALYSIS

The Fourier series of a **known** function f(x) in a given interval may be found by finding the Fourier coefficients. The method described cannot be employed when f(x) is not known explicitly, but defined through the values of the function at some equidistant points. In such a case, the integrals in Euler's formulae cannot be evaluated. Harmonic analysis is the process of finding the Fourier coefficients numerically.

To derive the relevant formulae for Fourier coefficients in Harmonic analysis, we employ the following result :

The mean value of a continuous function f(x) over the interval (a,b) denoted by **[f(x)]** is defined as

$$\left[f(x)\right] = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

The Fourier coefficients defined through Euler's formulae, (1), (2), (3) may be redefined as

$$a_0 = 2\left[\frac{1}{2l}\int_a^{a+2l} f(x)\,dx\right] = 2[f(x)]$$

$$a_n = 2\left[\frac{1}{2l}\int_a^{a+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx\right] = 2\left[f(x)\cos\left(\frac{n\pi x}{l}\right)\right]$$

$$b_n = 2\left[\frac{1}{2l}\int_a^{a+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx\right] = 2\left[f(x)\sin\left(\frac{n\pi x}{l}\right)\right]$$

Using these in (5), we obtain the Fourier series of f(x). The term $a_1\cos x + b_1\sin x$ is called the first harmonic or fundamental harmonic, the term $a_2\cos 2x + b_2\sin 2x$ is called the second harmonic and so on. The amplitude of the first harmonic is $\sqrt{a_1^2 + b_1^2}$ and that of second harmonic is $\sqrt{a_2^2 + b_2^2}$ and so on.

Examples

1. Find the first two harmonics of the Fourier series of f(x) given the following table :

x	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
f(x)	1.0	1.4	1.9	1.7	1.5	1.2	1.0

Note that the values of y = f(x) are spread over the interval $0 \le x \le 2\pi$ and f(0) = f(2π) = 1.0. Hence the function is periodic and so we omit the last value f(2π) = 0. We prepare the following table to compute the first two harmonics.

x^0	y = f(x)	cosx	cos2x	sinx	sin2x	ycosx	ycos2x	ysinx	ysin2x
0	1.0	1	1	0	0	1	1	0	0
60	1.4	0.5	-0.5	0.866	0.866	0.7	-0.7	1.2124	1.2124
120	1.9	-0.5	-0.5	0.866	-0.866	-0.95	-0.95	1.6454	-1.6454
180	1.7	-1	1	0	0	-1.7	1.7	0	0
240	1.5	-0.5	-0.5	-0.866	0.866	-0.75	-0.75	1.299	1.299
300	1.2	0.5	-0.5	-0.866	-0.866	0.6	-0.6	-1.0392	-1.0392
Total						-1.1	-0.3	3.1176	-0.1732

We have

$$a_n = 2\left[f(x)\cos\left(\frac{n\pi x}{l}\right)\right] = 2[y\cos nx]$$

$$b_n = 2\left[f(x)\sin\left(\frac{n\pi x}{l}\right)\right] = 2[y\sin nx]$$

as the length of interval= $2l = 2\pi$ or $l=\pi$

Putting, n=1,2, we get

$$a_1 = 2[y\cos x] = \frac{2\sum y\cos x}{6} = \frac{2(1.1)}{6} == -0.367$$

$$a_2 = 2[y\cos 2x] = \frac{2\sum y\cos 2x}{6} = \frac{2(-0.3)}{6} = -0.1$$

$$b_1 = [y\sin x] = \frac{2\sum y\sin x}{6} = 1.0392$$

$$b_2 = [y\sin 2x] = \frac{2\sum y\sin 2x}{6} = -0.0577$$

The first two harmonics are $a_1\cos x + b_1\sin x$ and $a_2\cos 2x + b_2\sin 2x$. That is $(-0.367\cos x + 1.0392\sin x)$ and $(-0.1\cos 2x - 0.0577\sin 2x)$

2. Express y as a Fourier series upto the third harmonic given the following values :

x	0	1	2	3	4	5
y	4	8	15	7	6	2

The values of y at x=0,1,2,3,4,5 are given and hence the interval of x should be $0 \le x < 6$. The length of the interval = 6-0 = 6, so that $2l = 6$ or $l = 3$.

The Fourier series upto the third harmonic is

$$y = \frac{a_0}{2} + \left(a_1\cos\frac{\pi x}{l} + b_1\sin\frac{\pi x}{l}\right) + \left(a_2\cos\frac{2\pi x}{l} + b_2\sin\frac{2\pi x}{l}\right) + \left(a_3\cos\frac{3\pi x}{l} + b_3\sin\frac{3\pi x}{l}\right)$$

or

$$y = \frac{a_0}{2} + \left(a_1\cos\frac{\pi x}{3} + b_1\sin\frac{\pi x}{3}\right) + \left(a_2\cos\frac{2\pi x}{3} + b_2\sin\frac{2\pi x}{3}\right) + \left(a_3\cos\frac{3\pi x}{3} + b_3\sin\frac{3\pi x}{3}\right)$$

Put $\theta = \frac{\pi x}{3}$, then

$$y = \frac{a_0}{2} + (a_1\cos\theta + b_1\sin\theta) + (a_2\cos 2\theta + b_2\sin 2\theta) + (a_3\cos 3\theta + b_3\sin 3\theta) \qquad (1)$$

We prepare the following table using the given values :

x	$\theta = \frac{\pi x}{3}$	**y**	**ycosθ**	**ycos2θ**	**ycos3θ**	**ysinθ**	**ysin2θ**	**ysin3θ**
0	0	04	4	4	4	0	0	0
1	60^0	08	4	-4	-8	6.928	6.928	0
2	120^0	15	-7.5	-7.5	15	12.99	-12.99	0
3	180^0	07	-7	7	-7	0	0	0
4	240^0	06	-3	-3	6	-5.196	5.196	0
5	300^0	02	1	-1	-2	-1.732	-1.732	0
Total		**42**	**-8.5**	**-4.5**	**8**	**12.99**	**-2.598**	**0**

$$a_0 = 2[f(x)] = 2[y] = \frac{2\sum y}{6} = \frac{1}{3}(42) = 14$$

$$a_1 = 2[y\cos\theta] = \frac{2}{6}(-8.5) = -2.833$$

$$b_1 = 2[y\sin\theta] = \frac{2}{6}(12.99) = 4.33$$

$$a_2 = 2[y\cos 2\theta] = \frac{2}{6}(-4.5) = -1.5$$

$$b_2 = 2[y\sin 2\theta] = \frac{2}{6}(-2.598) = -0.866$$

$$a_3 = 2[y\cos 3\theta] = \frac{2}{6}(8) = 2.667$$

$$b_3 = 2[y\sin 3\theta] = 0$$

Using these in (1), we get

$$y = 7 - 2,833\cos\left(\frac{\pi x}{3}\right) + (4.33)\sin\left(\frac{\pi x}{3}\right) - 1.5\cos\left(\frac{2\pi x}{3}\right) - 0.866\sin\left(\frac{2\pi x}{3}\right) + 2.667\cos\pi x$$

This is the required Fourier series upto the third harmonic.

3. The following table gives the variations of a periodic current A over a period T :

t(secs)	**0**	**T/6**	**T/3**	**T/2**	**2T/3**	**5T/6**	**T**
A (amp)	1.98	1.30	1.05	1.30	-0.88	-0.25	1.98

Show that there is a constant part of 0.75amp. in the current A and obtain the amplitude of the first harmonic.

Note that the values of A at t=0 and t=T are the same. Hence A(t) is a periodic function of period T. Let us denote $\theta = \left(\frac{2\pi}{T}\right)t$. We have

$$a_0 = 2[A]$$

$$a_1 = 2\left[A\cos\left(\frac{2\pi}{T}\right)t\right] = 2[A\cos\theta] \qquad (1)$$

$$b_1 = 2\left[A\sin\left(\frac{2\pi}{T}\right)t\right] = 2[A\sin\theta]$$

We prepare the following table:

t	$\theta = \frac{2\pi t}{T}$	A	cosθ	sinθ	Acosθ	Asinθ
0	0	1.98	1	0	1.98	0
T/6	60^0	1.30	0.5	0.866	0.65	1.1258
T/3	120^0	1.05	-0.5	0.866	-0.525	0.9093
T/2	180^0	1.30	-1	0	-1.30	0
2T/3	240^0	-0.88	-0.5	-0.866	0.44	0.7621
5T/6	300^0	-0.25	0.5	-0.866	-0.125	0.2165
Total		**4.5**			**1.12**	**3.0137**

Using the values of the table in (1), we get

$$a_0 = \frac{2\sum A}{6} = \frac{4.5}{3} = 1.5$$

$$a_1 = \frac{2\sum A\cos\theta}{6} = \frac{1.12}{3} = 0.3733$$

$$b_1 = \frac{2\sum A\sin\theta}{6} = \frac{3.0137}{3} = 1.0046$$

The Fourier expansion upto the first harmonic is

$$A = \frac{a_0}{2} + a_1\cos\left(\frac{2\pi t}{T}\right) + b_1\sin\left(\frac{2\pi t}{T}\right)$$

$$= 0.75 + 0.3733\cos\left(\frac{2\pi t}{T}\right) + 1.0046\sin\left(\frac{2\pi t}{T}\right)$$

The expression shows that A has a constant part 0.75 in it. Also the amplitude of the first harmonic is $\sqrt{a_1^2 + b_1^2}$ = 1.0717.

ASSIGNMENT :

1. The displacement y of a part of a mechanism is tabulated with corresponding angular movement x^0 of the crank. Express y as a Fourier series upto the third harmonic.

x^0	0	30	60	90	120	150	180	210	240	270	300	330
y	1.80	1.10	0.30	0.16	1.50	1.30	2.16	1.25	1.30	1.52	1.76	2.00

2. Obtain the Fourier series of y upto the second harmonic using the following table :

x^0	45	90	135	180	225	270	315	360
y	4.0	3.8	2.4	2.0	-1.5	0	2.8	3.4

3. Obtain the constant term and the coefficients of the first sine and cosine terms in the Fourier expansion of y as given in the following table :

x	**0**	**1**	**2**	**3**	**4**	**5**
y	9	18	24	28	26	20

4. Find the Fourier series of y upto the second harmonic from the following table :

x	**0**	**2**	**4**	**6**	**8**	**10**	**12**
Y	9.0	18.2	24.4	27.8	27.5	22.0	9.0

5. Obtain the first three coefficients in the Fourier cosine series for y, where y is given in the following table :

x	**0**	**1**	**2**	**3**	**4**	**5**
y	4	8	15	7	6	2

6. The turning moment T is given for a series of values of the crank angle $\theta^0 = 75^0$.

θ^0	**0**	**30**	**60**	**90**	**120**	**150**	**180**
T	0	5224	8097	7850	5499	2626	0

Obtain the first four terms in a series of sines to represent T and calculate T at $\theta = 75^0$.