Test Codes 2015

UGA (Multiple choice) and UGB (Short Answer Type)

Questions will include the following and related topics.

Algebra and number theory
Sets, operations on sets. Prime numbers, factorization of integers and divisibility. Rational and irrational numbers. Permutations and combinations, Binomial Theorem. Logarithms. Polynomials: relations between roots and coefficients, Remainder Theorem, Theory of quadratic equations and expressions. Arithmetic and geometric progressions. Inequalities involving arithmetic, geometric & harmonic means. Complex numbers.

Geometry
Class X level plane geometry. Geometry of 2 dimensions with Cartesian and polar coordinates, concept of a locus, equation of a line, angle between two lines, distance from a point to a line, area of a triangle, equations of circle, parabola, ellipse and hyperbola and equations of their tangents and normals, mensuration.

Trigonometry
Measures of angles, trigonometric and inverse trigonometric functions, trigonometric identities including addition formulae, solutions of trigonometric equations, properties of triangles, heights and distances.

Calculus
Sequences - bounded sequences, monotone sequences, limit of a sequence. Functions - one-one functions, onto functions. Limit, continuity and differentiability of functions of a single real variable. Derivatives and methods of differentiation, slope of a curve, tangents and normals, maxima and minima, use of calculus in sketching graphs of functions. Methods of integration, definite and indefinite integrals, evaluation of areas using integrals.

UGA - Sample Questions.

Instructions. UGA is a multiple choice examination. In each of the following questions, exactly one of the choices is correct. Please tick the correct answer. You get four marks for each correct answer, one mark for each unanswered question, and zero marks for each incorrect answer. You have two hours to answer these questions.

1. Define $a_n = (1^2 + 2^2 + \ldots + n^2)^n$ and $b_n = n^n (n!)^2$. Recall $n!$ is the product of the first n natural numbers. Then,
(a) $a_n < b_n$ for all $n > 1$
(b) $a_n > b_n$ for all $n > 1$
(c) $a_n = b_n$ for infinitely many n
(d) none of the above.

2. The last digit of $(2004)^5$ is:
(a) 4 (b) 8 (c) 6 (d) 2

3. If n is a positive integer such that $8n + 1$ is a perfect square, then
(a) n must be odd (b) n cannot be a perfect square
(c) $2n$ cannot be a perfect square (d) none of the above.

4. The coefficient of $a^3 b^4 c^5$ in the expansion of $(bc + ca + ab)^6$ is:
(a) $\frac{12!}{3!4!5!}$ (b) $\frac{6!}{3!}$ (c) 33 (d) $3 \cdot \left(\frac{6!}{3!3!}\right)$

5. If $\log_{10} x = 10^{\log_{100} 4}$, then x equals
(a) 4^{10} (b) 100 (c) $\log_{10} 4$ (d) none of the above.

6. Let $\mathbb{C}$ denote the set of all complex numbers. Define

$$A = \{(z, w) \mid z, w \in \mathbb{C} \text{ and } |z| = |w|\}$$
$$B = \{(z, w) \mid z, w \in \mathbb{C}, \text{ and } z^2 = w^2\}.$$

Then,
(a) $A = B$ (b) $A \subset B$ and $A \neq B$ (c) $B \subset A$ and $B \neq A$ (d) none of the above.

7. The set of all real numbers x such that $x^3(x + 1)(x - 2) \geq 0$ is:
(a) the interval $2 \leq x < \infty$ (b) the interval $0 \leq x < \infty$
(c) the interval $-1 \leq x < \infty$ (d) none of the above.

8. Let z be a non-zero complex number such that $\frac{z}{1+z}$ is purely imaginary. Then
(a) z is neither real nor purely imaginary (b) z is real
(c) z is purely imaginary (d) none of the above.

9. In the interval $(0, 2\pi)$, the function $\sin(\frac{1}{x^3})$
(a) never changes sign
(b) changes sign only once
(c) changes sign more than once, but finitely many times
(d) changes sign infinitely many times.

10. $\lim_{x \to 0} \frac{(e^x - 1)\tan^2 x}{x^3}$
(a) does not exist (b) exists and equals 0
(c) exists and equals $\frac{2}{3}$ (d) exists and equals 1.

11. Let $f_1(x) = e^x, f_2(x) = e^{f_1(x)}$ and generally $f_{n+1}(x) = e^{f_n(x)}$ for all $n \geq 1$. For any fixed n, the value of $\frac{d}{dx} f_n(x)$ is:
(a) $f_n(x)$ (b) $f_n(x) f_{n-1}(x)$
(c) $f_n(x) f_{n-1}(x) \ldots f_1(x)$ (d) $f_{n+1}(x) f_n(x) \ldots f_1(x) e^x$.

12. Let $f(x) = a_0 + a_1|x| + a_2|x|^2 + a_3|x|^3$, where a_0, a_1, a_2, a_3 are constants. Then
(a) $f(x)$ is differentiable at $x = 0$ whatever be a_0, a_1, a_2, a_3
(b) $f(x)$ is not differentiable at $x = 0$ whatever be a_0, a_1, a_2, a_3
(c) $f(x)$ is differentiable at $x = 0$ only if $a_1 = 0$
(d) $f(x)$ is differentiable at $x = 0$ only if $a_1 = 0, a_3 = 0$.

13. If $f(x) = \cos(x) - 1 + \frac{x^2}{2}$, then
(a) $f(x)$ is an increasing function on the real line
(b) $f(x)$ is a decreasing function on the real line
(c) $f(x)$ is increasing on the interval $-\infty < x \leq 0$ and decreasing on the interval $0 \leq x < \infty$
(d) $f(x)$ is decreasing on the interval $-\infty < x \leq 0$ and increasing on the interval $0 \leq x < \infty$.

14. The area of the region bounded by the straight lines $x = \frac{1}{2}$ and $x = 2$, and the curves given by the equations $y = \log_e x$ and $y = 2^x$ is
(a) $\frac{1}{\log_e 2}(4 + \sqrt{2}) - \frac{5}{2}\log_e 2 + \frac{3}{2}$ (b) $\frac{1}{\log_e 2}(4 - \sqrt{2}) - \frac{5}{2}\log_e 2$
(c) $\frac{1}{\log_e 2}(4 - \sqrt{2}) - \frac{5}{2}\log_e 2 + \frac{3}{2}$ (d) none of the above.

15. The number of roots of the equation $x^2 + \sin^2 x = 1$ in the closed interval $[0, \frac{\pi}{2}]$ is
(a) 0 (b) 1 (c) 2 (d) 3

16. The number of maps f from the set $\{1,2,3\}$ into the set $\{1,2,3,4,5\}$ such that $f(i) \le f(j)$ whenever $i < j$ is
(a) 60 (b) 50 (c) 35 (d) 30

17. Let a be a real number. The number of distinct solutions (x, y) of the system of equations $(x-a)^2 + y^2 = 1$ and $x^2 = y^2$, can only be
(a) $0, 1, 2, 3, 4$ or 5 (b) 0, 1 or 3 (c) $0, 1, 2$ or 4 (d) $0, 2, 3$, or 4

18. The set of values of m for which $mx^2 - 6mx + 5m + 1 > 0$ for all real x is
(a) $m < \frac{1}{4}$ (b) $m \ge 0$ (c) $0 \le m \le \frac{1}{4}$ (d) $0 \le m < \frac{1}{4}$.

19. A lantern is placed on the ground 100 feet away from a wall. A man six feet tall is walking at a speed of 10 feet/second from the lantern to the nearest point on the wall. When he is midway between the lantern and the wall, the rate of change in the length of his shadow is
(a) 3.6 ft./sec. (b) 2.4 ft./sec. (c) 3 ft./sec. (d) 12 ft./sec.

20. Let $n \ge 3$ be an integer. Assume that inside a big circle, exactly n small circles of radius r can be drawn so that each small circle touches the big circle and also touches both its adjacent small circles. Then, the radius of the big circle is:
(a) $r\,\mathrm{cosec}\frac{\pi}{n}$ (b) $r(1 + \mathrm{cosec}\frac{2\pi}{n})$
(c) $r(1 + \mathrm{cosec}\frac{\pi}{2n})$ (d) $r(1 + \mathrm{cosec}\frac{\pi}{n})$.

21. The digit in the units' place of the number $1! + 2! + 3! + \ldots + 99!$ is
(a) 3 (b) 0 (c) 1 (d) 7.

22. The value of $\lim_{n\to\infty} \frac{1^3+2^3+\ldots+n^3}{n^4}$ is:
(a) $\frac{3}{4}$ (b) $\frac{1}{4}$ (c) 1 (d) 4.

23. The function $x(\alpha - x)$ is strictly increasing on the interval $0 < x < 1$ if and only if
(a) $\alpha \ge 2$ (b) $\alpha < 2$ (c) $\alpha < -1$ (d) $\alpha > 2$.

24. For any integer $n \ge 1$, define $a_n = \frac{1000^n}{n!}$. Then the sequence $\{a_n\}$
(a) does not have a maximum
(b) attains maximum at exactly one value of n

(c) attains maximum at exactly two values of n
(d) attains maximum for infinitely many values of n.

25. The equation $x^3y + xy^3 + xy = 0$ represents
(a) a circle (b) a circle and a pair of straight lines
(c) a rectangular hyperbola (d) a pair of straight lines.

26. Let P be a variable point on a circle C and Q be a fixed point outside C. If R is the mid-point of the line segment PQ, then the locus of R is
(a) a circle (b) an ellipse
(c) a line segment (d) segment of a parabola.

27. Let $d_1, d_2, \ldots, d_k$ be all the factors of a positive integer n including 1 and n. If $d_1 + d_2 + \ldots + d_k = 72$, then $\frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_k}$ is:
(a) $\frac{k^2}{72}$ (b) $\frac{72}{k}$ (c) $\frac{72}{n}$ (d) none of the above.

28. A subset W of the set of real numbers is called a ring if it contains 1 and if for all $a, b \in W$, the numbers $a - b$ and ab are also in W. Let $S = \left\{\frac{m}{2^n} \mid m, n \text{ integers}\right\}$ and $T = \left\{\frac{p}{q} \mid p, q \text{ integers, } q \text{ odd}\right\}$. Then:

(a) neither S nor T is a ring (b) S is a ring T is not a ring.
(b) T is a ring S is not a ring. (d) both S and T are rings.

UGB - Sample Questions.

Instructions. All questions carry equal marks. You have two hours to solve 6 problems. Credit will be given to a partially correct answer. Do not feel discouraged if you cannot solve all the questions.

1. Find the sum of all distinct four digit numbers that can be formed using the digits 1, 2, 3, 4, 5, each digit appearing at most once.

2. Consider the squares of an 8×8 chessboard filled with the numbers 1 to 64 as in the figure below. If we choose 8 squares with the property that there is exactly one from each row and exactly one from each column, and add up the numbers in the chosen squares, show that the sum obtained is always 260.

1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16
17	18	19	20	21	22	23	24
25	26	27	28	29	30	31	32
33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48
49	50	51	52	53	54	55	56
57	58	59	60	61	62	63	64

3. An isosceles triangle with base 6 cms. and base angles 30^o each is inscribed in a circle. A second circle, which is situated outside the triangle, touches the first circle and also touches the base of the triangle at its midpoint. Find its radius.

4. Let $a_n = 1 \ldots 1$ with 3^n digits. Prove that a_n is divisible by $3a_{n-1}$.

5. If a circle intersects the hyperbola $y = 1/x$ at four distinct points $(x_i, y_i), i = 1, 2, 3, 4$, then prove that $x_1x_2 = y_3y_4$.

6. Show that the function $f(x)$ defined below attains a unique minimum for $x > 0$. What is the minimum value of the function? What is the value of x at which the minimum is attained?

$$f(x) = x^2 + x + \frac{1}{x} + \frac{1}{x^2} \qquad \text{for} \quad x \neq 0.$$

   Sketch on plain paper the graph of this function.

7. Let $S = \{1, 2, \ldots, n\}$. Find the number of unordered pairs $\{A, B\}$ of subsets of S such that A and B are disjoint, where A or B or both may be empty.

8. Find the maximum value of $x^2 + y^2$ in the bounded region, including the boundary, enclosed by $y = \frac{x}{2}, y = -\frac{x}{2}$ and $x = y^2 + 1$.

9. How many real roots does $x^4 + 12x - 5$ have?

10. Find the maximum among $1, 2^{1/2}, 3^{1/3}, 4^{1/4}, \ldots$.

11. For real numbers x, y and z, show that

$$|x| + |y| + |z| \leq |x + y - z| + |y + z - x| + |z + x - y|.$$

Hints and Answers.

There are also other ways to solve the problems apart from the ones sketched in the hints. Indeed, a student should feel encouraged if she finds a different way to solve some of these problems. All the Best!

Hints for UGA Sample Questions.

Q.1 **(b)**. Take the nth root of a_n and b_n and use A.M.$\geq$ G.M.
Q.2 **(a)**. As 2004 = 2000 +4, the last digits of $(2004)^5$ and 4^5 are equal.
Q.3 **(c)** If $8n+1=m^2$, then $2n$ is a product of two consecutive integers.
Q.4 **(d)** Use binomial expansion of $(bc+a(b+c))^6$.
Q.5 **(b)** Let $y=\log_{10} x$. Then $\log_{10} y=\log_{100} 4$. Hence $y=2$.
Q.6 **(c)** $z^2=w^2 \Rightarrow z=\pm w \Rightarrow B \subseteq A$. But $|i|=1$ and $i^2 \neq 1$.
Q.7 **(d)** Check for 'test points' -1, and 1.
Q.8 **(a)** Check (b) and (c) are false, and then that (a) is true.
Q.9 **(d)** $\sin\left(\frac{1}{x^3}\right)$ changes sign at the points $(n\pi)^{\frac{-1}{3}}$ for all $n \geq 1$.
Q.10 **(d)** Observe that $\frac{(e^x-1)\tan^2 x}{x^3}=\frac{(e^x-1)}{x}\cdot\frac{\sin^2 x}{x^2}\cdot\frac{1}{\cos^2 x}$.
Q.11 **(c)** Use induction and chain rule of differentiation.
Q.12 **(c)** Amongst $1,|x|,|x|^2,|x|^3$, only $|x|$ is not differentiable at 0.
Q.13 **(d)** Look at the derivative of f.
Q.14 **(c)** Compute the integral $\int_{1/2}^{2} 2^x dx-\int_{1/2}^{2}\log x dx$.
Q.15 **(b)** Draw graphs of $y=\cos x$ and $y=\pm x$ and find the number of points of intersections.
Q.16 **(c)** Compute the number of maps such that $f(3)=5$, $f(3)=4$ etc.. Alternatively, define $g:\{1,2,3\}\to\{1,2,\ldots,7\}$ by $g(i)=f(i)+(i-1)$. Then, g is a strictly increasing function and its image is a subset of size 3 of $\{1,2,\ldots 7\}$.
Q.17 **(d)** Draw graphs of $(x+y)(x-y)=0$ and $(x-a)^2+y^2=1$.
Q.18 **(d)** Calculate the discriminant (b^2-4ac) of the given quadratic.
Q.19 **(b)** Show that the height function is $\frac{60}{t}$.
Q.20 **(d)** Let s be distance between the centre of the big circle and the centre of (any) one of the small circles. Then there exists a right angle triangle with hypoteneuse s, side r and angle $\frac{\pi}{n}$.
Q.21 **(a)** The unit digit of all numbers $n!$ with $n \geq 5$ is 0.
Q.22 **(b)** Use the formula for $\sum_{i=1}^{n} i^3$.

Q.23 **(a)** Differentiate.
Q.24 **(c)** Find out the first values of n for which $\frac{a_{n+1}}{a_n}$ becomes < 1.
Q.25 **(d)** The equation is $xy(x^2+y^2+1)=0$.
Q.26 **(a)** Compute for $C = \left\{x^2+y^2=1\right\}$ and $Q=(a,0)$ for some $a>1$.
Q.27 **(c)** Multiply the given sum by n.
Q.28 (**d**) Verify using the given definition of a ring.

Hints for UGB Sample Questions.

Q.1 The answer is 399960. For each $x \in \{1,2,3,4,5\}$, there are 4! such numbers whose last digit is x. Thus the digits in the unit place of all the 120 numbers add up to $4!\,(1+2+3+4+5)$. Similarly the numbers at ten's place add up to 360 and so on. Thus the sum is $360\,(1+10+100+1000)$.
Q.2 Let the chosen entries be in the positions (i, a_i), $1 \leq i \leq 8$. Thus $a_1, \ldots, a_8$ is a permutation of $\{1, \ldots, 8\}$. The entry in the square corresponding to (i,j)th place is $i + 8\,(j-1)$. Hence the required sum is $\sum\limits_{i=1}^{8} (i + 8\,(a_j - 1))$.
Q.3 Radius is $\frac{3\sqrt{3}}{2}$. Use trigonometry.
Q.4 Observe that $a_n = a_{n-1}\left(1+t+t^2\right)$ where $t = 10^{3^n}$
Q.5 Substitute $y = \frac{1}{x}$ in the equation of a circle and clear denominator to get a degree 4 equation in x. The product of its roots is the constant term, which is 1.
Q.6 The function $f\,(x) - 4$ is a sum of squares and hence non-negative. So the minimum is 4 which is attained at $x = 1$.
Q.7 The number is $\frac{3^n+1}{2}$. An ordered pair (A, B) of disjoint subsets of S is determined by 3 choices for every element of S (either it is in A, or in B or in neither of them). Hence such pairs are 3^n in number. An unordered pair will be counted twice in this way, except for the case A and B both empty. Hence the number is $1 + \frac{3^n-1}{2}$.
Q.8 Answer is 5. The maximum is attained at points $(2,1)$ and $(2,-1)$.
Q.9 Answer is 2. Let f be the given polynomial. Then $f\,(0)$ is negative and f is positive as x tends to $\pm\infty$. Hence it has at least 2 real roots. Since the derivative of f is zero only at $\sqrt[3]{-3}$, it cannot have more than two real roots.
Q.10 Maximum is $\sqrt[3]{3}$. Either check the maximum of the function $x^{\frac{1}{x}}$, or compare $\sqrt[3]{3}$ with $\sqrt[n]{n}$.
Q.11 Rewrite the given inequality in terms of the new variables $\alpha = x+y-z$, $\beta = y+z-x$, $\gamma = x+z-y$, and use the triangle inequality.