ME623: Finite Element Methods in Engineering Mechanics

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Experiments versus simulations?

Qs. Sophisticated experiments can tell everything. Why do we need the FE method?

1: Experimental results are subject to interpretation. Interpretations are as good as the competence of the experimenter.

2. Experiments, especially sophisticated ones, can be expensive

3. There are regimes of mechanical material behaviour that experiments cannot probe.

4. Generality of behaviour is often not apparent from experiments.

Experiments and simulations are like two legs of a human being. You need both to walk and it does not matter which you use first!

A short history of FEA

1943: Richard Courant, a mathematician described a piecewise polynomial solution for the torsion problem of a shaft of arbitrary cross section. Even holes.
The early ideas of FEA date back to a 1922 book
by Hurwitz and Courant.

His work was not noticed by engineers
and the procedure was impractical at the time due to the lack of digital computers.

1888-1972: b in Lublintz Germany

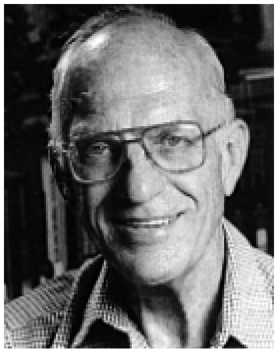
Student of Hilbert and Minkowski in Gottingen Germany

Ph.D in 1910 under Hilbert's supervision.

1934: moved to New York University, founded the Courant Institute

In the **1950**s: work in the aircraft industry introduced FE to practicing engineers.
A classic paper described FE work that was prompted by a need to analyze
delta wings, which are too short for beam theory to be reliable.

1960: The name "finite element" was coined by structural engineer Ray Clough
of the University of California



Professor emeritus of Structural Engineering at UC Berkley

Ph.D from MIT

Well known earthquake engineer

By **1963** the mathematical validity of FE was recognized and the
method was expanded from its structural beginnings to include heat transfer,
groundwater flow, magnetic fields, and other areas.

Large general-purpose FE software began to appear in the 1970s.
By the late **1980**s the software was available on microcomputers,
complete with color graphics and pre- and post-processors.
By the mid **1990**s roughly 40,000 papers and books about
FE and its applications had been published.

Books

- Concepts and applications of Finite element analysis: Cook, Malkus and Plesha, John Wiley and Sons, 2003.
- T.R. Chandrupatla and A.D. Belegundu, Introduction to Finite Elements in Engineering, Second Edition, Prentice-Hall, 1997.
- O. C. Zienkiewicz and R. L. Taylor, The Finite element method, vols 1 and 2, Butterworth Heinemann, 2000
- Klaus-Jurgen Bathe, Finite Element Procedures (Part 1-2), Prentice Hall, 1995.
- Daryl Logan, A First Course in Finite Element Method, Thomson, India Edition

Solving an engineering problem

Mathematical model: an
equation of motion

$$\frac{du}{dt} = f(t, u)$$

for $t > 0$ and $u = u_0$ at $t = 0$

Use

$$\left(\frac{du}{dt}\right)_i \simeq \frac{[u(t_{i+1}) - u(t_i)]}{t_{i+1} - t_i}$$

$$\Rightarrow u_{i+1} = u_i + \Delta t f(u_i, t_i)$$

Euler's explicit scheme or first order Runge Kutta scheme

A special case, a pendulum:

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

$\theta(0) = \theta_0$ and $\dot{\theta}(0) = v_0$.
Solution to this system is

$$\theta(t) = \frac{v_0}{\lambda}\sin\lambda t + \theta_0\cos\lambda t$$

where $\lambda = \sqrt{g/l}$

Write a MATLAB code to integrate the discretised equations of motion with different timesteps. Use l=1, g=10, initial velocity=0, position=45°.

Compare with the exact solution.

Alternately,

$$\begin{aligned}\dot{\theta} &= v\\ \dot{v} &= -\lambda^2\theta\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}\theta_{i+1} &= \theta_i + \Delta t v_i\\ v_{i+1} &= v_i - \Delta t\lambda^2\theta_i\end{aligned}$$

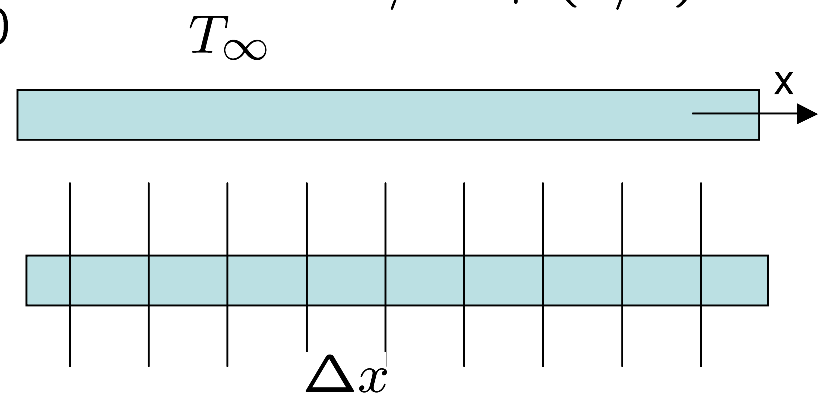
$$kA\frac{d^2T}{dx^2} + hP(T_\infty - T) = 0$$

At $x = 0$

$T = T_0$

At $x = L$

$dT/dx + (h/k)T = 0$



Using $T_\infty = 0$ arbitrarily

$$\left(\frac{d^2T}{dx^2}\right)_{x=x_i} \simeq \left(\frac{T_{i-1} - 2T_i + T_{i+1}}{(\Delta x)^2}\right)$$

$\Rightarrow$

$$-T_{i-1} + [2 + (m\Delta x)^2]T_i - T_{i+1} = 0$$

where $m = \sqrt{hP/kA}$

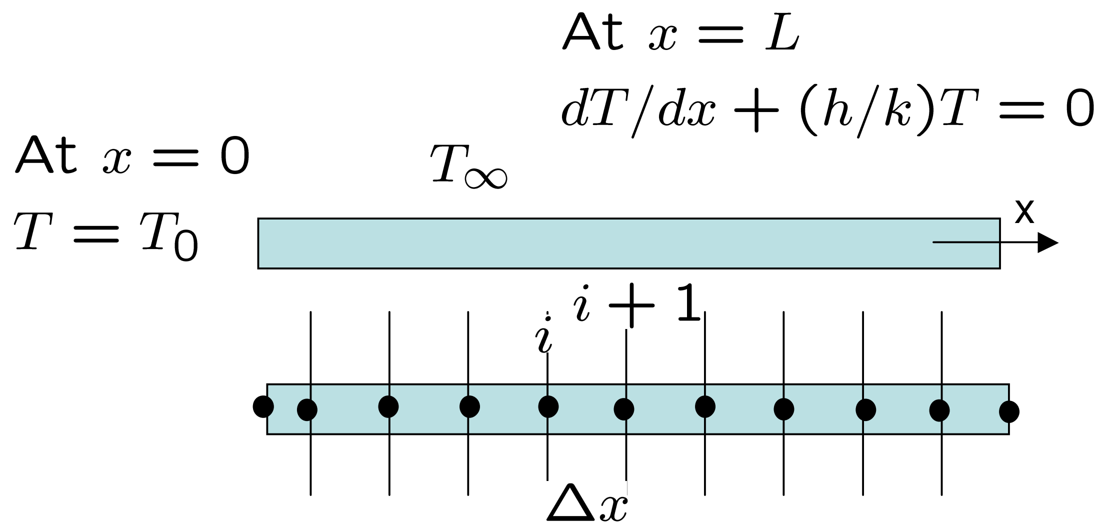
$$
\begin{aligned}
-T_0 + DT_1 - T_2 &= 0 \\
-T_1 + DT_2 - T_3 &= 0 \\
\dots\dots\dots\dots & \quad \dots \\
-T_{N-1} + DT_N - T_{N+1} &= 0
\end{aligned}
$$

Additionally,

$$
\begin{aligned}
\frac{T_{N+1} - T_N}{\Delta x} + \frac{h}{k} T_N &= 0 \\
\Rightarrow T_{N+1} &= \left(1 - \frac{h \Delta x}{k}\right) T_N
\end{aligned}
$$

Solve the above *tridiagonal* system for $T_\infty = 30^o$ C, $T_0 = 300^o$ C, $D = 2$, $h/k = 2$, diameter $d = 0.02$ m and $L = 0.05$ m. Use 4-10 nodes and plot the temperature versus x for all cases.

Alternately, the FEM idea.

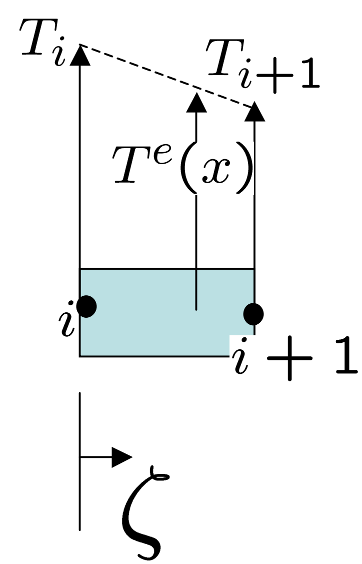


We can write

$$\mathcal{L} = \frac{d}{dT} + \frac{h}{k}$$

And approximate T in each element as:

$$\begin{aligned} T^1(x) &= N_1(\zeta)T_1 + N_2(\zeta)T_2 \\ T^2(x) &= N_2(\zeta)T_2 + N_2(\zeta)T_3 \\ \cdots & \quad \cdots \end{aligned}$$



Determine T_i on the basis of a *weak form*:

$$\sum_i \int_i^{(i+1)} w(\zeta)\mathcal{L}(T^e)d\zeta = 0$$

A Typical FE procedure

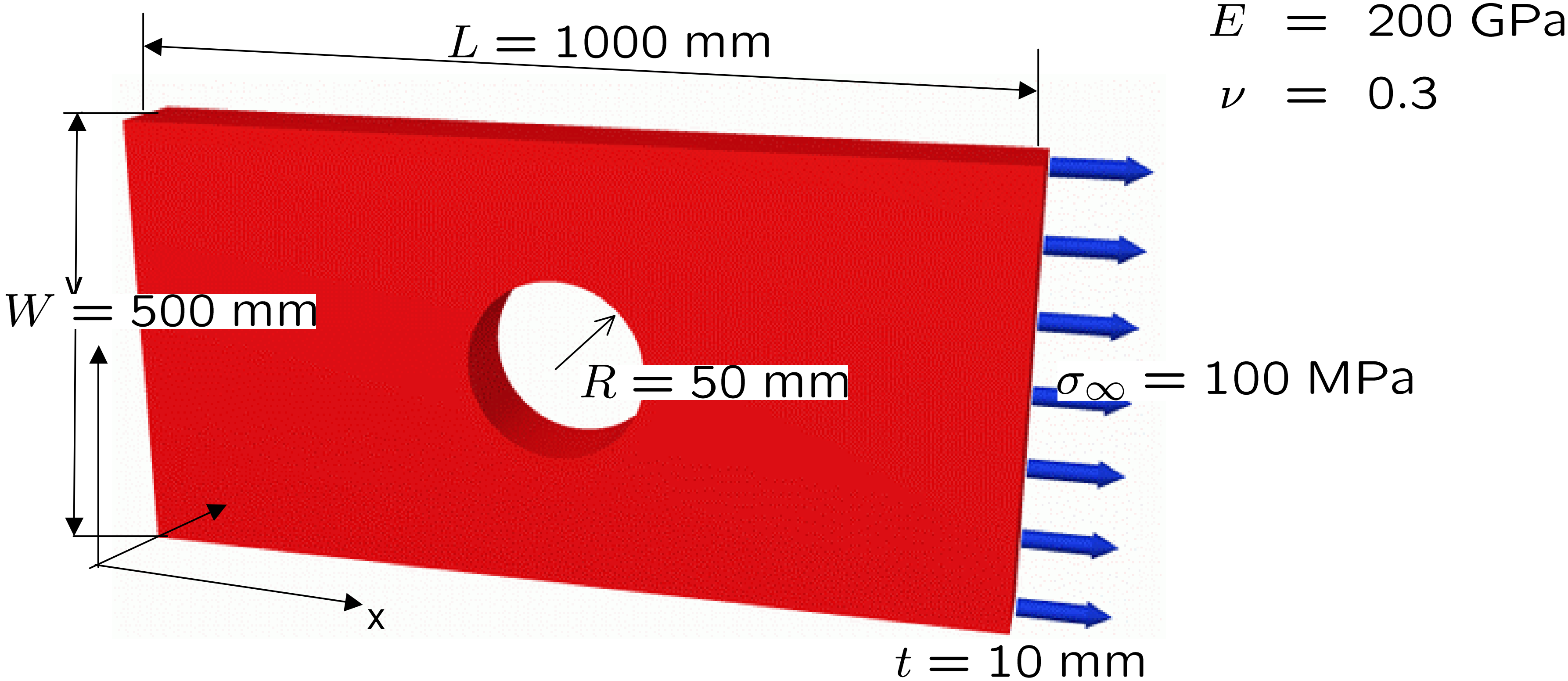


Plate with a hole.

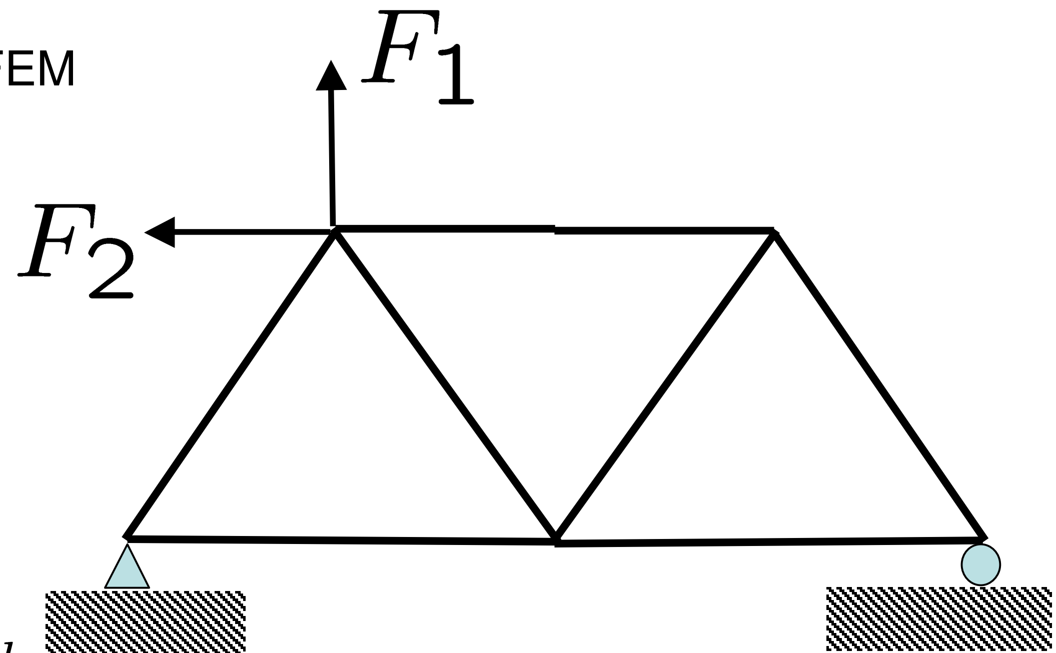
Step 1: Idealise

Plate thickness constant, loading is in the x-y plane ➔ Problem simplifies to 2-D

- **Step 2:** Create Geometry
- **Step 3**: Select Proper Elements
- **Step 4**: Mesh
- **Step 5**: Assign Material Properties
- **Step 6**: Apply boundary conditions
- **Step 7**: Solve
- **Step 8**: Visualise Results and post-process
- **Step 9:** Critically assess results

A simple example: another look at FEM

A 2-d truss with elements that can only withstand tension.



$$
\begin{aligned}
F &= \sigma A \\
&= EA\epsilon \\
&= EA\frac{l - l_0}{l_0} \\
&= k\delta
\end{aligned}
$$

Here,

$$k = \frac{AE}{l_0}$$

and

$$\delta = l - l_0.$$

Alternately, we can use the principle of minimum potential energy:

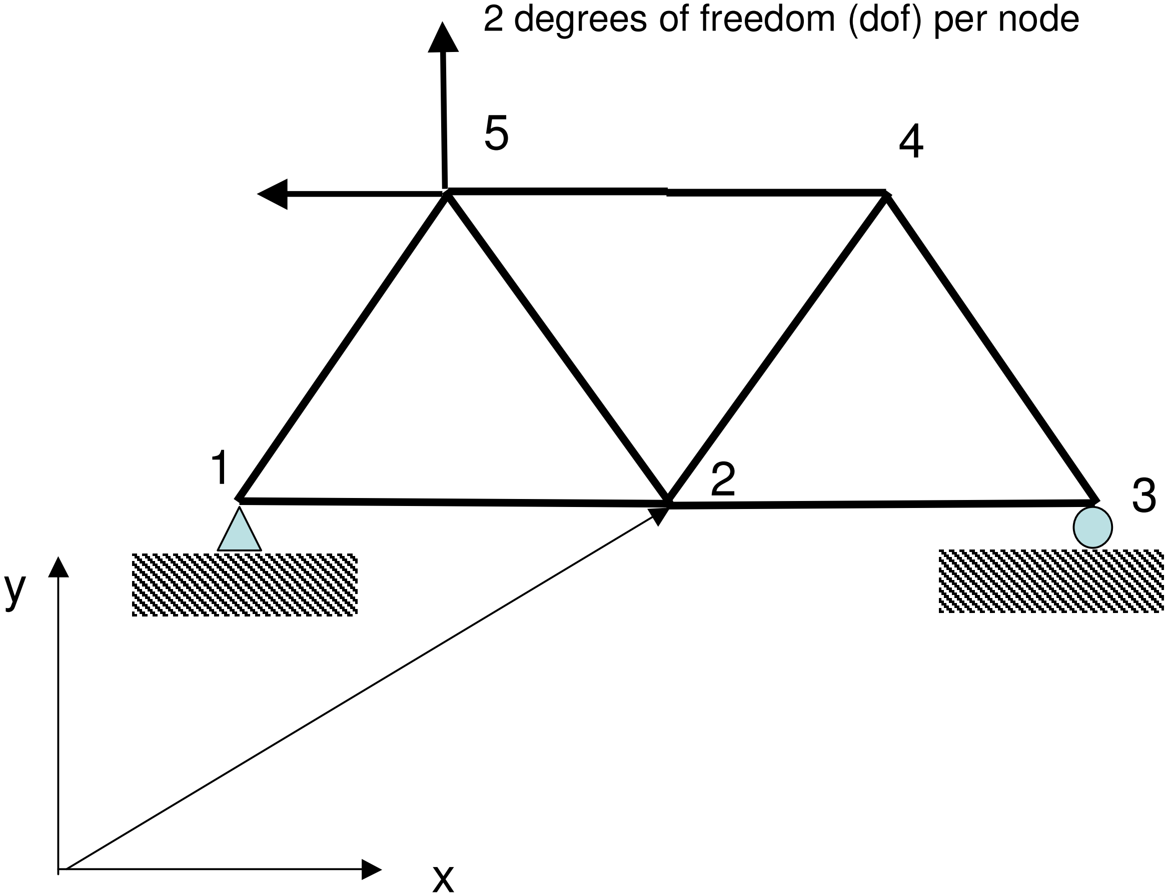
$$\frac{\partial}{\partial \delta}\Pi = 0$$

In case of a spring:

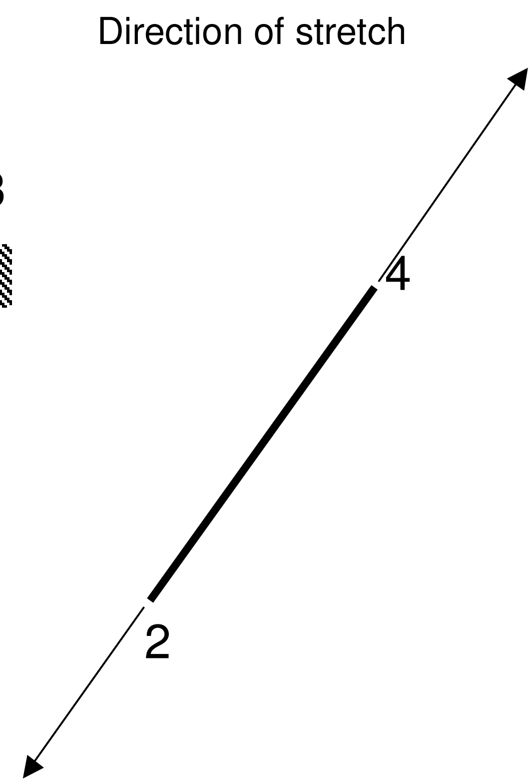
$$\frac{\partial}{\partial \delta}\Pi = \frac{\partial}{\partial \delta}\left(\frac{1}{2}k\delta^2 - F\delta\right) = 0$$

For a uniaxial bar

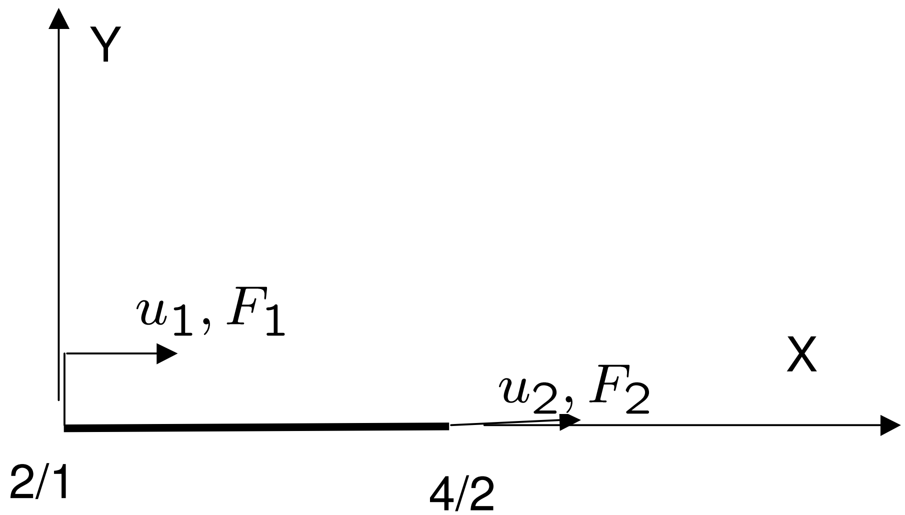
$$\Pi = \frac{1}{2}AE\left(\frac{\delta}{L}\right)^2 - F\delta$$

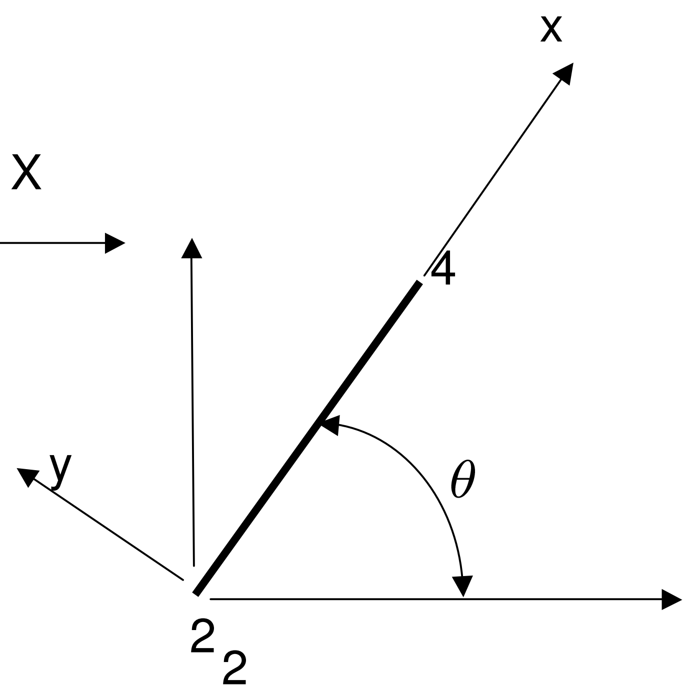


2 degrees of freedom (dof) per node
5
4
1
2
3
y
x



Direction of stretch
4
2





$$
\begin{aligned}
F_2 &= k(u_2 - u_1) \qquad (1)\\
F_1 &= k(-u_2 + u_1) \qquad (2)
\end{aligned}
$$

$$
\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{pmatrix} k & -k \\ -k & k \end{pmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}
$$

ndof : no. of dofs/node

nnod: no. of nodes/element

$$\begin{Bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \end{Bmatrix} = \begin{pmatrix} k & 0 & -k & 0 \\ 0 & 0 & 0 & 0 \\ -k & 0 & k & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

Local force vector (ndof X nnod,1)

Displacement vector (ndof X nnod, 1)

$$\boldsymbol{f} = \boldsymbol{K}_L \boldsymbol{u} \qquad (1)$$

Local stiffness matrix in local coordinate system (ndof X nnod, ndof X nnod)

$$u_2 = U_7 \cos\theta + U_8 \sin\theta \qquad (1)$$

$$v_2 = -U_7 \sin\theta + U_8 \cos\theta. \qquad (2)$$

$$\left\{ \begin{array}{c} u_2 \\ v_2 \end{array} \right\} = \left(\begin{array}{cc} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{array} \right) \left\{ \begin{array}{c} U_7 \\ U_8 \end{array} \right\}$$

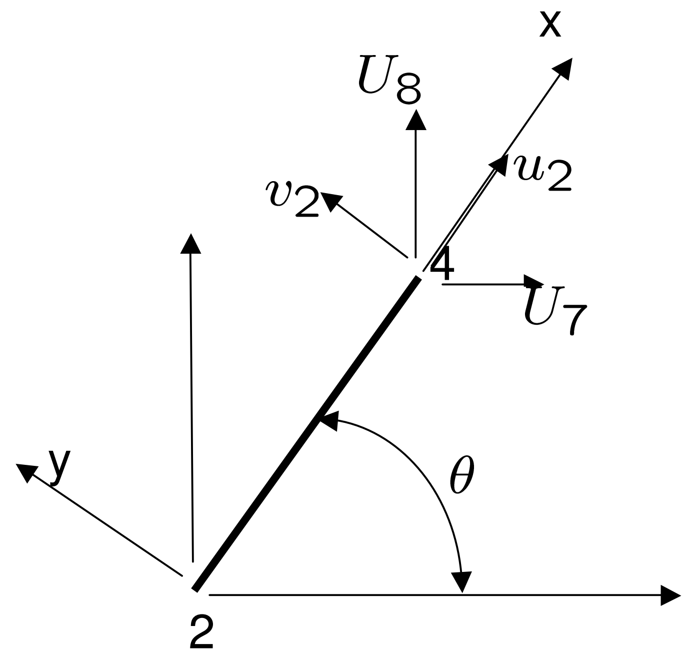
$$\left\{ \begin{array}{c} u_1 \\ v_1 \\ u_2 \\ v_2 \end{array} \right\} = \left(\begin{array}{cccc} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{array} \right) \left\{ \begin{array}{c} U_3 \\ U_4 \\ U_7 \\ U_8 \end{array} \right\}$$

$$\boldsymbol{u} = \boldsymbol{T}\boldsymbol{U}$$

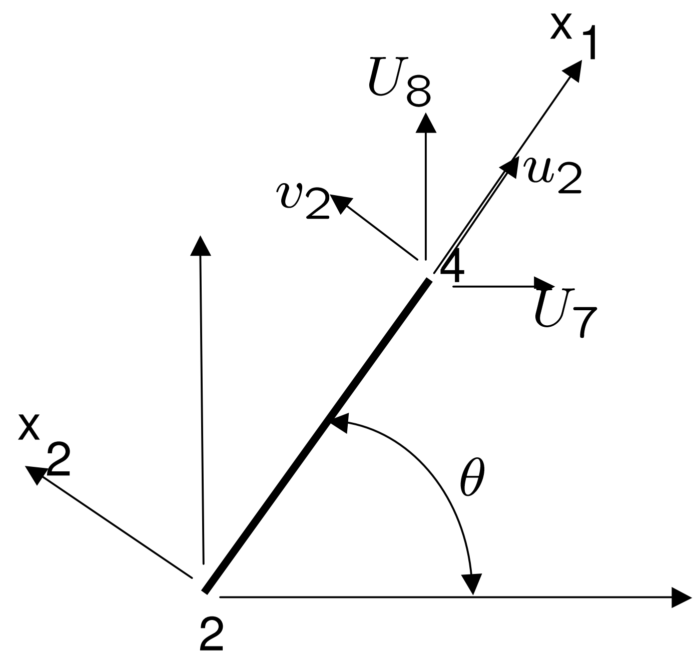
Similarly

$$\boldsymbol{f} = \boldsymbol{T}\boldsymbol{F}$$

Transformation matrix



$$\cos\theta_{ij} = \frac{\widehat{\boldsymbol{x}}_i \cdot \boldsymbol{e}_j}{|\widehat{\boldsymbol{x}}_i||\boldsymbol{e}_j|}$$



$$\boldsymbol{f} = \boldsymbol{K}_L \boldsymbol{u} \tag{1}$$

$$\boldsymbol{T}\boldsymbol{F} = \boldsymbol{K}_L \boldsymbol{T}\boldsymbol{U} \tag{2}$$

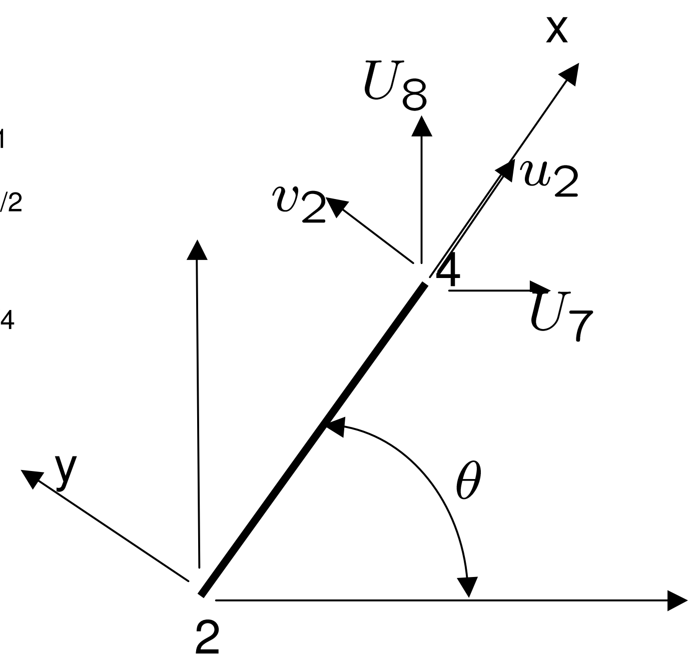
$$\boldsymbol{F} = \boldsymbol{T}^T \boldsymbol{K}_L \boldsymbol{T}\boldsymbol{U} \tag{3}$$

$$\boldsymbol{F} = \boldsymbol{K}\boldsymbol{U}$$

$$\boldsymbol{K} = \boldsymbol{T}^T \boldsymbol{K}_L \boldsymbol{T}$$

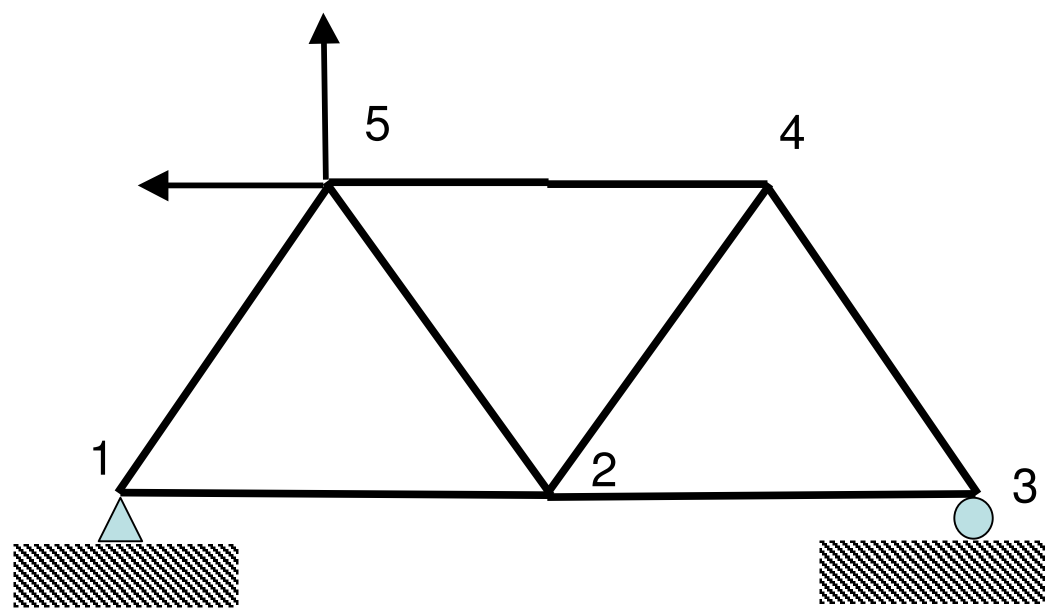
3/1 4/2 7/3 8/4

$$\begin{Bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \end{Bmatrix} = \boldsymbol{T}^T \begin{pmatrix} k & 0 & -k & 0 \\ 0 & 0 & 0 & 0 \\ -k & 0 & k & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \boldsymbol{T} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} \begin{matrix} \text{3/1} \\ \text{4/2} \\ \text{7/3} \\ \text{8/4} \end{matrix}$$

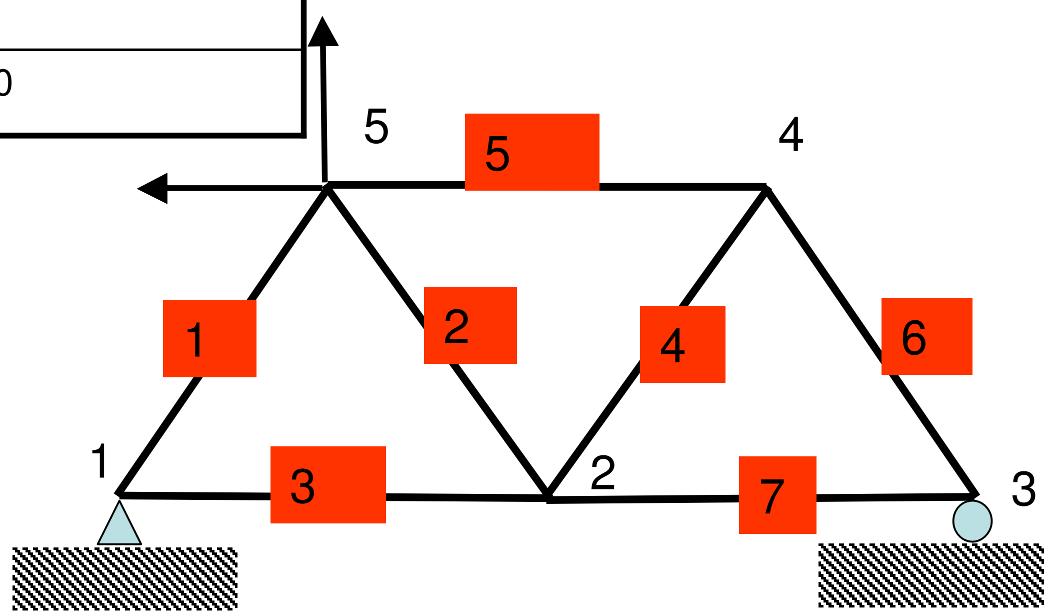


Eg: 2,2 → 4,4

3,1 → 7,3



Elem no	Local dof	Destination
1	1	1
	2	2
	3	9
	4	10
2	1	3
	2	4
	3	9
	4	10



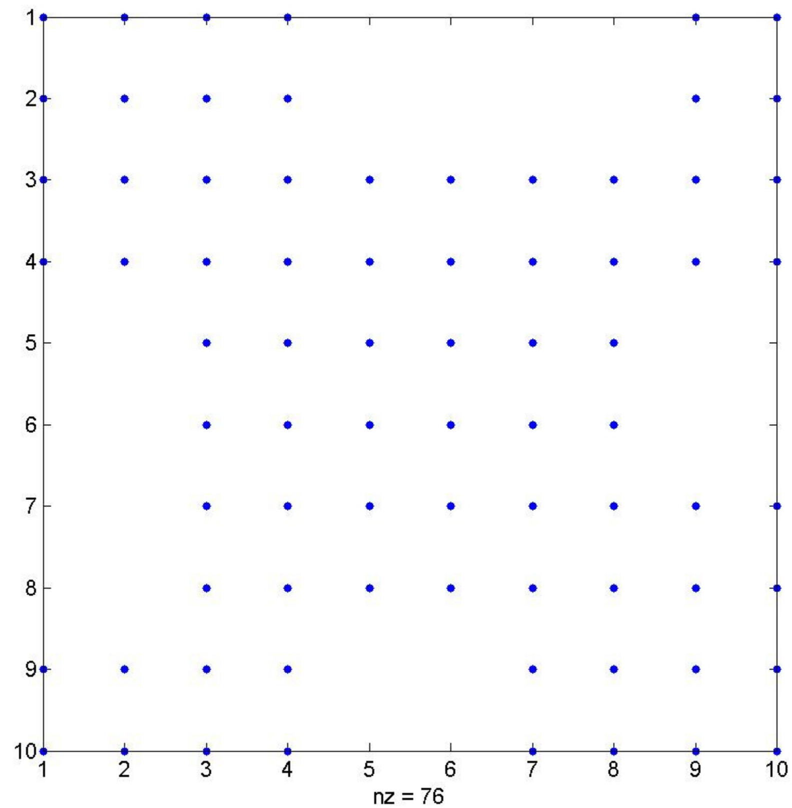
After assembling elements 1 and 2

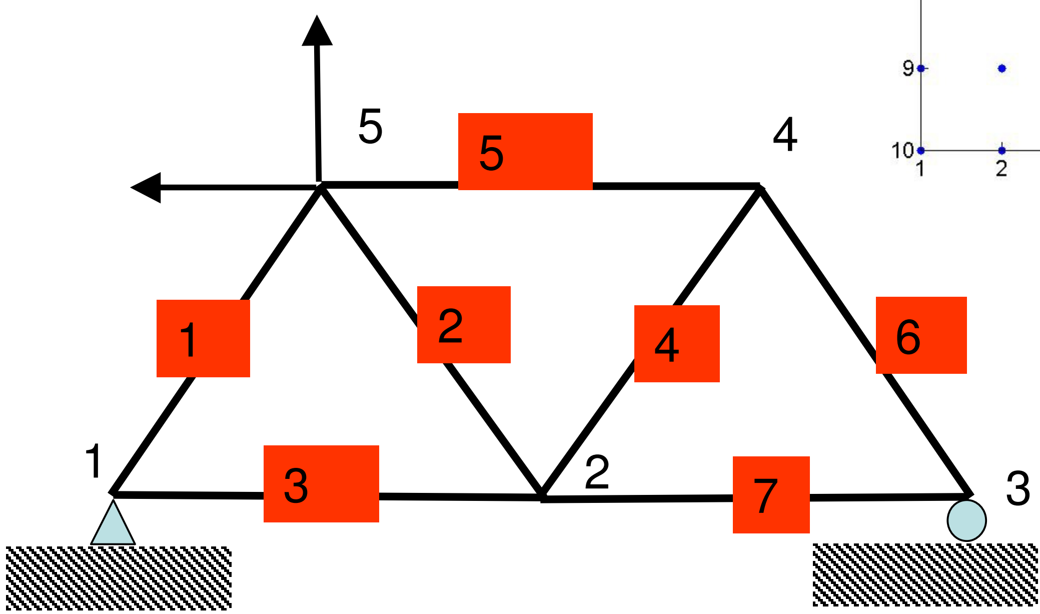
$$\begin{pmatrix} K^1_{11} & K^1_{12} & 0 & 0 & 0 & 0 & 0 & 0 & K^1_{13} & K^1_{14} \\ K^1_{21} & K^1_{22} & 0 & 0 & 0 & 0 & 0 & 0 & K^1_{23} & K^1_{24} \\ 0 & 0 & K^2_{11} & K^2_{12} & 0 & 0 & 0 & 0 & K^2_{13} & K^2_{14} \\ 0 & 0 & K^2_{21} & K^2_{22} & 0 & 0 & 0 & 0 & K^2_{23} & K^2_{24} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ K^1_{31} & K^1_{32} & K^2_{31} & K^2_{32} & 0 & 0 & 0 & 0 & K^1_{33} + K^2_{33} & K^1_{34} + K^2_{34} \\ K^1_{41} & K^1_{42} & K^2_{41} & K^2_{42} & 0 & 0 & 0 & 0 & K^1_{43} + K^2_{43} & K^1_{44} + K^2_{44} \end{pmatrix}$$

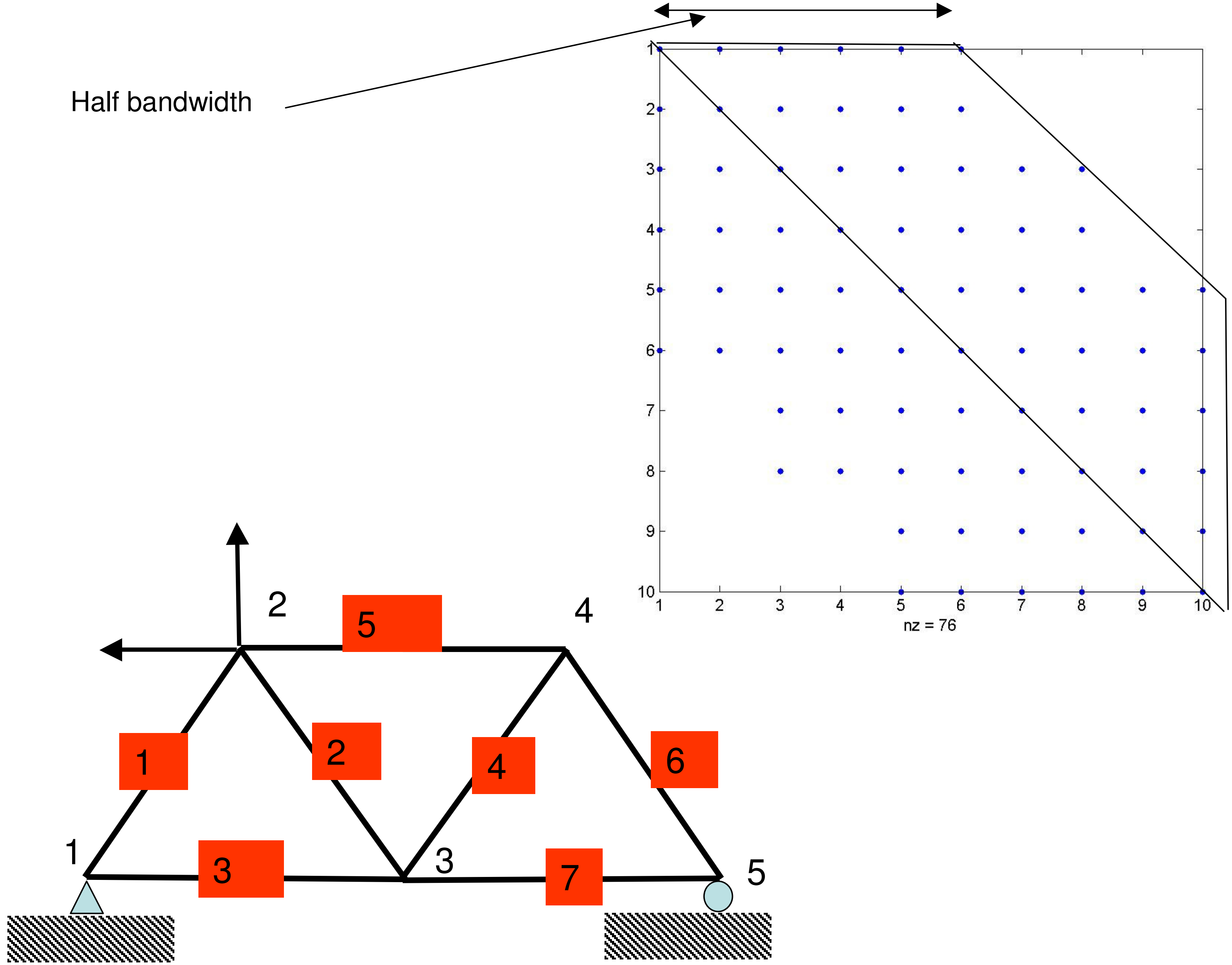
X+X	X+X	X	X					X	X
X+X	X+X	X	X					X	X
X	X	X+X+X	X+X+X			X	X	X	X
X	X	X+X+X	X+X+X			X	X	X	X
		X	X	X+X	X+X	X	X		
		X	X	X+X	X+X	X	X		
		X+X	X+X	X+X	X+X	X+X+X	X+X+X	X	X
		X+X	X+X	X+X	X+X	X+X+X	X+X+X	X	X
X	X	X	X			X	X	X+X+X	X+X+X
X	X	X	X			X	X	X+X+X	X+X+X

Stiffness matrix is symmetric, diagonally dominant, sparse and banded.

Sparsity pattern






Half bandwidth
1 2 3 4 5 6 7 8 9 10
nz = 76
1
2
3
4
5
1
2
3
4
5
6
7

Matlab function 1: Reading data from an input file

To be judged on generality and correctness

Read in nodal coordinates and element connectivities

Coordinates can be in 2d or 3d

Elements can have at most 20 nodes attached to them

Data will be written in comma separated form

User may want to read in either both coordinates and connectivity or either.

coord	
1,3.25,4.32,8.91	nodenum, x,y,z
2,6.56,7.11,11.32	
…..	
conn	
1,1,1,3,52,65	elemnum,matnum,n1,n2,n3,….
2,2,6,9,8,4	
….	

function [X,icon,nelm,nnode,nperelem,neltype, nmat,ndof] = ass1_group*n* (fname)

% read data from a file *fname*

Description of variables

X: global coordinates of the nodes, x, y, and z

icon: element connectivities

nelm: total number of elements

nnode: total number of nodes

nperelem: number of nodes per element for all elements

neltype: element type for all elements

nmat: material type for all elements

ndof: number of degrees of freedom/node for all elements

Submit by 13/1/2006

function [destination] = ass2_group*n* (X,icon,nelm,nnode,nperelem,neltype,ndof)

% create a destination array for the mesh data read in

Description of variables

destination(1:nelem,:): contains destination in the global stiffness matrix of all local degrees of freedom in an element

Submit by 18/1/2006

To be judged on correctness and speed

$$\boldsymbol{F} = \boldsymbol{K}\boldsymbol{U}$$

Global displacement vector

Global force vector

Global stiffness matrix

Notes:

Global stiffness matrix is singular i.e. it has zero eigenvalues

Hence it cannot be inverted!

Boundary conditions

Force specified: eg. dof 9 and 10 in our example

Displacement specified: eg. dof 1,2,and 6 in our example

Both forces and displacements cannot be specified at the same dof.

$$\boldsymbol{F} = \left\{ \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ F_1 \\ F_2 \end{array} \right\} \qquad \boldsymbol{U} = \left\{ \begin{array}{c} 0 \\ 0 \\ U_3 \\ U_4 \\ U_5 \\ 0 \\ U_7 \\ U_8 \\ U_9 \\ U_{10} \end{array} \right\}$$

Naïve approach for imposing displacement boundary conditions

$$\begin{pmatrix} L & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} & K_{17} & K_{18} & K_{19} & K_{1,10} \\ K_{21} & L & K_{23} & K_{24} & K_{25} & K_{26} & K_{27} & K_{28} & K_{29} & K_{2,10} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} & K_{37} & K_{38} & K_{39} & K_{3,10} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} & K_{47} & K_{48} & K_{49} & K_{4,10} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} & K_{57} & K_{58} & K_{59} & K_{5,10} \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & L & K_{67} & K_{68} & K_{69} & K_{6,10} \\ K_{71} & K_{72} & K_{73} & K_{74} & K_{75} & K_{76} & K_{77} & K_{78} & K_{79} & K_{7,10} \\ K_{81} & K_{82} & K_{83} & K_{84} & K_{85} & K_{86} & K_{87} & K_{88} & K_{89} & K_{8,10} \\ K_{91} & K_{92} & K_{93} & K_{94} & K_{95} & K_{96} & K_{97} & K_{98} & K_{99} & K_{9,10} \\ K_{10,1} & K_{10,2} & K_{10,3} & K_{10,4} & K_{10,5} & K_{10,6} & K_{10,7} & K_{10,8} & K_{10,9} & K_{10,10} \end{pmatrix}$$

L=a very large number. Also replace the corresponding dofs in the rhs vector by LXspecified displacement value

$$U_1 = L\delta_1 - \frac{K_{12}U_2 + \ldots K_{1,10}U_{10}}{L} \simeq \delta_1$$

The “proper” way of imposing displacement constraints

$$\begin{aligned} a_1x + b_1y + c_1z &= f_1 \\ a_2x + b_2y + c_2z &= f_2 \\ a_3x + b_3y + c_3z &= f_3 \end{aligned}$$

Suppose $y = \delta$ (known).

$$a_1x + c_1z = f_1 - b_1\delta$$

$$a_3x + c_3z = f_3 - b_3\delta$$

$$\begin{pmatrix} a_1 & 0 & c_1 \\ 0 & 1 & 0 \\ a_3 & 0 & c_3 \end{pmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{Bmatrix} f_1 \\ \delta \\ f_3 \end{Bmatrix} - \delta \begin{Bmatrix} b_1 \\ 0 \\ b_3 \end{Bmatrix}$$

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \end{pmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ \vdots \\ U_6 \\ \vdots \end{Bmatrix} = \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_6 \\ \vdots \end{Bmatrix} - \delta_1 \begin{Bmatrix} 0 \\ 0 \\ K_{15} \\ 0 \\ \vdots \end{Bmatrix} - \delta_2 \begin{Bmatrix} 0 \\ \vdots \\ K_{25} \\ 0 \\ \vdots \end{Bmatrix} - \delta_6 \begin{Bmatrix} 0 \\ \vdots \\ K_{65} \\ 0 \\ \vdots \end{Bmatrix}$$

Suppose dof k is specified

Traspose negative of the specified value X kth column to the right

Replace k th row and columns in the stiffness matrix to zero

Replace K(k,k) by 1

Set F(k)=specified value

Repeat above steps for all specified dofs.

Assignment 3: form local stiffness matrix for a truss element e oriented at an arbitrary angle to the global x-axis

function[stiff_loc_truss]=ass3_groupn(X,icon,e,spring_constant)

% programme to calculate stiffness matrix of a 2-noded truss element in the global X-Y system

Form stiffness in local coordinates

Find transformation matrix

Find stiff_loc_truss in global coordinates

To be judged on correctness only

Add to your data file reading programme: ass1_groupn.m

```
coor
1,2,0.0,0.0
2,2,1.0,0.0
3,2,1.0,1.0
4,2,0.0,1.0
5,2,0.0,2.0
6,2,1.0,2.0
7,2,2.0,1.0
conn
1,1,1,2,3,4
2,1,4,3,6,5
3,2,3,7
4,2,2,7
boun
1,1,0.0
1,2,0.0
2,2,0.0
4,1,0.0
5,1,0.0
force
5,2,1.0
6,2,1.0
7,2,3.0
```

Inputting specified dofs

Keyword: ‘bound’

Format: node no, dof no, value

Array: idisp(1:nnode,maxdof), specdisp(1:nnode,maxdof)

To be judged on correctness only

Inputting specified forces

Keyword: ‘force’

Format: node no, dof no., value

Array: iforce(1:nnode,maxdof), specforce(1:nnode,maxdof)

Assignment 5: Modify ass1_groupn.m accordingly
function
[X,icon,nelm,nnode,nperelem,neltype,nmat,ndof,idisp,specdisp,iforce,specforce] =
ass1_group*n* (fname)
% read data from a file *fname*

Assignment 6: Assemble stiffness matrix for element e

function[stiffness_dummy] = ass4_groupn(icon,destination,stiff_loc_truss,e)

% programme to assemble local stiffness matrix of element e onto the global stiffness

Add stiffness of e to the global stiffness

To be judged on correctness and speed

Assignment 7: the main programme

```
[X,icon,nelm,nnode,nperelem,neltype,nmat,ndof,idisp,specdisp,iforce,specforce] =
ass1_groupn (fname);
destination = ass2_groupn (X,icon,nelm,nnode,nperelem,neltype,ndof);
spring_constant=1;
for e=1:nelm
    stiff_loc_truss=ass3_groupn(X,icon,e,spring_constant);
    stiffness_dummy = ass4_groupn(icon,destination,stiff_loc_truss,e);
    stiffness_global=stiffness_global+stiffness_dummy;
end
```

To be judged on correctness only

To be judged on correctness and speed

Assignment 8:

function[modified_siffness_matrix, modified_rhs]=
ass6_groupn(icon,nelm,nnode,nperelem,ndof,idisp,specdisp,iforce,specforce,stiffmat)

% function to modify stiffness matrix and rhs vector according to specified forces and

% dofs

for i=1,no. of nodes

for j=1,number of dofs for node i

modify stiffness matrix and rhs vector to accommodate the specified values of the dofs and forces.

end

end

Assignments 2-8: due on February 6, 2006.

Solving the equations

$$\boldsymbol{F} = \boldsymbol{K}\boldsymbol{U}$$

$$\boldsymbol{U} = \boldsymbol{K}^{-1}\boldsymbol{F}$$

K: requires huge storage, largest component of a FE code

Strategies:

Use sparsity: K_sparse=sparse(K);

Matlab command U=inv(K_sparse)*F;

Direct methods: Gauss elimination

$$\begin{pmatrix} 18 & -6 & -6 & 0 \\ -6 & 12 & 0 & -6 \\ -6 & 0 & 12 & -6 \\ 0 & -6 & -6 & 12 \end{pmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} 60 \\ 0 \\ 20 \\ 0 \end{Bmatrix}$$

Original problem

$$\begin{pmatrix} 18 & -6 & -6 & 0 \\ 0 & 10 & -2 & -6 \\ 0 & -2 & 10 & -6 \\ 0 & -6 & -6 & 12 \end{pmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} 60 \\ 20 \\ 40 \\ 0 \end{Bmatrix}$$

1st elimination:

Row2- Row 1*(-6/18)

Row3-Row1*(-6/18)

$$\begin{pmatrix} 18 & -6 & -6 & 0 \\ 0 & 10 & -2 & -6 \\ 0 & 0 & 9.6 & -7.2 \\ 0 & 0 & -7.2 & 8.4 \end{pmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} 60 \\ 20 \\ 44 \\ 12 \end{Bmatrix}$$

2nd elimination

Row3-Row2*(-2/10)

Row4-Row2*(-6/10)

$$\begin{pmatrix} 18 & -6 & -6 & 0 \\ 0 & 10 & -2 & -6 \\ 0 & 0 & 9.6 & -7.2 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} 60 \\ 20 \\ 44 \\ 45 \end{Bmatrix}$$

3rd elimination:

Row4-Row3*(-7.2/9.6)

Back substitution:

$$
\begin{aligned}
U_4 &= 45/3 \\
U_3 &= (44 + 7.2U_4)/9.6 = 15.83 \\
U_2 &= (20 + 2U_3 + 6U_4)/10 = 14.17 \\
U_1 &= (60 + 6U_2 + 6U_3)/18 = 13.33
\end{aligned}
$$

Suggestion:

To clearly get a feel of how Gauss elimination works, try and write this function in Matlab:

```
function[U] = gauss_elimination(K,F)
```

Where U=inv(K)*F;

Can you utilise the diagonality and sparsity of K to speed up the solution?

L U Decomposition

$$A = \begin{pmatrix} 18 & -6 & -6 & 0 \\ -6 & 12 & 0 & -6 \\ -6 & 0 & 12 & -6 \\ 0 & -6 & -6 & 12 \end{pmatrix}$$

We can generate a lower triangular matrix

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1/3 & 1 & 0 & 0 \\ -1/3 & 1/5 & 1 & 0 \\ 0 & -3/5 & -3/4 & 0 \end{pmatrix}$$

just like we generated the upper triangular one

$$L = \begin{pmatrix} 18 & -6 & -6 & 0 \\ 0 & 10 & -2 & -6 \\ 0 & 0 & 9.6 & -7.2 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

And can show that

$$A = LU$$

$$
\begin{aligned}
\boldsymbol{Ax} &= \boldsymbol{b} \\
\boldsymbol{LUx} &= \boldsymbol{b}
\end{aligned}
$$

First solve

$$\boldsymbol{Ly} = \boldsymbol{b}$$

where

$$y_i = \frac{1}{l_{ii}} \left[b_i - \sum_{j=1}^{i-1} l_{ij} y_j \right]$$

Then solve

$$\boldsymbol{Ux} = \boldsymbol{y}$$

so that by usual back-substitution

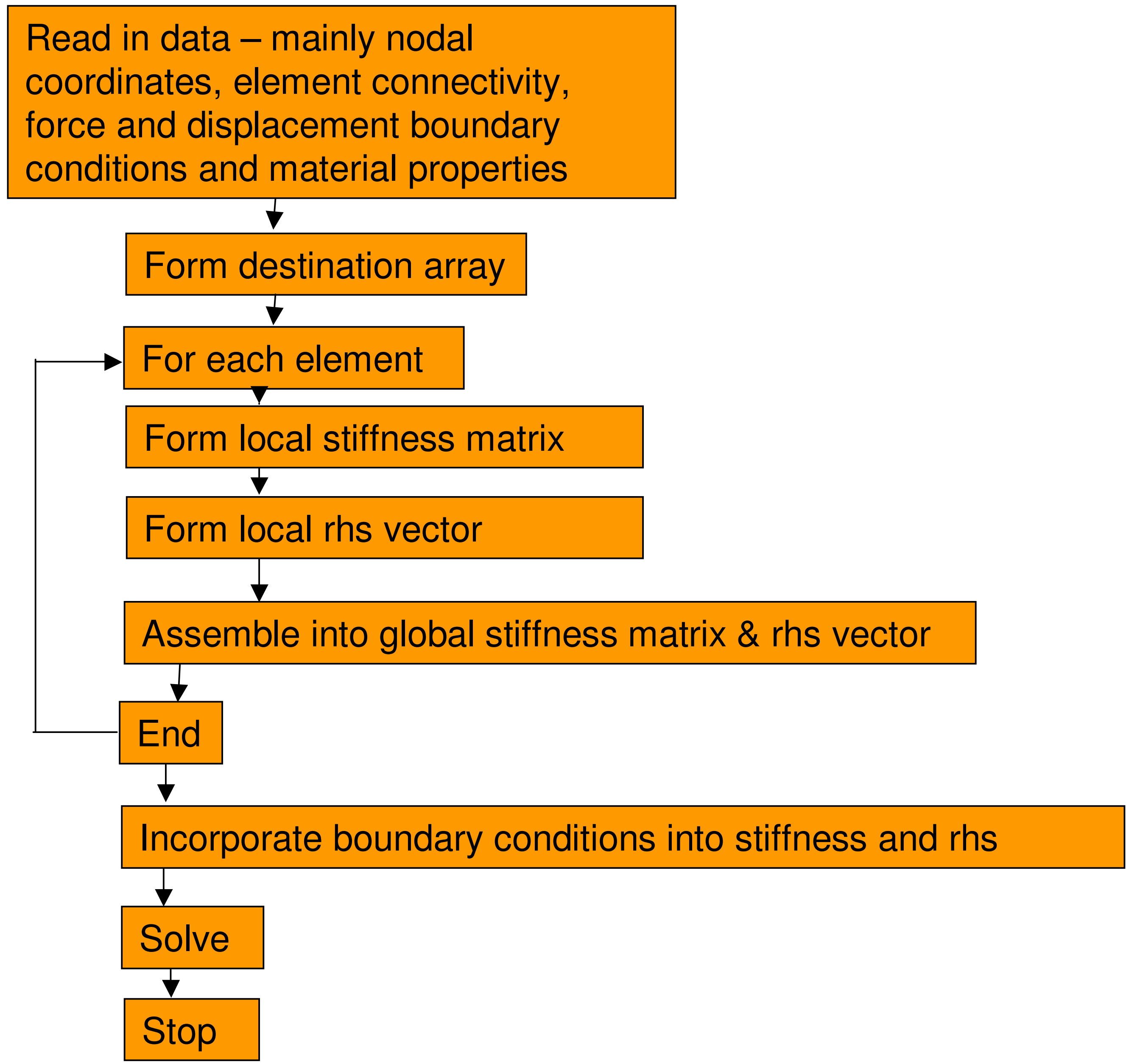
$$x_i = \frac{1}{u_{ii}} \left[y_i - \sum_{j=i+1}^{N} u_{ij} x_j \right]$$

Regular Gauss elimination (or LU decomposition) takes $\sim N^3$ operations, while solving solving $\boldsymbol{LUx} = \boldsymbol{b}$ takes $\sim N^2$ operations

How does the solution procedure work?

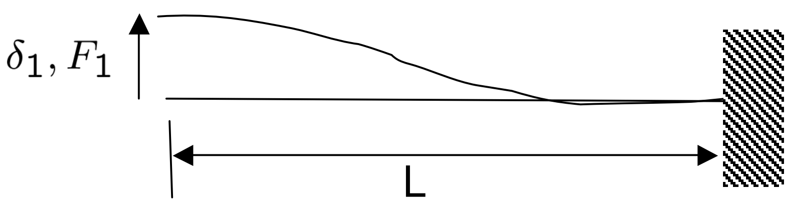
1. Multiply first row of $\boldsymbol{K}$ by K_{21}/K_{11}
2. Subtract first row from the second
3. Similarly subtract $(K_{21}/K_{11})F_1$ from F_2
4. Repeat procedures with K_{i1}/K_{11}. U_1 is eliminated!
5. Repeat above with the factor K_{mn}/K_{nn} for row n with $m \in n+1, N$
6. Backsubstitute

The FEM scheme of things



Stiffness matrix from basics!

$$\theta_1 = \theta_2 = \delta_2 = 0$$



$$\begin{aligned} EI\frac{d^4w}{dx^4} &= p(x) = 0 \\ w &= \frac{1}{6}C_1x^3 + \frac{1}{2}C_2x^2 + C_3x + C_4 \end{aligned}$$

At $x = 0$ $\quad \frac{dw}{dx} = 0 \Rightarrow C_3 = 0$

At $x = L$ $\quad \frac{dw}{dx} = 0, w = 0 \Rightarrow$

$$C_1L = -2C_2, C_4 = -\frac{1}{6}C_2L^2$$

$$\begin{aligned} \frac{d^2w}{dx^2} &= \frac{M(x)}{EI} \\ \frac{dM}{dx} &= V \\ \Rightarrow \qquad & EI\frac{d^3w}{dx^3} = V. \end{aligned} \tag{1}$$

At $x = 0$

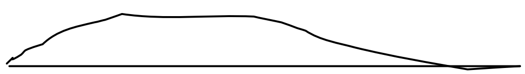
$$\frac{d^3w}{dx^3} = \frac{F_1}{EI} \Rightarrow C_1 = \frac{F_1}{EI}$$

Thus,

$$K_{11} = \frac{F_1}{\delta_1} = \frac{F_1}{w(x=0)} = \frac{12EI}{L^3}$$

Intuitive but not easy

$$\boldsymbol{K}_L = \frac{EI}{L}\begin{pmatrix} 12/L^2 & -6/L & -12/L^2 & -6/L \\ & 4 & 6/L & 2 \\ & & 12/L^2 & 6/L \\ & & & 4 \end{pmatrix}$$



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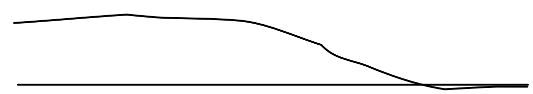

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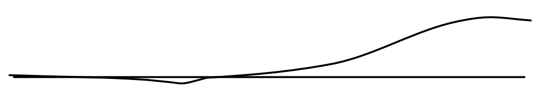

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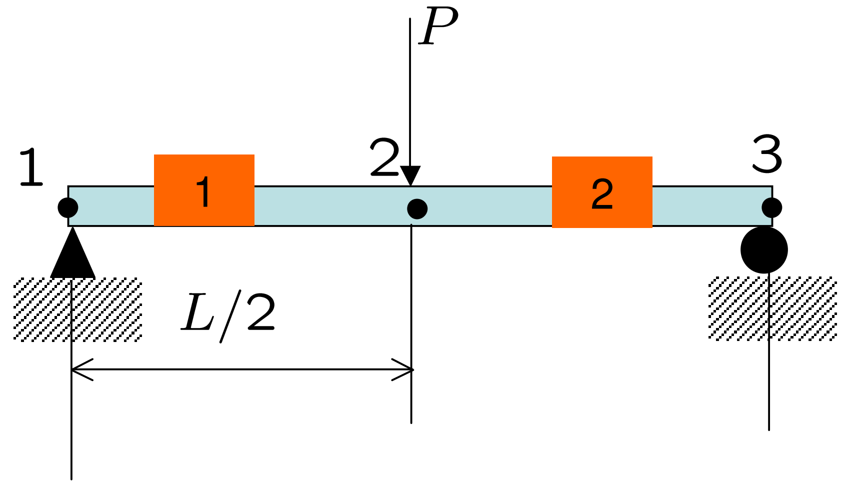


For each element

$$\boldsymbol{K}_L^{1/2} = \frac{8EI}{L^3} \begin{pmatrix} 12 & 6(L/2) & -12 & 6(L/2) \\ 6L/2 & 4(L/2)^2 & -6(L/2) & 2(L/2)^2 \\ -12 & -6(L/2) & 12 & -6(L/2) \\ 6(L/2) & 2(L/2)^2 & -6(L/2) & 4(L/2)^2 \end{pmatrix}$$

The assembled global stiffness

$$\boldsymbol{K}_G = \frac{8EI}{L^3}\begin{pmatrix} 12 & 6(L/2) & -12 & 6(L/2) & 0 & 0 \\ 6L/2 & 4(L/2)^2 & -6(L/2) & 2(L/2)^2 & 0 & 0 \\ -12 & -6(L/2) & 12+12 & -6(L/2)-6(L/2) & -12 & 6(L/2) \\ 6(L/2) & 2(L/2)^2 & -6(L/2)+6(L/2) & 4(L/2)^2+4(L/2)^2 & -6(L/2) & 2(L/2)^2 \\ 0 & 0 & -12 & -6(L/2) & 12 & -6(L/2) \\ 0 & 0 & 6(L/2) & 2(L/2)^2 & -6(L/2) & 4(L/2)^2 \end{pmatrix}$$

The global force vector

$$\boldsymbol{F} = \begin{Bmatrix} 0 \\ 0 \\ -P \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

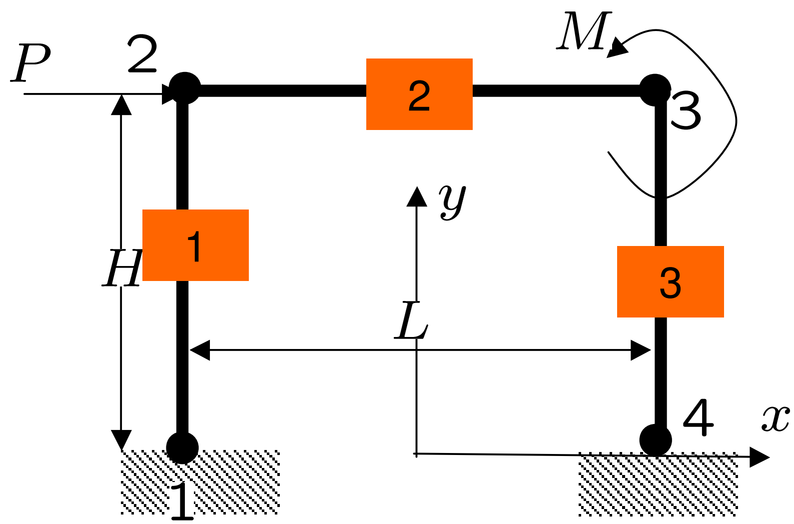
The global displacement vector

$$\boldsymbol{F} = \begin{Bmatrix} w_1 = 0 \\ \theta_1 \\ w_2 \\ \theta_2 \\ w_3 = 0 \\ \theta_3 \end{Bmatrix}$$

Finally we solve

$$\frac{8EI}{L^3}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4(L/2)^2 & -6(L/2) & 2(L/2)^2 & 0 & 0 \\ 0 & -6(L/2) & 12+12 & -6(L/2)-6(L/2) & 0 & 6(L/2) \\ 0 & 2(L/2)^2 & -6(L/2)+6(L/2) & 4(L/2)^2+4(L/2)^2 & 0 & 2(L/2)^2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 6(L/2) & 2(L/2)^2 & 0 & 4(L/2)^2 \end{pmatrix}\begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \\ w_3 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -P \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

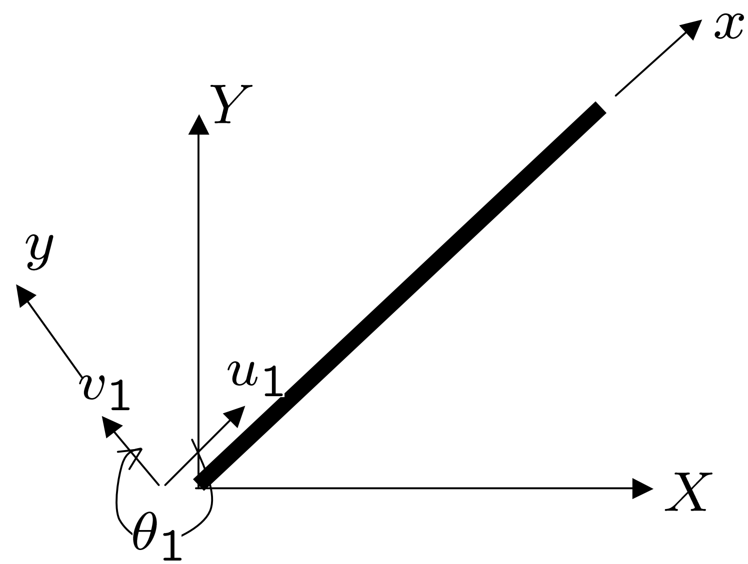
Beams in 2-d



The local stiffness for any element (say 2)

$$\boldsymbol{K}_L^2 = \begin{pmatrix} AE/L & 0 & 0 & -AE/L & 0 & 0 \\ 0 & 12EI/L^3 & 6EI/L^2 & 0 & -12EI/L^3 & 6EI/L^2 \\ 0 & 6EI/L^2 & 4EI/L & 0 & -6EI/L^2 & 2EI/L \\ -AE/L & 0 & 0 & AE/L & 0 & 0 \\ 0 & -12EI/L^3 & -6EI/L^2 & 0 & 12EI/L^3 & -6EI/L^2 \\ 0 & -6EI/L^2 & -4EI/L & 0 & 6EI/L^2 & -2EI/L \end{pmatrix}$$

Transformation Matrix for a 2-d beam element

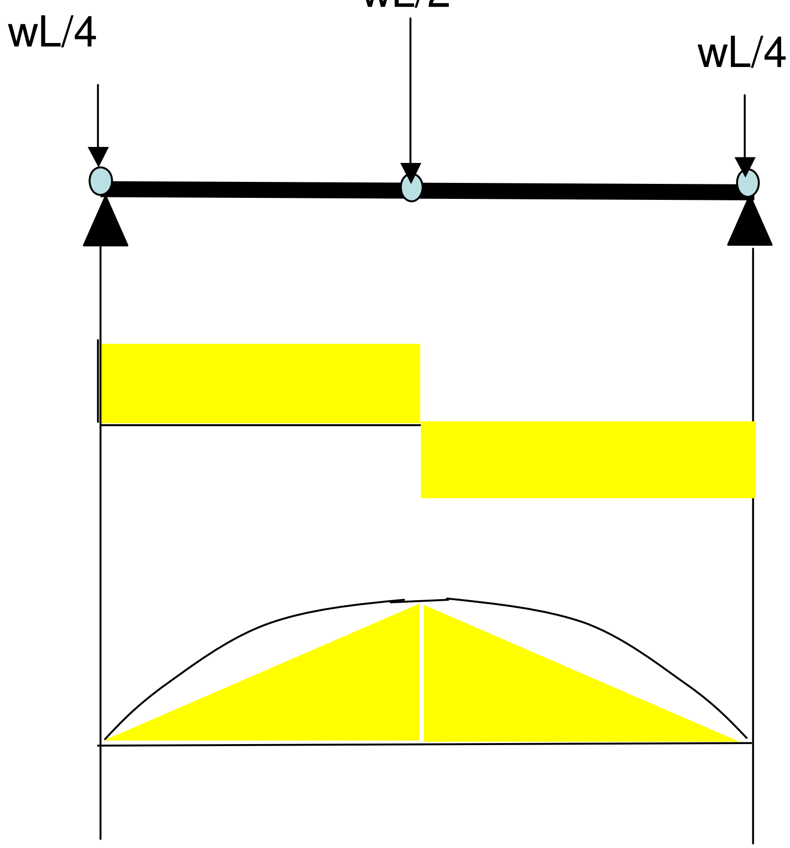


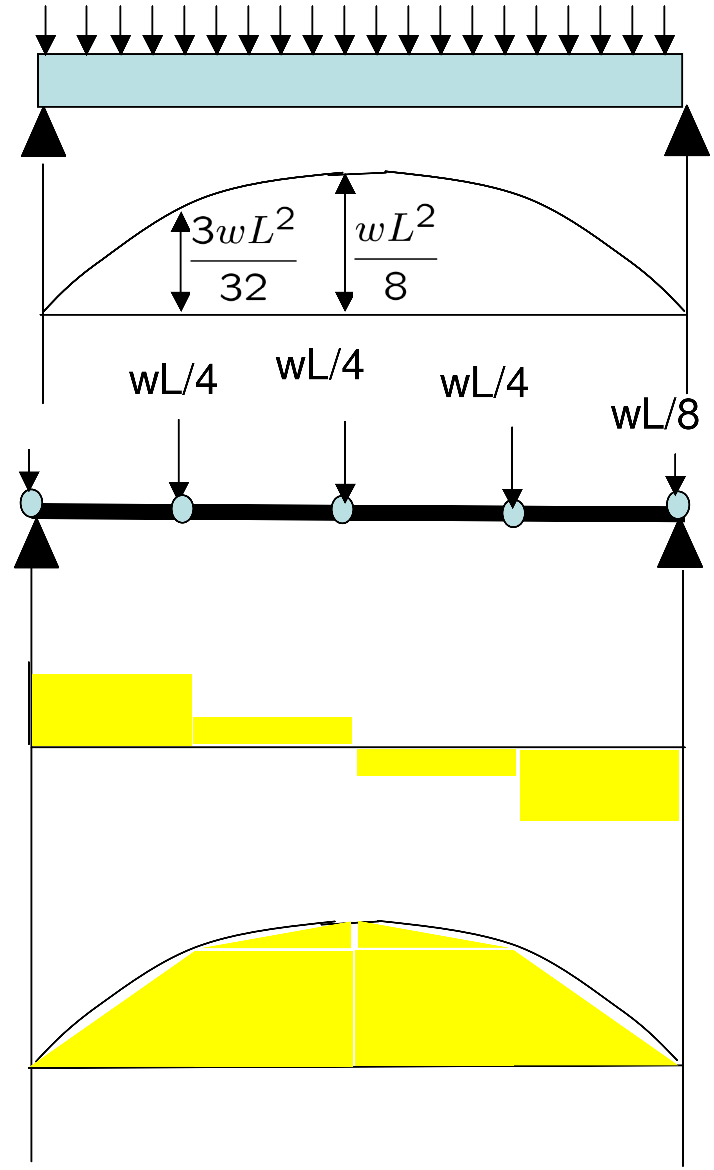
$$\boldsymbol{u}_1 = \left\{ \begin{array}{c} u_1 \\ v_1 \\ \theta_1 \end{array} \right\} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \left\{ \begin{array}{c} U_1 \\ V_1 \\ \Theta_1 \end{array} \right\} = \boldsymbol{t}\boldsymbol{U}$$

$$\boldsymbol{u} = \begin{pmatrix} \boldsymbol{t} & 0 \\ 0 & \boldsymbol{t} \end{pmatrix} \boldsymbol{U} = \boldsymbol{T}\boldsymbol{U}$$

$$\boldsymbol{K}_G = \boldsymbol{T}^T \boldsymbol{K}_L \boldsymbol{T}$$

How do we solve a beam problem with distributed loads?





Small strain, linear elasticity.

Generalised Hooke's law

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl} + \sigma_{ij}^{0}$$

In case of static equilibrium, the stress tensor satisfies

$$\begin{aligned} \frac{\partial \sigma_{ij}}{\partial x_j} + b_i &= 0 \text{ From balance of Linear momentum} \\ \sigma_{ij} &= \sigma_{ji} \text{ From balance of angular momentum} \end{aligned}$$

Using x, y, z, the independent components of stress are $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{yz}, \sigma_{zx}$

C_{ijkl} has the following symmetries:

$$\begin{aligned} C_{ijkl} &= C_{jikl} \\ C_{ijkl} &= C_{ijlk} \\ C_{ijkl} &= C_{klij} \end{aligned}$$

Number of independent elastic constants = 21

Number of elastic constants when there exists one plane of symmetry = 13

Number of elastic constants when there exists three mutually perpendicular planes of symmetry = 9

Number of independent elastic constants for an Isotropic material = 2

$$\left\{ \begin{array}{c} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{array} \right\} = \begin{pmatrix} C_{xxxx} & C_{xxyy} & C_{xxzz} & C_{xxxy} & C_{xxyz} & C_{xxzx} \\ & C_{yyyy} & C_{yyzz} & C_{yyxy} & C_{yyyz} & C_{yyzx} \\ & & C_{zzzz} & C_{zzxy} & C_{zzyz} & C_{zzzx} \\ & & & C_{xyxy} & C_{xyyz} & C_{xyzx} \\ & & & & C_{yzyz} & C_{yzzx} \\ & & & & & C_{zxzx} \end{pmatrix} \left\{ \begin{array}{c} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ 2\epsilon_{xy} \\ 2\epsilon_{yz} \\ 2\epsilon_{zx} \end{array} \right\}$$

$$\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\epsilon}$$

For an isotropic elastic material

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

$\lambda = \nu E/(1+\nu)(1-2\nu)$ and $\mu = E/2(1+\nu)$

$$
\begin{aligned}
C_{xxxx} &= C_{yyyy} = C_{zzzz} = \lambda + 2\mu \\
C_{xxyy} &= C_{yyzz} = C_{zzxx} = \lambda \\
C_{xyxy} &= C_{yzyz} = C_{zxzx} = \mu
\end{aligned}
$$

Thus

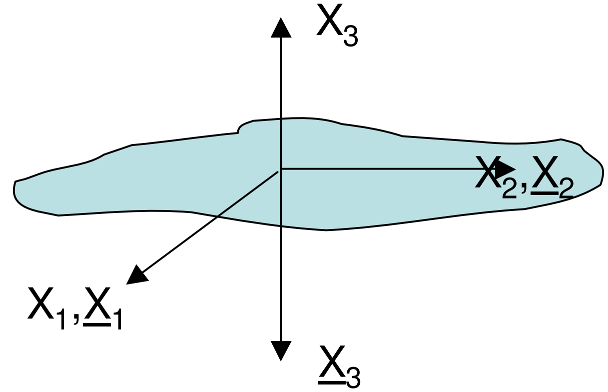
$$
\boldsymbol{C} = \begin{pmatrix}
\lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\
\lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\
\lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\
0 & 0 & 0 & \mu & 0 & 0 \\
0 & 0 & 0 & 0 & \mu & 0 \\
0 & 0 & 0 & 0 & 0 & \mu
\end{pmatrix}
$$

Tensor transformation

$$\bar{\sigma}_{ij} = l_{ik} l_{jl} \sigma_{kl}$$

where $l_{ij} = \cos(x_i, \bar{x}_j)$
Alternately,

$$\bar{\boldsymbol{\sigma}} = \boldsymbol{L} \boldsymbol{\sigma} \boldsymbol{L}^T$$



For one plane of symmetry $(x_1 - x_2)$

$$\begin{aligned}
\bar{\sigma}_{23} &= -\sigma_{23} \\
\bar{\sigma}_{31} &= -\sigma_{31} \\
\bar{\epsilon}_{23} &= -\epsilon_{23} \\
\bar{\epsilon}_{31} &= -\epsilon_{31}
\end{aligned}$$

$\bar{\sigma}_{11} = \sigma_{11} \Rightarrow$

$$\sigma_{11} = C_{1111}\epsilon_{11} + C_{1122}\epsilon_{22} + C_{1133}\epsilon_{33} + 2C_{1112}\epsilon_{12} - 2C_{1123}\epsilon_{23} - 2C_{1131}\epsilon_{31}$$

But

$$\sigma_{11} = C_{1111}\epsilon_{11} + C_{1122}\epsilon_{22} + C_{1133}\epsilon_{33} + 2C_{1112}\epsilon_{12} + 2C_{1123}\epsilon_{23} + 2C_{1131}\epsilon_{31}$$

Thus

$$\begin{aligned} C_{1123} &= 0 \\ C_{1131} &= 0 \end{aligned}$$

Similarly

$$\begin{aligned} C_{2223} &= 0 \\ C_{2231} &= 0 \\ C_{3323} &= 0 \\ C_{3331} &= 0 \end{aligned}$$

Thus for a laminated composite for example,

$$\left\{\begin{array}{c}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{xy}\\ \sigma_{yz}\\ \sigma_{zx}\end{array}\right\} = \left(\begin{array}{cccccc} C_{xxxx} & C_{xxyy} & C_{xxzz} & C_{xxxy} & 0 & 0 \\ & C_{yyyy} & C_{yyzz} & C_{yyxy} & 0 & 0 \\ & & C_{zzzz} & C_{zzxy} & 0 & 0 \\ & & & C_{xyxy} & 0 & 0 \\ & & & & C_{yzyz} & C_{yzzx} \\ & & & & & C_{zxzx} \end{array}\right) \left\{\begin{array}{c}\epsilon_{xx}\\ \epsilon_{yy}\\ \epsilon_{zz}\\ 2\epsilon_{xy}\\ 2\epsilon_{yz}\\ 2\epsilon_{zx}\end{array}\right\}$$

Note for pure shear,

$$\sigma_{11} = 2C_{xxxy}\epsilon_{xy}$$

Also, when all shear strains are zero

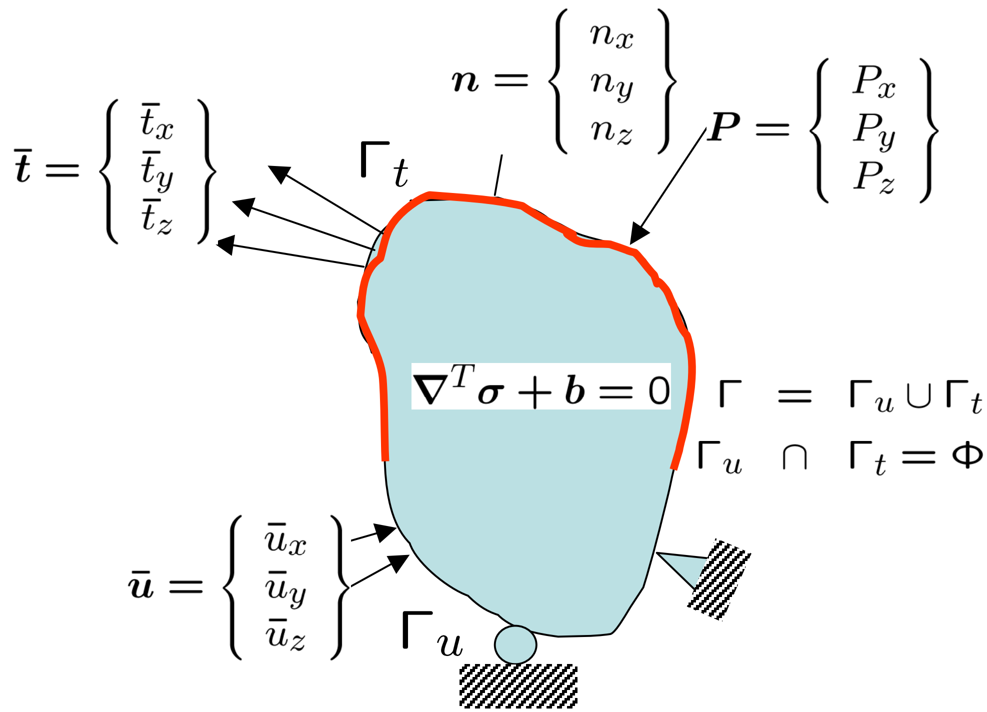
$$\sigma_{12} = C_{xxxy}\epsilon_{xx}$$

Principal axes of stress and strain are not the same.

Strain displacement relationships

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

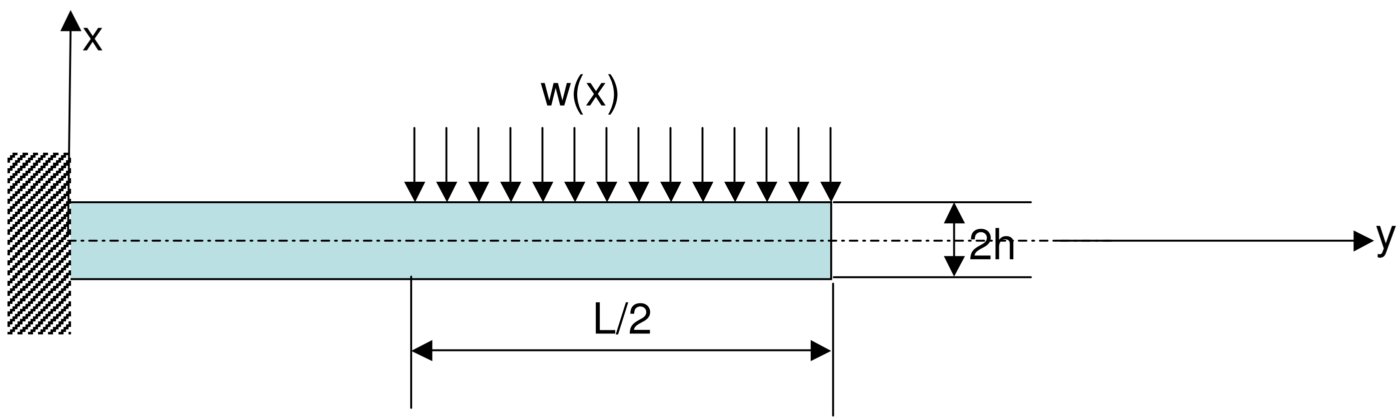
$$\left\{ \begin{array}{c} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ 2\epsilon_{xy} \\ 2\epsilon_{yz} \\ 2\epsilon_{zx} \end{array} \right\} = \left\{ \begin{array}{c} \frac{\partial u_x}{\partial x} \\ \frac{\partial u_y}{\partial y} \\ \frac{\partial u_z}{\partial z} \\ \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \\ \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \\ \frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z} \end{array} \right\} = \left(\begin{array}{ccc} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{array} \right) \left\{ \begin{array}{c} u_x \\ u_y \\ u_z \end{array} \right\}$$



On Γ_t

$$\bar{t}_i = \sigma_{ji} n_j$$

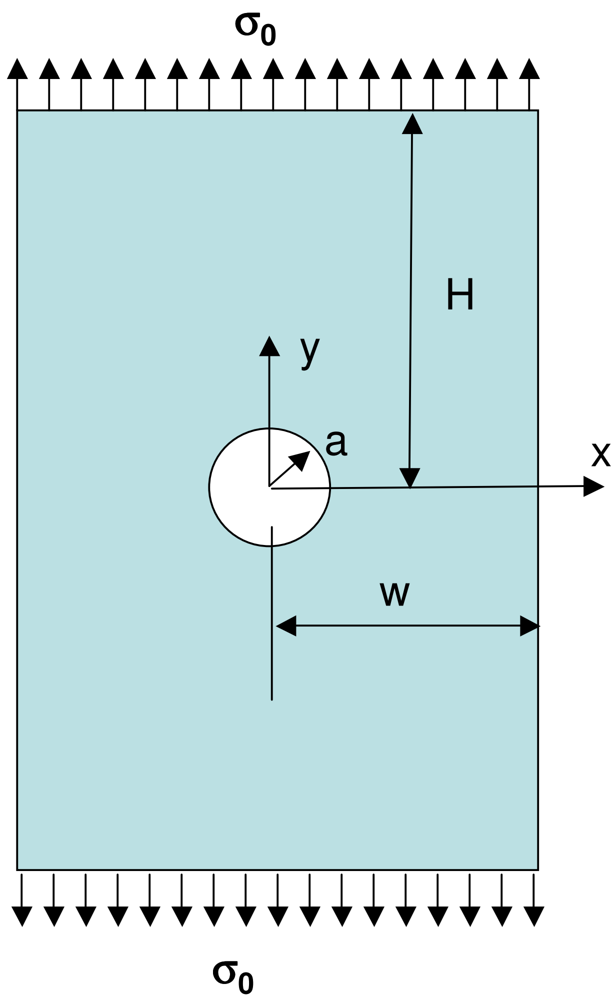
$$\bar{\boldsymbol{t}} = \begin{Bmatrix} \sigma_{xx}n_x + \sigma_{xy}n_y + \sigma_{xz}n_z \\ \sigma_{yx}n_x + \sigma_{yy}n_y + \sigma_{yz}n_z \\ \sigma_{zx}n_x + \sigma_{zy}n_y + \sigma_{zz}n_z \end{Bmatrix}$$



At $x = 0$, $u_x = u_y = 0$

At $x = L$, $t_x = 0 \Rightarrow \sigma_{xx} = 0$

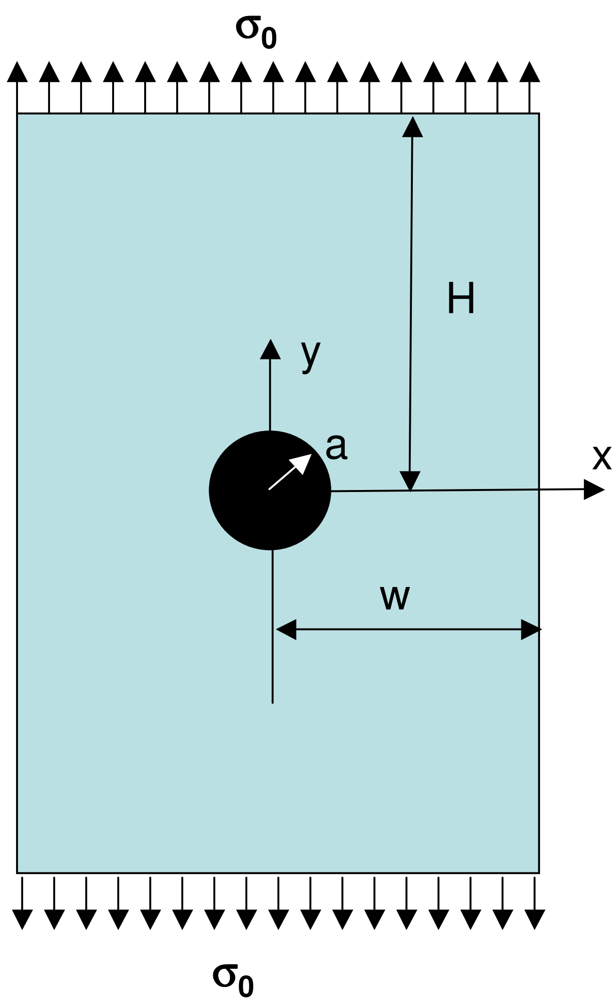
$L/2 \leq x \leq L, y = h$, $t_y = \sigma_{yy} = -w(x)$



At $y = \pm H \;\; \sigma_{yy} = \sigma_0$

At $x = \pm W \;\; \sigma_{xx} = 0$

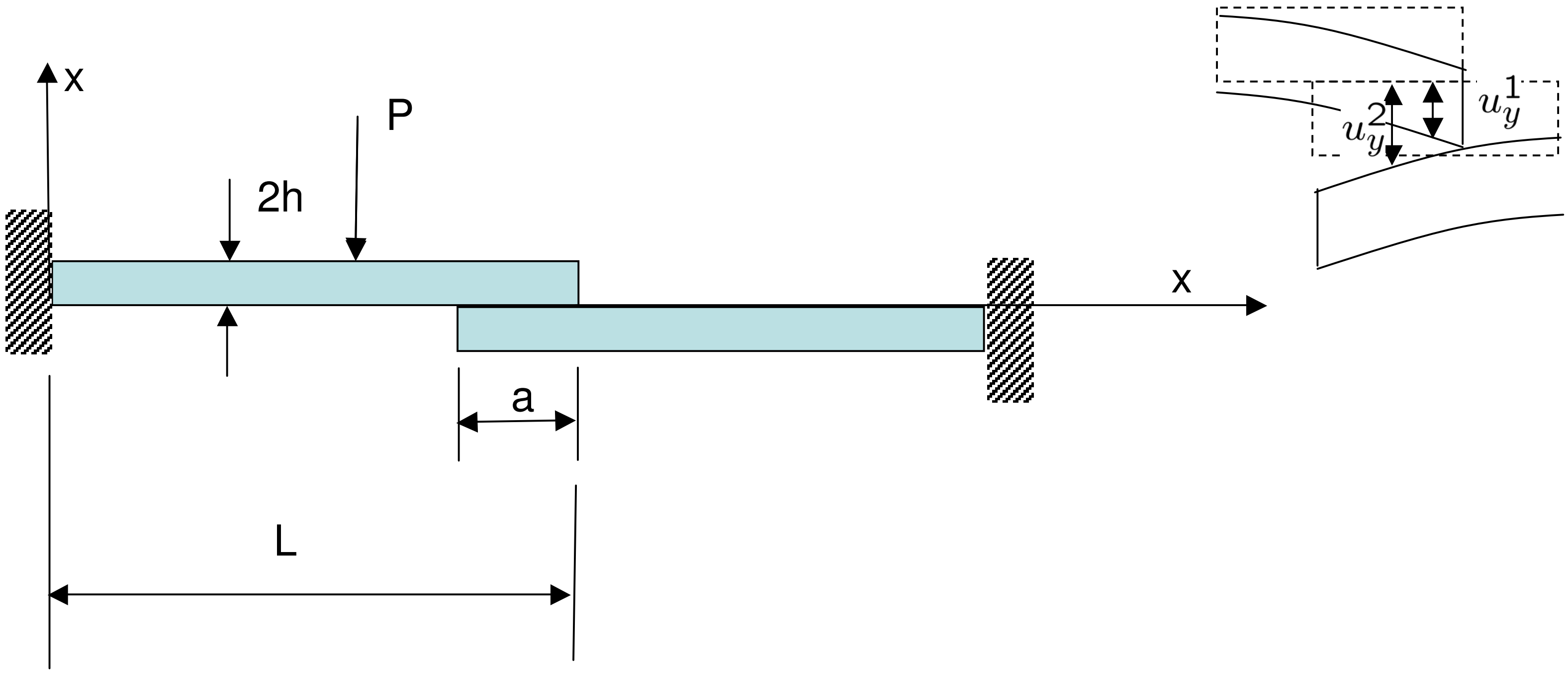
At $r = a \;\; \boldsymbol{t} = 0 \Rightarrow \sigma_{rr} = \sigma_{r\theta} = 0$



At $y = \pm H \;\; \sigma_{yy} = \sigma_0$

At $x = \pm W \;\; \sigma_{xx} = 0$

At $r = a \;\; \boldsymbol{u} = 0 \Rightarrow u_r = u_\theta = 0$



At $x = 0$ $u_x = u_y = 0$

At $x = 2L - a$, $u_x = u_y = 0$

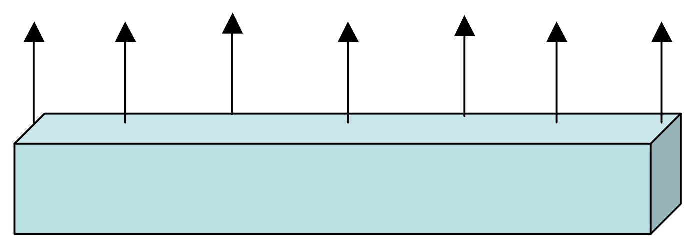
At $x = b$, $F_y = P$

Over $L - a \leq x < L, y = 0$ $u_y^2 - u_y^1 \geq 0$

Plane strain

$$u_z \simeq 0, \frac{\partial}{\partial z} \simeq 0$$

$$\left\{ \begin{array}{c} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ 2\epsilon_{xy} \\ 2\epsilon_{yz} \\ 2\epsilon_{zx} \end{array} \right\} = \left\{ \begin{array}{c} \frac{\partial u_x}{\partial x} \\ \frac{\partial u_y}{\partial y} \\ \frac{\partial u_z}{\partial z} \\ \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \\ \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \\ \frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z} \end{array} \right\} = \begin{pmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & 0 \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \left\{ \begin{array}{c} u_x \\ u_y \\ u_z \end{array} \right\}$$



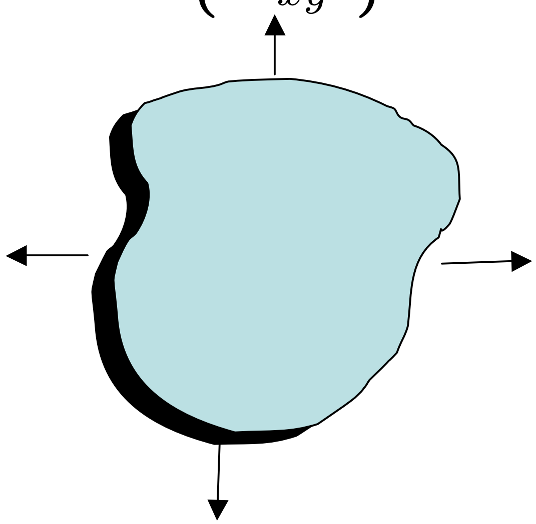
$$
\begin{aligned}
\sigma_{xx} &= C_{xxxx}\epsilon_{xx} + C_{xxyy}\epsilon_{yy} + 2C_{xxxy}\epsilon_{xy} \\
\sigma_{yy} &= C_{yyxx}\epsilon_{xx} + C_{yyyy}\epsilon_{yy} + 2C_{yyxy}\epsilon_{xy} \\
\sigma_{xy} &= C_{xyxx}\epsilon_{xx} + C_{xyyy}\epsilon_{yy} + 2C_{xyxy}\epsilon_{xy}
\end{aligned}
$$

$$
\left\{ \begin{array}{c} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{array} \right\} = \frac{E}{(1+\nu)(1-2\nu)} \left(\begin{array}{ccc} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & (1-2\nu)/2 \end{array} \right) \left\{ \begin{array}{c} \epsilon_{xx} \\ \epsilon_{yy} \\ 2\epsilon_{xy} \end{array} \right\}
$$

Plane stress

$$
\begin{aligned}
\sigma_{xx} &= C_{xxxx}\epsilon_{xx} + C_{xxyy}\epsilon_{yy} + C_{xxzz}\epsilon_{zz} + 2C_{xxxy}\epsilon_{xy} \\
\sigma_{yy} &= C_{yyxx}\epsilon_{xx} + C_{yyyy}\epsilon_{yy} + C_{yyzz}\epsilon_{zz} + 2C_{yyxy}\epsilon_{xy} \\
\sigma_{xy} &= C_{xyxx}\epsilon_{xx} + C_{xyyy}\epsilon_{yy} + C_{xyzz}\epsilon_{zz} + 2C_{xyxy}\epsilon_{xy}
\end{aligned}
$$

$$
\left\{ \begin{array}{c} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{array} \right\} = \frac{E}{1-\nu^2} \left(\begin{array}{ccc} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{array} \right) \left\{ \begin{array}{c} \epsilon_{xx} \\ \epsilon_{yy} \\ 2\epsilon_{xy} \end{array} \right\}
$$



Axisymmetry

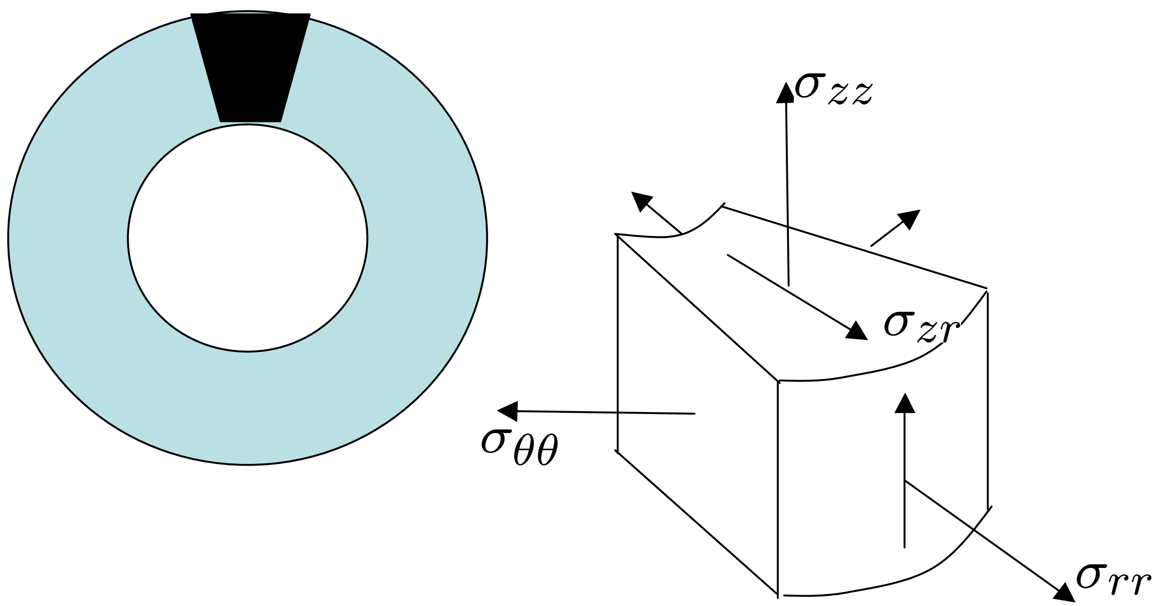
$$\frac{\partial}{\partial \theta} \simeq 0, u_\theta \simeq 0$$

$$\begin{aligned}
\epsilon_{rr} &= \frac{\partial u_r}{\partial r} \\
\epsilon_{r\theta} &= \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) \simeq 0 \\
\epsilon_{\theta\theta} &= \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \simeq \frac{u_r}{r}
\end{aligned}$$

$$\begin{aligned}
\epsilon_{zz} &= \frac{\partial u_z}{\partial z} \\
\epsilon_{rz} &= \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \\
\epsilon_{z\theta} &\simeq 0
\end{aligned}$$

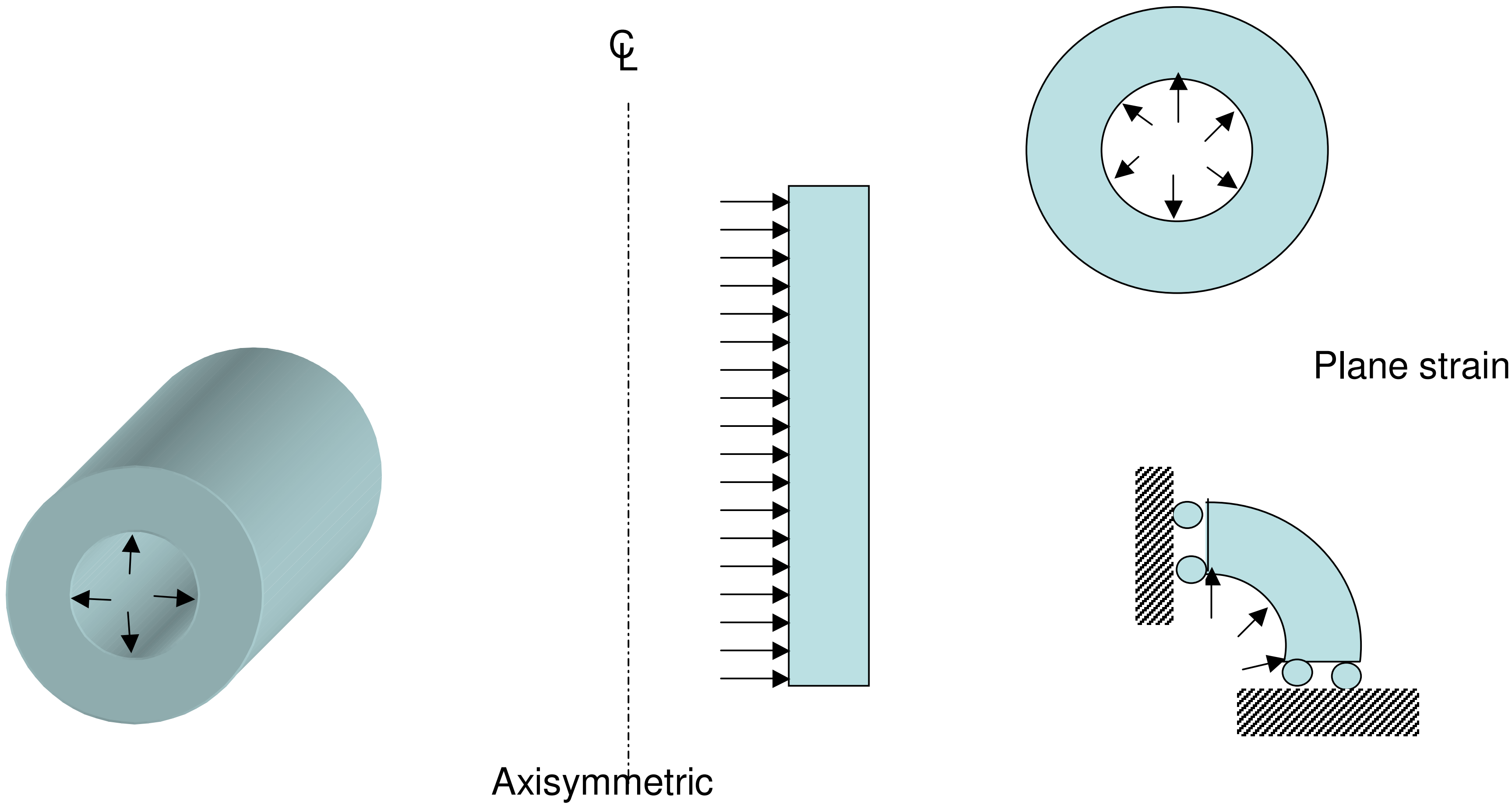
Axisymmetry

Geometry as well as loading should be axisymmetric!

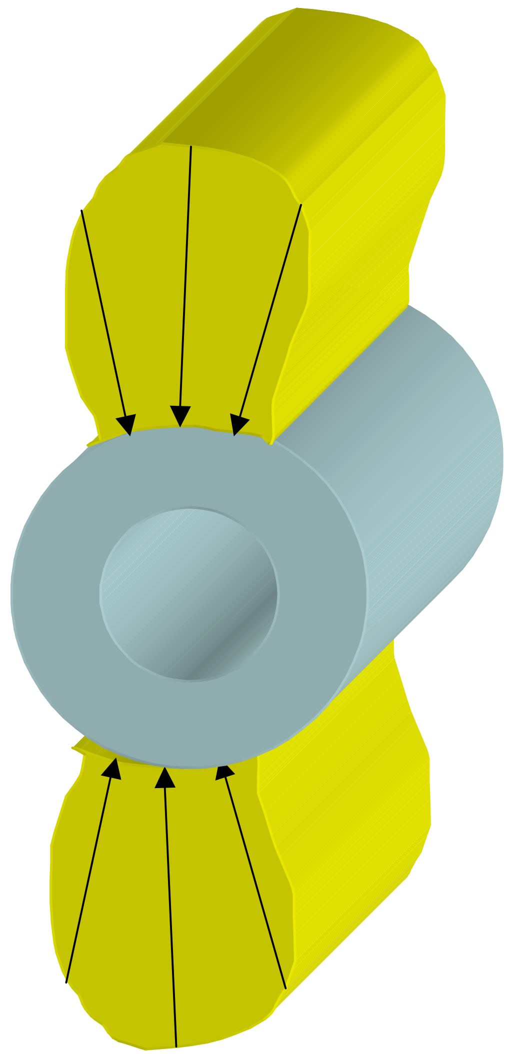


$$\left\{ \begin{array}{c} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \\ \sigma_{zr} \end{array} \right\} = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)} \left(\begin{array}{cccc} 1 & f & f & 0 \\ & 1 & f & 0 \\ & & 1 & 0 \\ & & & g \end{array} \right) \left\{ \begin{array}{c} \epsilon_{rr} \\ \epsilon_{\theta\theta} \\ \epsilon_{zz} \\ \epsilon_{zr} \end{array} \right\}$$

$$\begin{aligned} f &= \frac{\nu}{1-\nu} \\ g &= \frac{1-2\nu}{2(1-\nu)} \end{aligned}$$

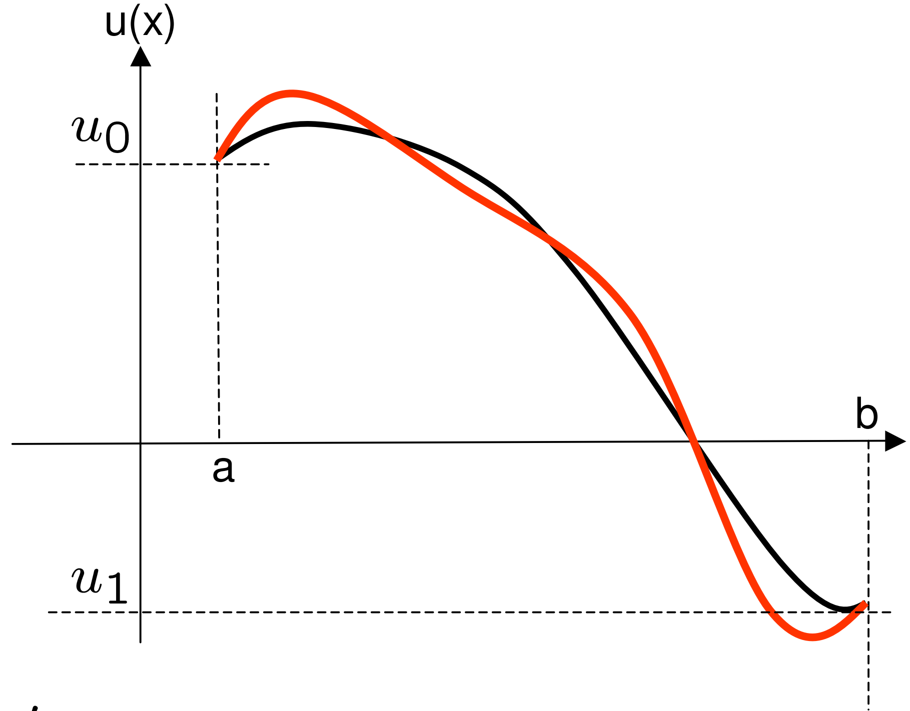


CL
Plane strain
Axisymmetric



Axisymmetric geometry, Non-axisymmetric loading

Digression: A little bit of variational calculus



$$J[u] = \int_a^b F(x, u, u')dx$$

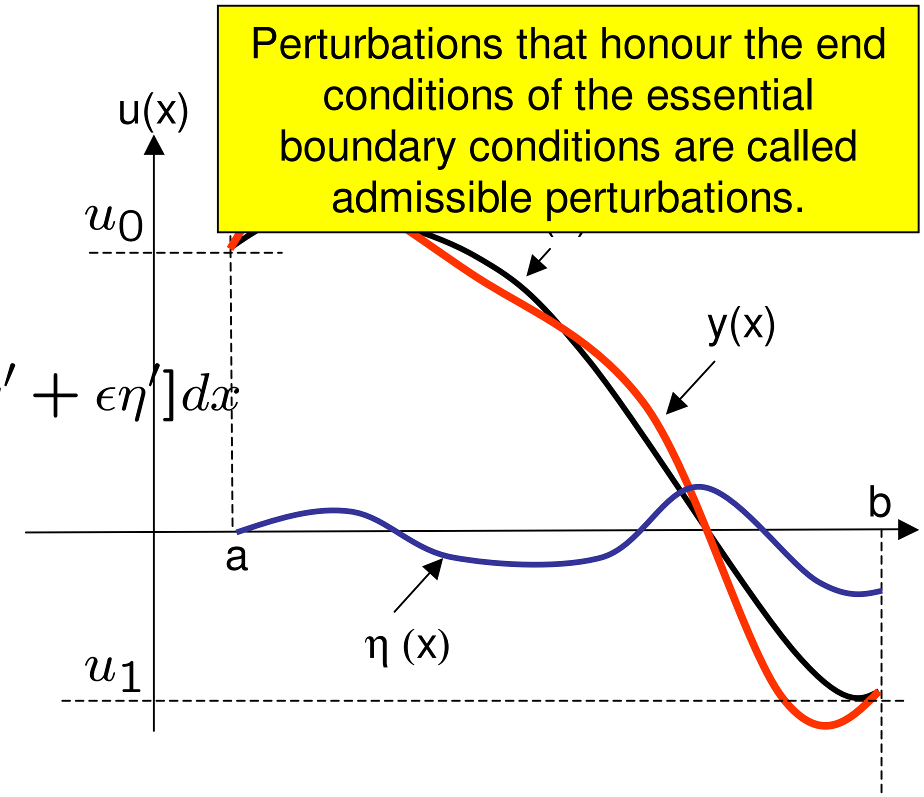
Boundary conditions

$$u(a) = u_0, u(b) = u_1$$

$$u(x) = y(x) + \epsilon\eta(x)$$

$$\eta(a) = \eta(b) = 0$$

$$J[y+\epsilon\eta] = \int_a^b F[x, y+\epsilon\eta(x), y' + \epsilon\eta']dx$$



Let $J[y+\epsilon\eta] = \phi(\epsilon)$

Since $\phi(\epsilon)$ is minimum at $\epsilon = 0$

$$\phi'(0) = 0$$

$$\phi'(\epsilon) \;=\; \int_a^b \left[F_u(x, y+\epsilon\eta, y'+\epsilon\eta') - \frac{d}{dx}F_{u'}(x, y+\epsilon\eta, y'+\epsilon\eta') \right] \eta(x)dx$$

$$+F_{u'}(x, y+\epsilon\eta, y'+\epsilon\eta')\eta(x)|_a^b$$

$$0 = \phi'(0) = \int_a^b \left[F_y(x, y, y') - \frac{d}{dx} F_{y'}(x, y, y') \right] \eta(x) dx$$

$\Rightarrow$

$$F_y(x, y, y') - \frac{d}{dx} F_{y'}(x, y, y') = 0$$

Euler equation

as $\eta(x)$ is arbitrary.

Examples: $J[u] = \int_a^b (1 + u'^2) dx = \min, u(a) = 0, u(b) = 1$

$\Rightarrow$

$$2u'' = 0$$

Again,

$$I[u] = \int_0^{\pi/2} [u'^2 - u^2] dx, u(0) = 0, u(\pi/2) = 1$$

is minimised by the curve $u = \sin x$.

Euler equation for several independent variables

$$J[u] = \int\int_G F(x, y, u, u_x, u_y)dxdy$$

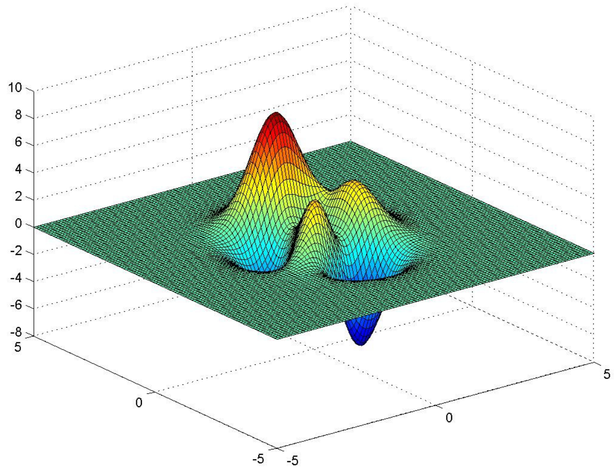
Prescribed properties:

$u(x, y), u_x(x, y), u_y(x, y), u_{xx}(x, y), u_{xy}(x, y), u_{yy}(x, y)$ continuous in G

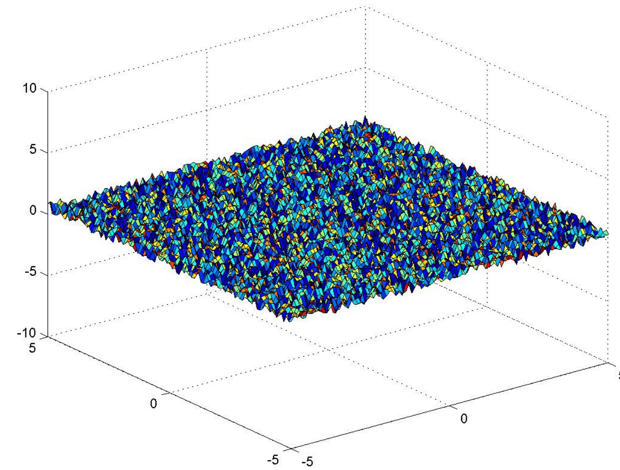
$u(x, y)$ prescribed on C

$$u(x, y) = v(x, y) + \epsilon\eta(x, y)$$

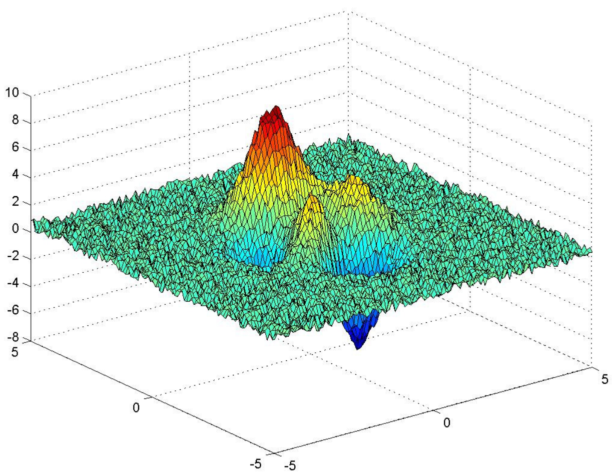
$\eta(x, y) = 0$ on C



v(x,y)



η(x,y)



u(x,y)

$$\phi(\epsilon) = J[v + \epsilon\eta] \Rightarrow$$

$$\frac{d\phi}{d\epsilon}(0) = 0$$

i.e

$$0 = \int\int_G \left\{F_v\eta + F_{v_x}\eta_x + F_{v_y}\eta_y\right\} dxdy$$

$$\int\int_G F_{v_x}\eta_x dxdy = \int\int_G \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial v_x}\eta\right) dxdy - \int\int_G \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial v_x}\right)\eta dxdy$$

$$= \int_{\partial G}\frac{\partial F}{\partial v_x}\eta n_x dS - \int\int_G \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial v_x}\right)\eta dxdy = -\int\int_G \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial v_x}\right)\eta dxdy$$

This is zero on the boundary

Finally,

$$\frac{\partial F}{\partial v} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial v_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial v_y}\right) = 0$$

Necessary condition for a function v(x,y) minimising or maximising the functional J

Example :

$$J[u] = \int_V (u_x^2 + u_y^2) dV$$

and $\bar{u} = f(x, y)$ on ∂D.

Application of the Euler equation yields

$$u_{xx} + u_{yy} = 0$$

Laplace equation

Solving Laplace equation with given boundary conditions is equivalent to minimising the variational statement

Weak anchoring

$$J[v] = \int\int_G F(x, y, v, v_x, v_y)dxdy + \int_C \gamma(v)dS$$

In this case, $\eta(x, y) \neq 0$ on C. Therefore,

$$\int\int_G \left\{\frac{\partial F}{\partial v} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial v_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial v_y}\right)\right\}\eta dV$$

$$\int_C \left[n_x\frac{\partial F}{\partial v_x} + n_y\frac{\partial F}{\partial v_y} + \frac{\partial \gamma}{\partial u}\right]\eta dS = 0$$

Thus,

$$\frac{\partial F}{\partial v} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial v_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial v_y}\right) = 0$$

$$n_x\frac{\partial F}{\partial v_x} + n_y\frac{\partial F}{\partial v_y} + \frac{\partial \gamma}{\partial u} = 0$$

First variation

Let

$$\epsilon\eta = \delta u$$

be the *variation of* u.
δu represents and admissible perturbation of u.

For a functional $F(x, u, u')$, due to an admissible variation in u we have

$$\begin{aligned}
\Delta F &= F(x, u + \epsilon\eta, u' + \epsilon\eta') - F(x, u, u') \\
&= \epsilon\eta \frac{\partial F}{\partial u} + \epsilon\eta' \frac{\partial F}{\partial u'} + \frac{(\epsilon\eta)^2}{2} \frac{\partial^2 F}{\partial u^2} + \\
&\quad \frac{(\epsilon\eta)(\epsilon\eta')}{2} \frac{\partial^2 F}{\partial u \partial u'} + \frac{(\epsilon\eta')^2}{2} \frac{\partial^2 F}{\partial u'^2} + \ldots - F(x, u, u')
\end{aligned}$$

Therefore, the first variation of F is

$$\delta F = \frac{\partial F}{\partial u} \delta u + \frac{\partial F}{\partial u'} \delta u'$$

Note:

First variation

$$\delta F = \frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u'$$

and, Total derivative

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial u}du + \frac{\partial F}{\partial u'}du'$$

$\Rightarrow$ x is not varied during a perturbation from $u + \delta u$.

Variations follow the laws of differentiation:

$$\begin{aligned}
\delta(F_1 \pm F_2) &= \delta F_1 \pm \delta F_2 \\
\delta(F_1 F_2) &= \delta F_1 F_2 + F_1 \delta F_2 \\
\delta(F_1)^n &= n(F_1)^{n-1} \delta F_1
\end{aligned}$$

and also, for $G = G(x, y, z, u, v, w, u_x, u_y, \ldots)$

$$\delta G = \delta_u G + \delta_v G + \delta_w G$$

where eg. $\delta_u G$ is

$$\delta_u G = \frac{\partial G}{\partial u}\delta u + \frac{\partial G}{\partial u_x}\delta u_x + \frac{\partial G}{\partial u_y}\delta u_y + \ldots$$

Thus, if

$$I[u] = \int_a^b F(x, u, u')dx,$$

we have

$$\delta I = \int_a^b \left(\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u' \right) dx$$

or

$$\delta I = \int_a^b \left(\frac{\partial F}{\partial u} - \frac{d}{dx}\frac{\partial F}{\partial u'} \right) \delta u dx + \frac{\partial F}{\partial u'}\delta u \bigg|_a^b$$

$\Rightarrow$ Euler equation can be written as

$$\delta I = 0$$

Natural and Essential boundary conditions

Consider

$$I[u] = \int_a^b F(x,u,u')dx - Q_a u(a) - Q_b u(b)$$

The necessary condition to attain a minimum is

$$\int_a^b \eta \left[\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'} \right) \right] dx + \left(\frac{\partial F}{\partial u'} \eta \right) \Big|_a^b - Q_a \eta(a) - Q_b \eta(b) = 0$$

The above will be satisfied if

1. Euler equation is satisfied
2. $\eta(a) = 0$, $\eta(b) = 0$
3. $\eta(a) = 0$,$\partial F/\partial u'|_b - Q_b = 0$
4. $\eta(b) = 0$,$-\partial F/\partial u'|_b - Q_a = 0$
5. $-\partial F/\partial u'|_a - Q_a = 0$, $\partial F/\partial u'|_b - Q_b = 0$

Example: Strong form to weak form

Let us consider a differential equation in one variable (from Reddy, pp58)

$$-\frac{d}{dx}\left[a(x)\frac{du}{dx}\right] = f(x) \quad \text{for } 0 < x < L$$

with

$$u(0) = 0, \; \left(a\frac{du}{dx}\right)\bigg|_{x=L} = Q_L$$

Devise the *weighted residual* statement

$$\int_0^L w(x)\left[-\frac{d}{dx}\left(a\frac{du}{dx}\right) - f\right] dx = 0$$

so that

$$\int_0^L \left\{w\left[-\frac{d}{dx}\left(a\frac{du}{dx}\right)\right] - wf\right\} dx = \int_0^L \left(a\frac{dw}{dx}\frac{du}{dx} - wf\right) dx - \left[wa\frac{du}{dx}\right]_0^L = 0$$

Thus

$$\int_0^L \left(a\frac{dw}{dx}\frac{du}{dx} - wf \right) dx - \left(wa\frac{du}{dx}n_x \right)_{x=0} - \left(wa\frac{du}{dx}n_x \right)_{x=L} =$$

$$\Pi[w,u] \;=\; \underbrace{\int_0^L \left(a\frac{dw}{dx}\frac{du}{dx} - wf \right) dx - w(L)Q_L}_{} = 0$$

Weak/Variational form corresponding to the deq

If

$$B(w,u) = \int_0^L a\frac{du}{dx}\frac{dw}{dx}dx,$$

and

$$l(w) = \int_0^L wf dx + w(L)Q_L$$

We have

$$\Pi[w,u] = B(w,u) - l(w) = 0$$

Replace w by the variation in u, i.e δu, so that

$$B(\delta u, u) - l(\delta u) = 0$$

Under certian restricted forms of $B(w,v)$ we can have

$$B(\delta u, u) = \frac{1}{2}\delta[B(u,u)]$$

eg. in the case at hand

$$B(\delta u, u) = \int_0^L a\frac{d\delta u}{dx}\frac{du}{dx} = \frac{1}{2}\delta\int_0^L a\frac{du}{dx}\frac{du}{dx}dx$$

and

$$l(\delta u) = \delta\left[\int_0^L uf dx + u(L)Q_L\right]$$

Summary of the variational statement: Minimise functional $I[u]$ such that

$$\delta\Pi = 0 = B(\delta u, u) - l(\delta u)$$

for

$$\delta^2\Pi = B(\delta u, \delta u) > 0, \text{ for } \delta u \neq 0$$

Generalised forms of the 1d case

$$\mathcal{A} = \mathcal{L}\boldsymbol{u} + \boldsymbol{b} = 0$$

Linear, differential operator

Self adjointness implies that

$$\int_V \boldsymbol{\psi}^T \mathcal{L}\boldsymbol{\gamma} dV = \int_V \boldsymbol{\gamma}^T \mathcal{L}\boldsymbol{\psi} dV + \text{boundary terms}$$

Self adjointness allows

$$\Pi = \int_V \left[\frac{1}{2}\boldsymbol{u}^T \mathcal{L}\boldsymbol{u} + \boldsymbol{u}^T \boldsymbol{b}\right] dV + \text{boundary terms}$$

Example (from Zienkiewicz and Taylor)

$$\nabla^2\phi + c\phi + Q = 0$$

c and Q are functions of position only.

$$\mathcal{L} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + c, \boldsymbol{b} = Q$$

The operator is self adjoint as

$$\int_V \psi \left\{ \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} \right\} dV$$

$$\int_V \psi \frac{\partial^2\phi}{\partial x^2} dV = \int_{\partial V} \psi \frac{\partial\phi}{\partial x} n_x dS - \int_V \frac{\partial\psi}{\partial x}\frac{\partial\phi}{\partial x} dV$$

Similarly $$\int_V \phi \frac{\partial^2\psi}{\partial x^2} dV = \int_{\partial V} \phi \frac{\partial\psi}{\partial x} n_x dS - \int_V \frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial x} dV$$

Thus,

$$\int_V \psi \nabla^2\phi dV = \int_V \phi \nabla^2\psi dV + \text{boundary terms}$$

Thus the variational principle corresponsding to this equation becomes:

$$\Pi = \int_V \left\{ \frac{1}{2}\phi \left[\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + c\phi \right] + Q\phi \right\} dV$$

or, applying Gauss law,

$$\Pi = \int_V \left[\frac{1}{2} \left(\frac{\partial \phi}{\partial x} \right)^2 + \frac{1}{2} \left(\frac{\partial \phi}{\partial y} \right)^2 + \frac{1}{2} c\phi^2 - Q\phi \right] dV + \text{boundary terms}$$

Alternately, once we know that a variational principle exists:

$$\int_V w\left[\nabla^2\phi + c\phi + Q\right] dV = 0$$

leads to

$$\int_V \left(\frac{\partial w}{\partial x}\frac{\partial \phi}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial \phi}{\partial y} + cw\phi - wQ\right) dxdy - \oint_{\partial V} w\left(\frac{\partial \phi}{\partial x}n_x + \frac{\partial \phi}{\partial y}n_y\right) ds = 0$$

$\Rightarrow$

$$B(w,\phi) = \int_V \left(\frac{\partial w}{\partial x}\frac{\partial \phi}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial \phi}{\partial y} + cw\phi\right) dxdx$$

$$l(w) = \int_V wQdxdy + \oint_{\partial V} w\bar{q}ds$$

$\Rightarrow$

$$\Pi = \frac{1}{2}\int_V \left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + c\phi^2\right] dxdy - \int_V \phi Q dxdy - \oint_{\partial V} \bar{q}\phi ds$$

Another example (from Reddy, p69)

Equations governing bending of a Timoshenko beam

$$
\begin{aligned}
-\frac{d}{dx}\left[S\left(\frac{dw}{dx}+\phi_x\right)\right]+c_f w &= q \\
-\frac{d}{dx}\left(D\frac{d\phi_x}{dx}\right)+S\left(\frac{dw}{dx}+\phi_x\right) &= 0
\end{aligned}
$$

S shear stiffness, D bending stiffness, w transverse deflection, c_f foundation modulus, ϕ_x rotation

$$
\mathcal{L} = \begin{pmatrix} -S\frac{d^2}{dx^2}+c_f & -S\frac{d}{dx} \\ -S\frac{d}{dx} & -D\frac{d^2}{dx^2}+S \end{pmatrix}
$$

$\mathcal{L}$ is self adjoint and linear

$$
\boldsymbol{b} = \left\{ \begin{array}{c} -q \\ 0 \end{array} \right\}
$$

Instead of using the formula, it is better to approach in the following manner (now that we know that a variational principle exists)

$$\int_0^L v_1 \left\{ -\frac{d}{dx}\left[S\left(\frac{dw}{dx}+\phi_x\right)\right] + c_f w - q \right\} dx = 0$$
$$\int_0^L v_2 \left[-\frac{d}{dx}\left(D\frac{d\phi_x}{dx}\right) + S\left(\frac{dw}{dx}+\phi_x\right)\right] dx = 0$$

which yields

$$\int_0^L \left[\frac{dv_1}{dx} S\left(\frac{dw}{dx}+\phi_x\right) + c_f v_1 w - v_1 q\right] dx -$$
$$v_1(L)\left[S\left(\frac{dw}{dx}+\phi_x\right)\right]_{x=L} + v_1(0)\left[S\left(\frac{dw}{dx}+\phi_x\right)\right]_{x=0} = 0$$
$$\int_0^L \left[D\frac{dv_2}{dx}\frac{d\phi_x}{dx} + v_2 S\left(\frac{dw}{dx}+\phi_x\right)\right] dx$$
$$-v_2(L)\left[D\frac{d\phi_x}{dx}\right]_{x=L} + v_2(0)\left[D\frac{d\phi_x}{dx}\right]_{x=0} = 0$$

Boundary conditions:

$$w(0) = \phi_x(0) = 0, \left[S\left(\frac{dw}{dx} + \phi_x\right)\right]_{x=L} = F_0, \left[D\frac{d\phi_x}{dx}\right]_{x=L} = M_0$$

$\Rightarrow$

$$v_1(0) = v_2(0) = 0$$

Combining into a single expression

$$\begin{aligned}
& B((v_1, v_2), (w, \phi_x)) - l(v_1, v_2) = \\
& \int_0^L \left[\left(\frac{dv_1}{dx} + v_2\right) S\left(\frac{dw}{dx} + \phi_x\right) + \frac{dv_2}{dx} d\frac{d\phi_x}{dx} + c_f v_1 w\right] \\
- \quad & \int_0^L v_1 q dx v_1(L) F_0 - v_2(L) M_0 = 0
\end{aligned}$$

Thus

$$\begin{aligned}
\Pi(w, \phi_x) = \quad & B((w, \phi_x), (w, \phi_x)) - l(w, \phi_x) = \\
& \int_0^L \left[S\left(\frac{dw}{dx} + \phi_x\right)^2 + D\left(\frac{d\phi_x}{dx}\right)^2 + c_f w^2\right] \\
- \quad & \int_0^L w q dx + w(L) F_0 - \phi_x(L) M_0
\end{aligned}$$

Principle of minimum potential energy

$$\sigma_{ij} = \frac{\partial W}{\partial \epsilon_{ij}}$$

We assume the existence of W

$$\Pi = \int_V W dV - \int_V \bar{b}_i u_i dV - \int_S \bar{t}_i u_i dS$$

Internal energy

$$\Pi = \int_V W dV - \int_V \bar{\boldsymbol{b}} \cdot \boldsymbol{u} dV - \int_S \boldsymbol{t} \cdot \boldsymbol{u} dS.$$

Additionally, $\boldsymbol{u} = \bar{u}$ on Γ_u

Potential of the loads

Thus, first variation of Π vanishes, i.e.

$$\delta\Pi = 0$$

Finding a displacement field satisfying the boundary conditions is equivalent to minimising the above variational statement, I.e making its first variation go to zero.

For an elastic system,

$$\Pi = \int_V \left\{ \frac{1}{2} \sigma_{ij} \epsilon_{ij} - \bar{b}_i u_i \right\} dV - \int_{\Gamma_t} \bar{t}_i u_i dS - \sum F_i u_i$$

with

$$u_i = \bar{u}_i, \delta u_i = 0 \quad \text{on } \Gamma_u$$

Moreover, assuming isotropy

$$\sigma_{ij} = 2\mu\epsilon_{ij} + \lambda\delta_{ij}\epsilon_{kk}$$

Thus, $\delta\Pi = 0 \Rightarrow$

$$\int_V \left\{ \sigma_{ij} \delta\epsilon_{ij} - \bar{b}_i \delta u_i \right\} dV - \int_{\Gamma_t} \bar{t}_i \delta u_i dS = 0$$

In linear elasticity, we deal with functionals of the form

$$\Pi = \frac{1}{2}B(\boldsymbol{w}, \boldsymbol{u}) - l(\boldsymbol{w})$$

eg.

$$\begin{aligned} B(\boldsymbol{w}, \boldsymbol{u}) &= \int_V \left\{ C_{ijkl} w_{k,l} u_{i,j} \right\} dV \\ l(\boldsymbol{w}) &= \int_V \bar{b}_i w_i dV + \int_{\Gamma_t} \bar{t}_i w_i dS \end{aligned}$$

$B(\boldsymbol{u}, \boldsymbol{v})$ is called a 'bilinear form' having the general form

$$\int_V \{ \boldsymbol{p}(\boldsymbol{x}) \boldsymbol{u} \cdot \boldsymbol{w} + \boldsymbol{q}(\boldsymbol{x}) \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{w} \} dV$$

We have already seen in cases with one independent variable that bilinear forms obey

$$B(\delta u, u) = \frac{1}{2} \delta B(u, u)$$

Summary of variational principles

A variational statement is a functional Π of the form

$$\Pi[\boldsymbol{u}] = \int_V \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{u}, \nabla \boldsymbol{u}) dV \text{ on } V$$

with $\boldsymbol{u} = \bar{\boldsymbol{u}}$ on part of $\Gamma = \partial V$.

Minimising a functional is equivalent to finding $\boldsymbol{u}$ such that

$$\delta \Pi = 0,$$

which is same as solving the corresponding Euler equations.

We are particularly interested in functionals of the form

$$\Pi[\boldsymbol{u}] = \frac{1}{2} B(\boldsymbol{u}, \boldsymbol{u}) - l(\boldsymbol{u})$$

as the potential energy of an elastic body (as well as many other problems in physics) has this form.

It can be shown that this form is equivalent to an equation of the type

$$\mathcal{L}\boldsymbol{u} + \boldsymbol{b} = 0$$

where $\mathcal{L}$ is a linear, self adjoint operator.

General schemes for minimising functionals of the form $\Pi = \frac{1}{2}B(\boldsymbol{w},\boldsymbol{u}) - l(\boldsymbol{w})$

Rayleigh-Ritz scheme:

$$\phi^h(x) = \sum_{j=1}^{N} c_j \phi_j(x) + \phi_0$$

$\Rightarrow$

$$B(\phi_i, \sum_{j=1}^{N} c_j \phi_j + \phi_0) = l(\phi_i)$$

Linearity of B leads to

$$\sum_{j=1}^{N} B(\phi_i, \phi_j) c_j = l(\phi_i) - B(\phi_i, \phi_0)$$

or

$$\sum_{j=1}^{N} K_{ij} c_j = F_i$$

or

$$\boldsymbol{K}\boldsymbol{c} = \boldsymbol{F}$$

Alternately,

$$\Pi = \frac{1}{2}B\left(\left\{\sum_{j=1}^{N} c_j\phi_j + \phi_0\right\}, \left\{\sum_{j=1}^{N} c_j\phi_j + \phi_0\right\}\right) - l(\sum_{j=1}^{N} c_j\phi_j + \phi_0)$$

Minimising:

$$\frac{\partial \Pi}{\partial c_i} = 0 \quad i \in [1, N]$$

Note that for a scalar function in one variable

$$B(u, v) = \int_a^b \left\{p(x)uv + q(x)\frac{du}{dx}\frac{dv}{dx}\right\} dx$$

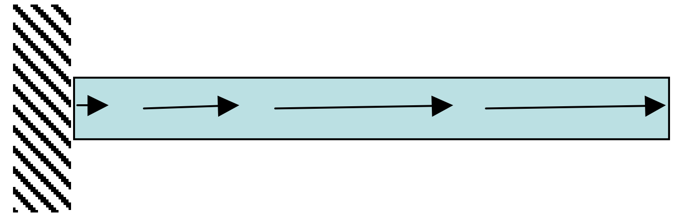
and

$$l(w) = \int_a^b wf dx + w(a)Q_a + w(b)q_b$$

Leads to the same equations

$$\sum_{j=1}^{N} K_{ij}c_j = F_i$$

An example of the application of the Rayleigh Ritz technique



Bar of c/s A, modulus E loaded by a body force cx where c has units of force/area

$$\Pi = \int_0^L \frac{1}{2} E \left(\frac{du}{dx}\right)^2 A dx - \int_0^L ucx dx$$

Also, $u = 0$ at $x = 0$.

Differential equation corresponding to the problem is

$$\sigma_{x,x} + \frac{cx}{A} = 0 \text{ or } AEu_{,xx} + cx = 0$$

Exact solution

$$u = \frac{c}{6AE}(3L^2 x - x^3)$$

Assume,

$$u = a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

First try: use just $u = a_1 x$

$$\Pi = \frac{AEL}{2} a_1^2 - \frac{cL^3}{3} a_1$$

$d\Pi / d a_1$ yields $a_1 = cL^2 / 3AE$. Thus

$$u = \frac{cL^2}{3AE} x, \sigma = \frac{cL^2}{3A}$$

Second try: 2 term solution, i.e use $\partial\Pi/\partial a_1 = 0$ and $\partial\Pi/\partial a_2 = 0$

$$AEL \begin{pmatrix} 1 & L \\ L & 4L^2/3 \end{pmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \frac{cL^2}{12} \begin{Bmatrix} 4 \\ 3L \end{Bmatrix}$$

This gives

$$u = \frac{cL}{12AE}(7Lx - 3x^2)$$

Using $u = a_1 x + a_2 x^2 + a_3 x^3$ gives

$$a_1 = \frac{cL^2}{2AE}, a_2 = 0, a_3 = -\frac{c}{6AE}$$

i.e the correct solution

This is the Rayleigh-Ritz technique where the problem is reduced to that of determining a few constant coefficients.

Weighted residual techniques

Again (in a 1-d example) assume for a differential equation

$$\mathcal{L}\phi - b = 0$$

a solution of the form

$$\phi^h = \phi_0 + \sum_{j=1}^{N} c_j \phi_j$$

such that the *residual* is

$$R = \mathcal{L}\phi^h - b \neq 0$$

Then the *weighted residual* is

$$\int_V \psi_i R(x, c_j) dx = 0$$

where ψ_j are *weight functions*

In general, for a equation of the form

$$\mathcal{L}\boldsymbol{u} - \boldsymbol{b} = 0$$

assume

$$\boldsymbol{u}^h = \sum_{j=1}^{N} c_j \boldsymbol{u}_j + \boldsymbol{u}_0$$

we have

$$\int_V \boldsymbol{\psi}_i \cdot \boldsymbol{R}(\boldsymbol{x}, c_j) dV = 0$$

As the operator $\boldsymbol{\mathcal{L}}$ is linear,

$$\boldsymbol{\mathcal{L}}\boldsymbol{u}^h = c_j \boldsymbol{\mathcal{L}}\boldsymbol{u}_j + \boldsymbol{\mathcal{L}}\boldsymbol{u}_0$$

and thus,

$$\sum_{j=1}^{N} \left[\int_V \boldsymbol{\psi}_i \cdot \boldsymbol{\mathcal{L}}\boldsymbol{u}_j dV \right] \boldsymbol{c} = \int_V \boldsymbol{\psi}_i \cdot [\boldsymbol{b} - \boldsymbol{\mathcal{L}}\boldsymbol{u}_0] dV$$

$\Rightarrow \boldsymbol{K}\boldsymbol{c} = \boldsymbol{F}$

$$K_{ij} = \int_V \boldsymbol{\psi}_i \cdot \boldsymbol{\mathcal{L}}\boldsymbol{u}_j dV, F_i = \int_V \boldsymbol{\psi}_i \cdot [\boldsymbol{b} - \boldsymbol{\mathcal{L}}\boldsymbol{u}_0] dV$$

$\boldsymbol{\psi}_i \neq \boldsymbol{u}_i$ Petrov-Galerkin scheme

$\boldsymbol{\psi}_i = \boldsymbol{u}_i$ Galerkin scheme

$\boldsymbol{\psi}_i = \delta(\boldsymbol{x} - \boldsymbol{x}_i)$ Collocation scheme

$\boldsymbol{\psi}_i = \boldsymbol{\mathcal{L}} \boldsymbol{u}_i$ Least square scheme

Weighted Residual techniques: an example

$$-\frac{d^2u}{dx^2} - u + x^2 = 0$$

with $u(0) = 0, u'(1) = 1.$
Assume

$$u^h = \sum c_j \phi_j + \phi_0$$

so that

$$\begin{aligned} \phi_j(0) &= 0, \phi_0(0) = 0 \\ \phi_0'(1) &= 1, \phi_j'(1) = 0 \end{aligned}$$

Assume $\phi_0 = a + bx \Rightarrow \phi_0(x) = x$
$\phi_1 = a + bx + cx \Rightarrow \phi_1(x) = x(2 - x)$
$\phi_2 = a + cx^2 + dx^3, \Rightarrow \phi_2(x) = x^2\,(1 - 2/3x)$

Thus

$$R(x) = c_1(2-2x+x^2)+c_2(-2+4x-x^2+\frac{2}{3}x^3)-x+x^3$$

1. **Petrov-Galerkin:** Let $\psi_1 = x, \psi_2 = x^2$ $\Rightarrow \int_0^1 xRdx = 0$, $\int_0^1 x^2Rdx = 0 \Rightarrow u^h =$ $1.302x - 0.173x^2 - 0.0147x^3$

2. **Galerkin:** $\psi_i = \phi_i \Rightarrow u^h = 1.289x - 0.1398x^2 - 0.00325x^3$

3. **Least Squares:** $\psi_i = \partial R/\partial c_i \Rightarrow u^h = 1.26x - 0.08x^2 - 0.003325x^3$

4. **Collocation:** $R(1/3) = 0, R(2/3) = 0 \Rightarrow$ $u^h = 1.36x - 0.13x^2 - 0.00342x^3$

The general way for minimising the potential energy

$$\Pi = \int_V W dV - \int_V \boldsymbol{b} \cdot \boldsymbol{u} dV - \int_S \boldsymbol{t} \cdot \boldsymbol{u} dS.$$

Additionally, $\boldsymbol{u} = \bar{u}$ on Γ_u

$$\begin{aligned} \widehat{u}_i &= u_i + \epsilon \eta_i \\ \widehat{u}_i &= u_i + \delta u_i \end{aligned}$$

← A variation in the displacement

$$\widehat{\epsilon}_{ij} = \epsilon_{ij} + \frac{\partial \delta u_i}{\partial x_j} = \epsilon_{ij} + \delta \epsilon_{ij}$$

$$\delta W = \sigma_{ij} \delta \epsilon_{ij}$$

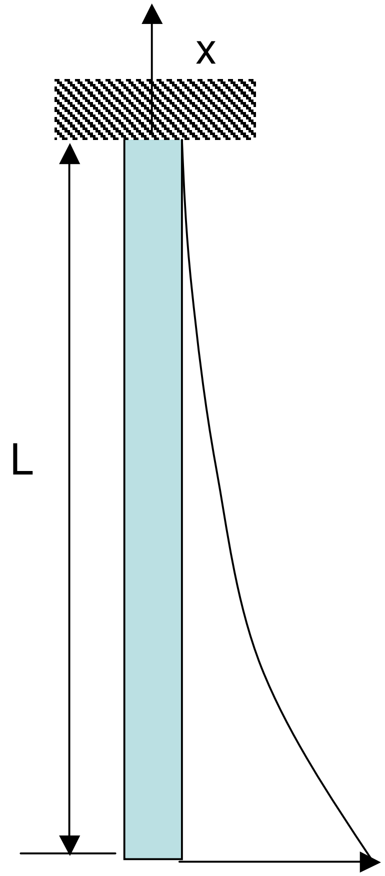
$\Rightarrow$

$$\delta \Pi = \int_V \sigma_{ij} \delta \epsilon_{ij} dV - \int_V b_i \delta u_i dV - \int_S t_i \delta u_i dS = 0$$

$$\int_V \delta \boldsymbol{\epsilon}^T \boldsymbol{\sigma} dV = \int_V \boldsymbol{b} \cdot \delta \boldsymbol{u} dV + \int_S \boldsymbol{t} \cdot \delta \boldsymbol{u} dS$$

Virtual work equation

A 1-d problem with exact solution



$$u(L) = -\frac{\rho g L^2}{2}$$

$$\frac{d\sigma_x}{dx} - \rho g = 0$$

$$\epsilon_x = E\sigma_x = E\frac{du}{dx}$$

$$\sigma_x = \rho g x + C_1$$

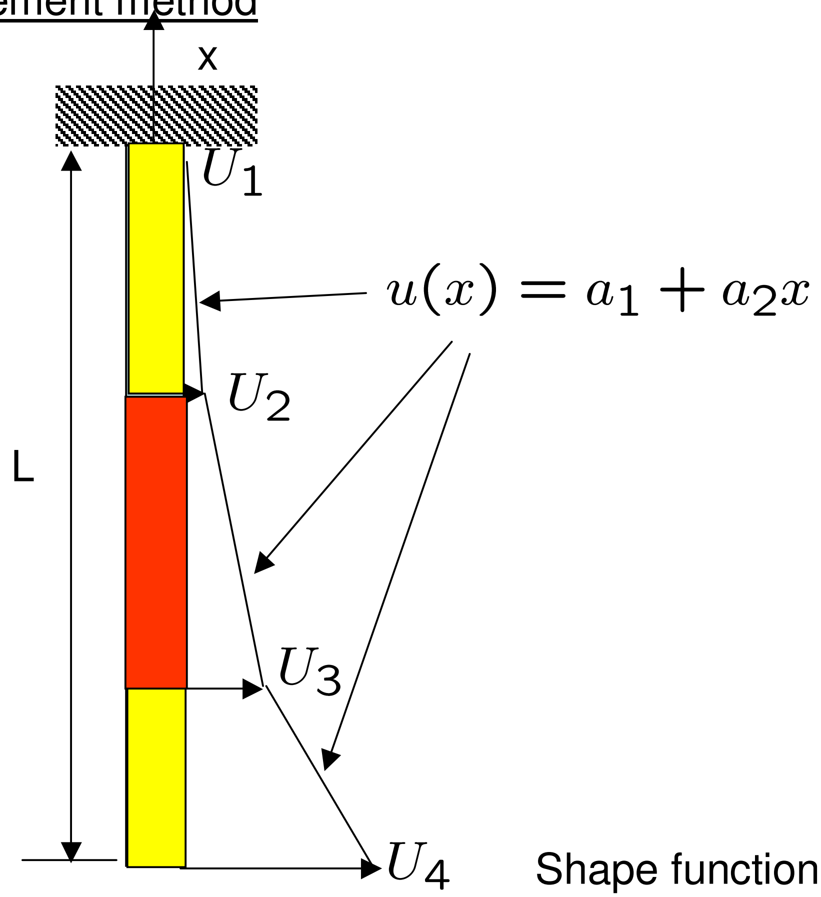
$$u = \frac{\rho g x^2}{2E} + \frac{C_1 x}{E} + C_2$$

At $x = 0,\ u = 0, \Rightarrow\ C_2 = 0$

At $x = -L,\ \sigma_x = 0 \Rightarrow\ C_1 = \rho g L$

$$u(x) = \frac{\rho g x^2}{2E} + \frac{\rho g L x}{E}$$

Piecewise linear approximation: <u>the assumed displacement method</u>



$$u(x) = \langle\ 1 \quad x\ \rangle \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}$$

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \end{pmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}$$

$$\boldsymbol{U} = \boldsymbol{A}\boldsymbol{a}$$

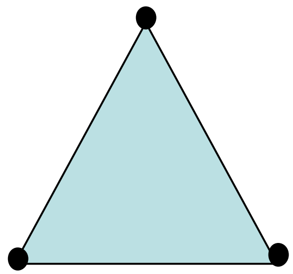
$$u(x) = \underbrace{\langle\ 1 \quad x\ \rangle \boldsymbol{A}^{-1}}_{\boldsymbol{N}} \boldsymbol{U}$$

Shape function matrix

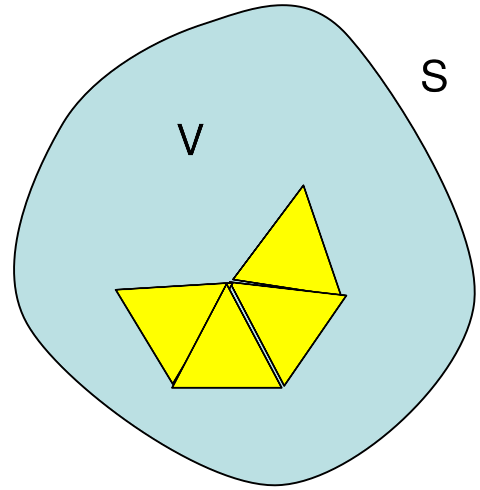
$$\frac{du}{dx} = \langle\ 0 \quad 1\ \rangle \boldsymbol{A}^{-1}\boldsymbol{U} = \boldsymbol{B}\boldsymbol{U}$$

Strain displacement matrix

2-d domain, triangular elements



Linear triangle, 3 nodes/element, 2 dofs/node



$$\left\{ \begin{array}{c} u_x(x,y) \\ u_y(x,y) \end{array} \right\} = \left(\begin{array}{cccccc} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{array} \right) \left\{ \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ b_1 \\ b_2 \\ b_3 \end{array} \right\}$$

$$
\left\{\begin{array}{c} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{array}\right\} = \begin{pmatrix} 1 & x_1 & y_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_1 & y_1 \\ 1 & x_2 & y_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_2 & y_2 \\ 1 & x_3 & y_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_3 & y_3 \end{pmatrix} \left\{\begin{array}{c} a_1 \\ a_2 \\ a_3 \\ b_1 \\ b_2 \\ b_3 \end{array}\right\}
$$

Thus,

$$
\begin{aligned}
\boldsymbol{u}(x,y) &= \begin{pmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{pmatrix} \boldsymbol{a} \\
\boldsymbol{U} &= \boldsymbol{A}\boldsymbol{a} \\
\boldsymbol{u}(x,y) &= \begin{pmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{pmatrix} \boldsymbol{A}^{-1}\boldsymbol{U} = \boldsymbol{N}\boldsymbol{U}
\end{aligned}
$$

Shape functions

$$
\boldsymbol{\epsilon} = \left\{ \begin{array}{c} \epsilon_{xx} \\ \epsilon_{yy} \\ 2\epsilon_{xy} \end{array} \right\} = \left\{ \begin{array}{c} \frac{\partial u_x}{\partial x} \\ \frac{\partial u_y}{\partial y} \\ \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \end{array} \right\} = \begin{pmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{pmatrix} \left\{ \begin{array}{c} u_x \\ u_y \end{array} \right\}
$$

$$
= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \boldsymbol{a} = \boldsymbol{BU}
$$

Strain-displacement matrix

The Virtual work equation reads

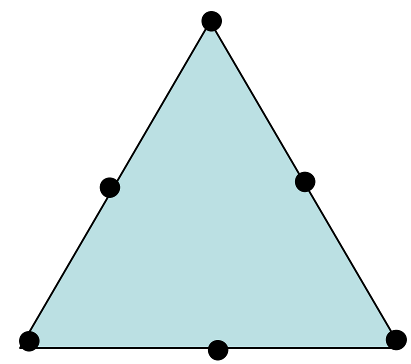
$$\begin{aligned}
\int_V \delta\boldsymbol{\epsilon}^T \boldsymbol{\sigma} dV &= \int_V \boldsymbol{b}\cdot\delta\boldsymbol{u} dV + \int_S \boldsymbol{t}\cdot\delta\boldsymbol{u} dS \\
\int_{V_e} \delta\boldsymbol{u}^T \boldsymbol{B}^T \boldsymbol{C}\boldsymbol{B}\boldsymbol{U} dV &= \int_{V_e} \delta\boldsymbol{u}^T \boldsymbol{N}^T \boldsymbol{b} + \int_{S_e} \delta\boldsymbol{u}^T \boldsymbol{N}^T \boldsymbol{t} dS \\
\boldsymbol{K}_L \boldsymbol{U} &= \boldsymbol{f}_b + \boldsymbol{f}_s
\end{aligned}$$

Where

$$\begin{aligned}
\boldsymbol{K}_L &= \int_{V_e} \boldsymbol{B}^T \boldsymbol{C}\boldsymbol{B} dV \\
\boldsymbol{f}_b &= \int_{V_e} \boldsymbol{N}^T \boldsymbol{b} dV \\
\boldsymbol{f}_s &= \int_{V_e} \boldsymbol{N}^T \boldsymbol{t} dV
\end{aligned}$$

Strains are constant within an element: Constant strain triangle (CST)

Quadratic traingle (Linear strain triangle)



$$
\begin{aligned}
u(x,y) &= a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2 \\
v(x,y) &= a_7 + a_8 x + a_9 y + a_{10} x^2 + a_{11} xy + a_{12} y^2
\end{aligned}
$$

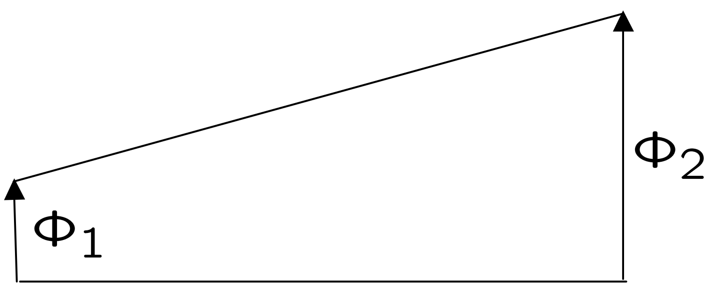
$$
\begin{aligned}
\epsilon_{xx} &= a_2 + 2a_4 x + a_5 y \\
\epsilon_{yy} &= a_9 + a_{11} x + 2a_{12} y \\
2\epsilon_{xy} &= (a_3 + a_8) + (a_5 + 2a_{10})x + (2a_6 + a_{11})y
\end{aligned}
$$

For a field variable ϕ

$$\phi(x,y) = \boldsymbol{N}\boldsymbol{\phi}_e$$

$\Rightarrow N_i = 1$ at dof i and 0 elsewhere.

Definition: C^m continuity $\Rightarrow$ derivatives upto the mth are interelement continuous

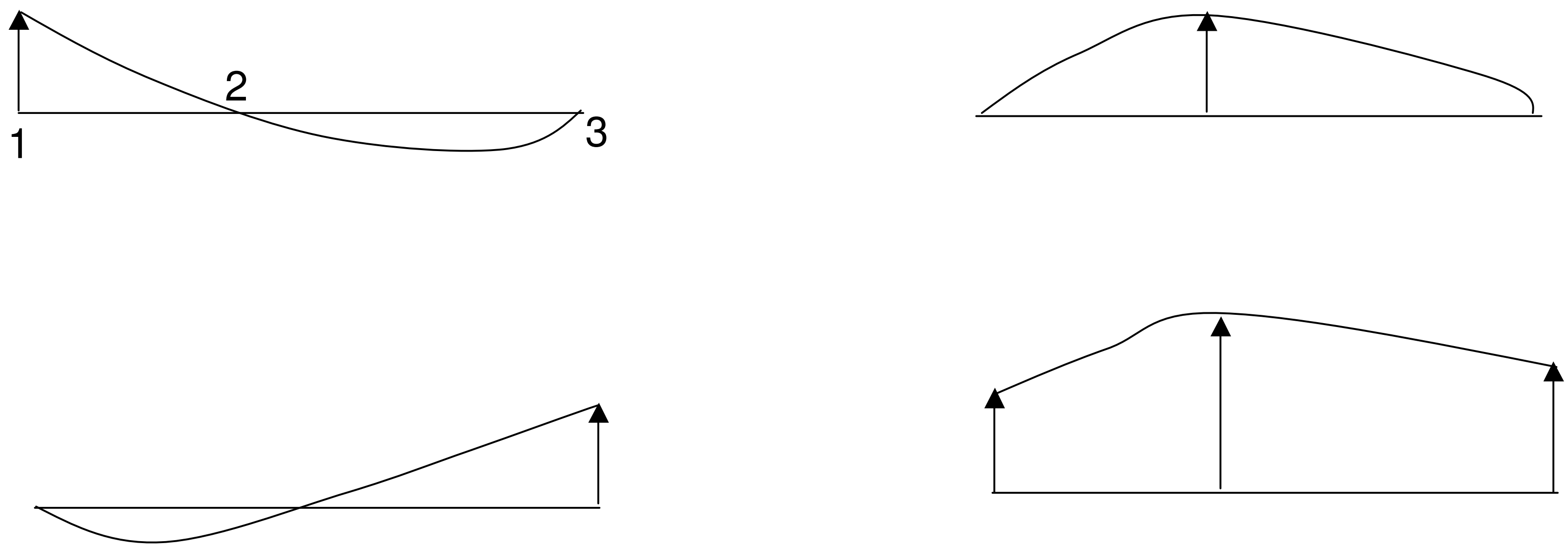


$$\phi = \left\langle \frac{x_2 - x}{x_2 - x_1} \quad \frac{x - x_1}{x_2 - x_1} \right\rangle \begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix}$$

C^0 interpolation

The intepolation functions are same as the shape functions we got for the 1-d example.

Quadratic interpolation in 1-d

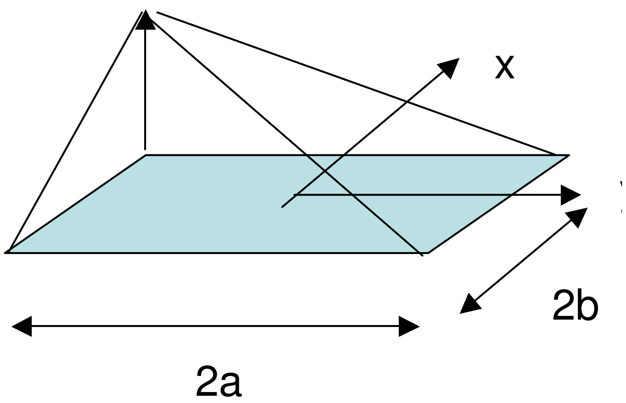


$$
\begin{aligned}
N_1 &= \frac{(x_2 - x)(x_3 - x)}{(x_2 - x_1)(x_3 - x_1)} \\
N_2 &= \frac{(x_1 - x)(x_3 - x)}{(x_2 - x_1)(x_3 - x_2)} \\
N_3 &= \frac{(x_1 - x)(x_2 - x)}{(x_1 - x_3)(x_2 - x_3)}
\end{aligned}
$$

General interpolation formula in 1-d

$$N_k = \frac{(x_1 - x)(x_2 - x)\dots(x_{k-1} - x)(x_{k+1} - x)\dots(x_n - x)}{(x_1 - x_k)(x_2 - x_k)\dots(x_{k-1} - x_k)(x_{k+1} - x_k)\dots(x_n - x_k)}$$

Bilinear rectangle (through interpolation formulas)



$$N_1 = \frac{(a-x)(b-y)}{4ab}$$

$$N_2 = \frac{(a+x)(b-y)}{4ab}$$

$$N_3 = \frac{(a+x)(b+y)}{4ab}$$

$$N_4 = \frac{(a-x)(b+y)}{4ab}$$

Thus, directly

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4$$

and

$$v = N_1 v_1 + N_2 v_2 + N_3 v_3 + N_4 v_4$$

This is equivalent to assuming a displacement variation with <1 x y xy> terms

Assignment 9: Assemble stiffness matrix for CST element e

function[stiffness_dummy] = ass9_groupn(icon,destination,E,nu,e)

% programme to assemble local stiffness matrix of CST element e onto the global stiffness

E=1000;nu=0.3; % units N and m

Formulate stiffness matrix

Add stiffness of e to the global stiffness

Assignment 10: Assemble stiffness matrix for LST element e

function[stiffness_dummy] = ass9_groupn(icon,destination,E,nu,e)

% programme to assemble local stiffness matrix of LST element e onto the global stiffness

E=1000;nu=0.3; % units N and m

Formulate stiffness matrix

Add stiffness of e to the global stiffness

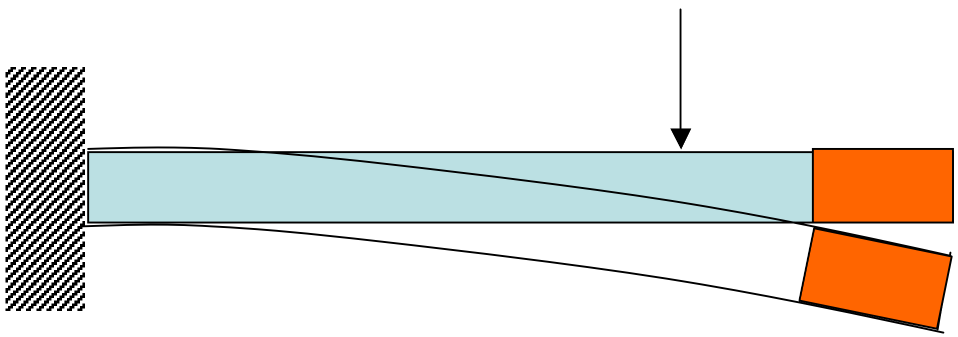
To be judged on correctness. This element routine will be plugged into your main programme and checked on a real problem.

Completeness, compatibility etc….

Requirements of a FE model

•*Monotonic convergence*

•*Completeness* ➔ Element must represent rigid body modes + constant strain states



Element must rotate and translate as a rigid body.

$$\boldsymbol{K}\boldsymbol{\phi} = \lambda\boldsymbol{\phi}$$

The solution consisits of eigen-pairs $(\lambda_1, \boldsymbol{\phi}_1)\ldots(\lambda_N, \boldsymbol{\phi}_N)$
Characteristic equation is

$$p(\lambda) = \det(\boldsymbol{K} - \lambda\boldsymbol{I})$$

For distinct eigenvalues, $\boldsymbol{\phi}_i$ and $\boldsymbol{\phi}_j$ are orthogonal.

$$\begin{aligned} \boldsymbol{K}\boldsymbol{\phi}_i &= \lambda_i\boldsymbol{\phi}_i \\ \boldsymbol{K}\boldsymbol{\phi}_j &= \lambda_j\boldsymbol{\phi}_j \end{aligned}$$

Thus,

$$\begin{aligned} \boldsymbol{\phi}_j^T\boldsymbol{K}\boldsymbol{\phi}_i &= \lambda_i\boldsymbol{\phi}_j^T\boldsymbol{\phi}_i \\ \boldsymbol{\phi}_i^T\boldsymbol{K}\boldsymbol{\phi}_j &= \lambda_j\boldsymbol{\phi}_i^T\boldsymbol{\phi}_j \end{aligned}$$

As we have distinct eigenvalues $(\lambda_i - \lambda_j)\boldsymbol{\phi}_i^T\boldsymbol{\phi}_j = 0 \Rightarrow \boldsymbol{\phi}_i^T\boldsymbol{\phi}_j = 0$

$$\phi_i^T \phi_j = \delta_{ij} \qquad \text{orthonormality}$$

Thus,

$$\boldsymbol{K}\boldsymbol{\Phi} = \boldsymbol{\Phi}\boldsymbol{\Lambda} \qquad \text{Generalised eigenproblem}$$

where,

$$\boldsymbol{\Lambda} = \mathrm{diag}(\lambda_i)$$

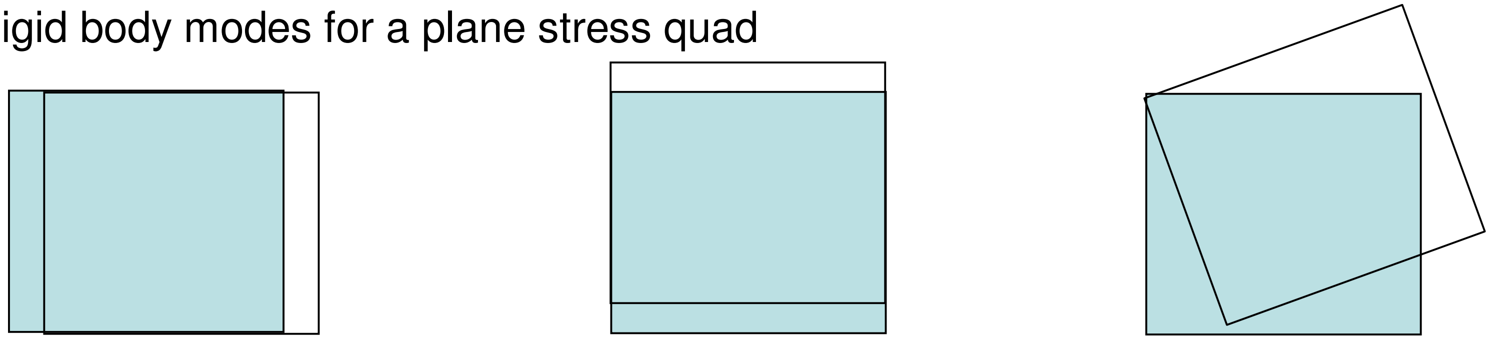
and

$$\boldsymbol{\Phi} = [\phi_1 \; \phi_2 \; \dots \; \phi_N]$$

Therefore,

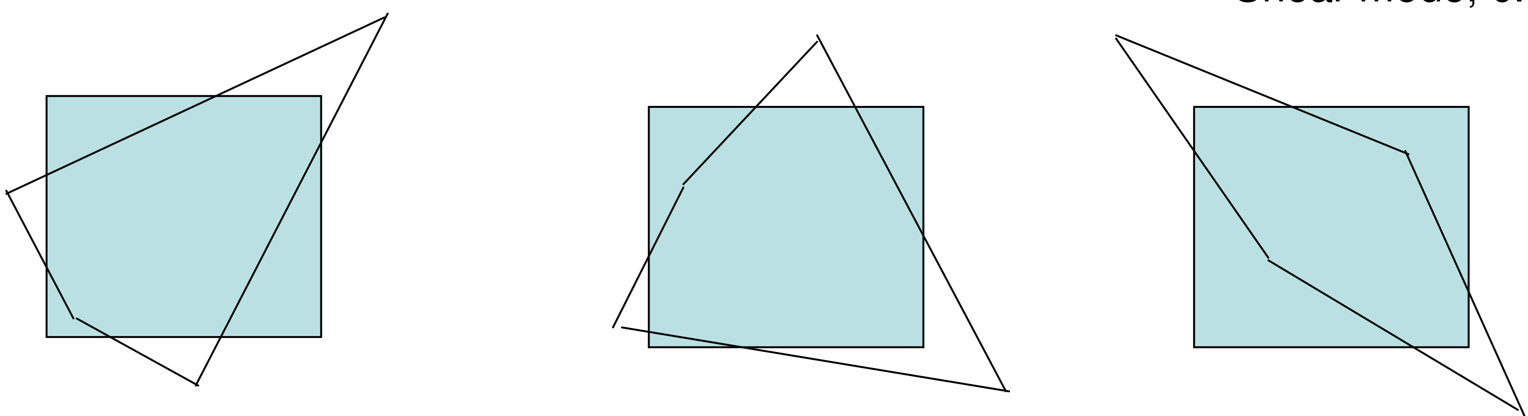
$$\boldsymbol{\Phi}^T \boldsymbol{K} \boldsymbol{\Phi} = \boldsymbol{\Lambda}$$

Rigid body modes for a plane stress quad

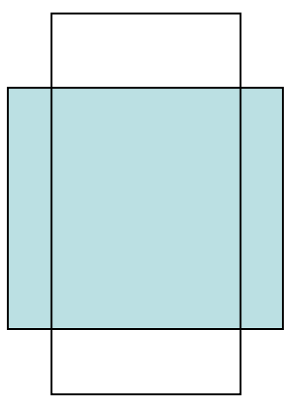


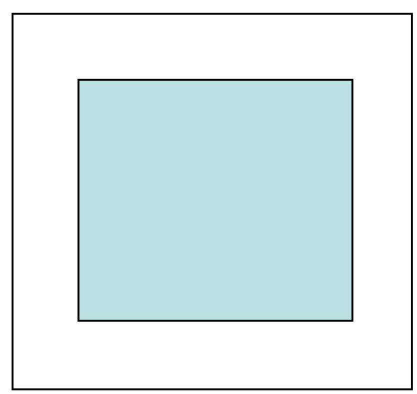
Flexural modes, 0.495, 0.495

Shear mode, 0.769



Stretch mode, 0.769





Uniform extension mode 1.43

Strains in pure bending

$$\epsilon_x = -\frac{\theta_b y}{2a}, \;\; \epsilon_y = \nu\frac{\theta_b y}{2a} \;\; \gamma_{xy} = 0$$

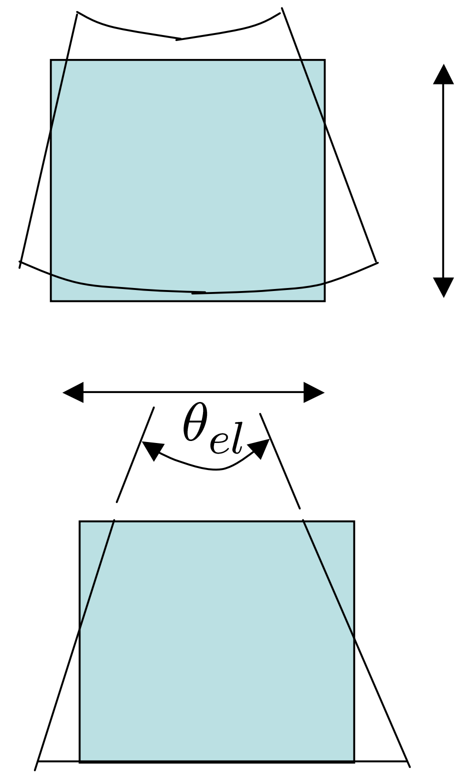
When a Q4 element is bent, the element strains are

$$\epsilon_x = -\frac{\theta_{el} y}{2a}, \;\; \epsilon_y = 0 \;\; \gamma_{xy} = \frac{\theta_{el} x}{2a}$$

Energy stored are

$$U_b = \frac{M_b\theta_b}{2} \;\; U_{el} = \frac{M_{el}\theta_{el}}{2}$$

If $\theta_{el} = \theta_b$ then $M_{el} > M_b$.



Parasitic shear makes the Q4 element unusable in bending

For $M_b = M_{el}$

$$\frac{\theta_{el}}{\theta_b} = \frac{1-\nu^2}{1+\frac{1-\nu}{2}\left(\frac{a}{b}\right)^2}$$

Parasitic shear gets worse with (a/b) ratio. This condition is known as shear locking.

For completeness we require:

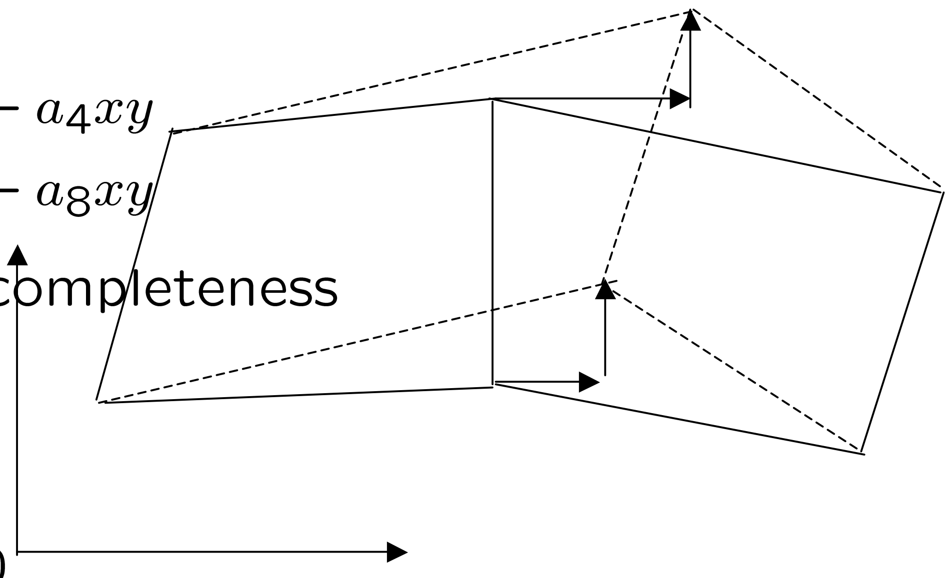
Rigid body modes to be represented (in 2-d two translations and one rotation)

Constant strain states must be represented (in 2-d two normal and one shear strain)

Interelement continuity of displacements must be satisfied.

For a Q4 element

$$u = a_1 + a_2 x + a_3 y + a_4 xy$$
$$v = a_5 + a_6 x + a_7 y + a_8 xy$$



Elements are compatible. For completeness

$$a_1 \neq 0$$
$$a_5 \neq 0$$
$$a_6 = -a_3 \neq 0$$

Constant strain states are represented.

More on convergence

$$\Pi = \frac{1}{2}\boldsymbol{U}^T\boldsymbol{K}\boldsymbol{U} - \boldsymbol{U}^T\boldsymbol{R}$$

leading to

$$\boldsymbol{K}\boldsymbol{U} = \boldsymbol{R}$$

Thus $\Pi = -\frac{1}{2}\boldsymbol{U}^T\boldsymbol{R}$ while $\mathcal{U} = \frac{1}{2}\boldsymbol{U}^T\boldsymbol{R}$

Also $\boldsymbol{K}$ is positive definite if there are no zero eigenvalues

Since $\mathcal{U} > 0$

Starting from a trial function and always adding more terms to it (equivalent to h → 0 in FEM) will always make the potential go towards a lower value.

Decrease in the total potential implies increase in the strain energy. Thus predicted stiffness is always higher than actual in FE analysis.

Spatially anisotropic elements depend on coordinate orientations.

Rate of convergence depends on the completeness of the polynomial used.

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & x & & y & & \\
 & x^2 & & xy & & y^2 & \\
x^3 & & x^2y & & xy^2 & & y^3
\end{array}
$$

Pascal's triangle

An element of size h with a complete displacement expansion of order c can represent displacement variations of that order exactly.
For arbitraray displacements error $\simeq O(h^{c+1})$
Stress error $\simeq O(h^{c+1-1})$

Nodal field variable interpolated as:

$$\phi = N_1\phi_1 + N_2\phi_2 + \dots$$

while coordinates are interpolated as

$$x = \hat{N}_1 x_1 + \hat{N}_2 x_2 + \dots$$

If $N_i = \hat{N}_i \Rightarrow$ isoparametric
If degree($\hat{N}_i$) $<$ degree(N_i) $\Rightarrow$ subparametric
If degree($\hat{N}_i$) $>$ degree(N_i) $\Rightarrow$ superparametric

First proposed by: B.M. Irons, Engineering applications of numerical integration in stiffness methods, AIAA Journal, v4, n11, 1966, 2035-2037.

The Q4 element revisited

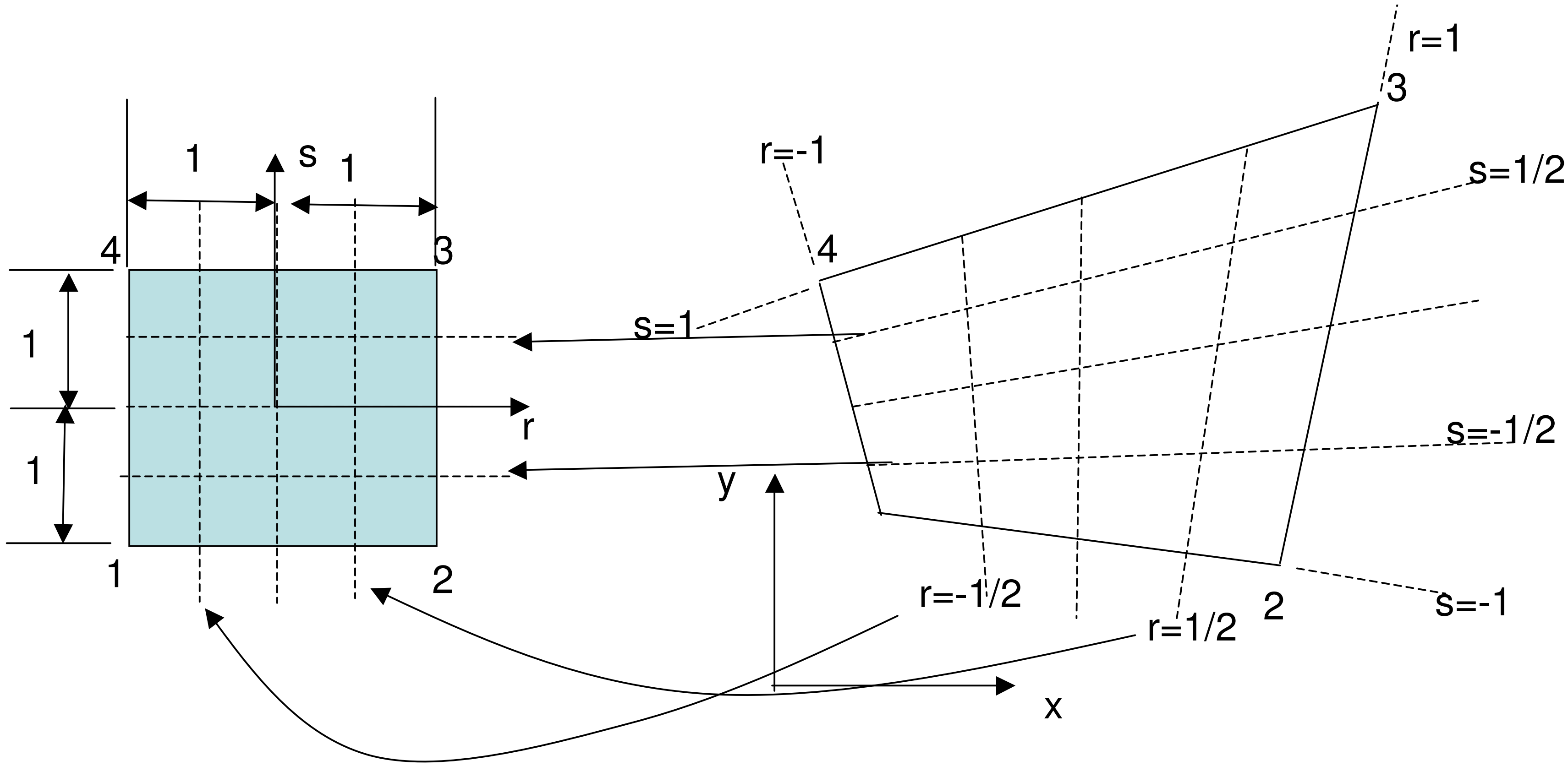
$$\boldsymbol{x} = \left\{ \begin{array}{c} x \\ y \end{array} \right\} = \left\{ \begin{array}{c} \sum_i N_i x_i \\ \sum_i N_i y_i \end{array} \right\}$$

$$= \begin{pmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{pmatrix} \left\{ \begin{array}{c} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \end{array} \right\}$$

$$= \boldsymbol{N}\boldsymbol{X}$$

Under iso-parametric mapping

$$\boldsymbol{u} = \boldsymbol{N}\boldsymbol{U}$$



Parent element

Physical element

Calculation of gradients in 2-d

$$\frac{\partial()}{\partial r} = \frac{\partial()}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial()}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial()}{\partial s} = \frac{\partial()}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial()}{\partial y}\frac{\partial y}{\partial s}$$

Thus

$$\left\{ \begin{array}{c} \frac{\partial}{\partial r} \\ \\ \frac{\partial}{\partial s} \end{array} \right\} = \underbrace{\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{pmatrix}}_{\mathbf{J}} \left\{ \begin{array}{c} \frac{\partial}{\partial x} \\ \\ \frac{\partial}{\partial y} \end{array} \right\}$$

J Jacobian matrix

$$\boldsymbol{J} = \begin{pmatrix} \sum_i N_{i,r} x_i & \sum_i N_{i,r} y_i \\ \\ \sum_i N_{i,s} x_i & \sum_i N_{i,s} y_i \end{pmatrix}$$

$$\left\{ \begin{array}{c} \frac{\partial}{\partial x} \\ \\ \frac{\partial}{\partial y} \end{array} \right\} = \boldsymbol{J}^{-1} \left\{ \begin{array}{c} \frac{\partial}{\partial r} \\ \\ \frac{\partial}{\partial s} \end{array} \right\}$$

$$J = \det \boldsymbol{J} = J_{11} J_{22} - J_{21} J_{12}$$

$$\boldsymbol{\Gamma} = \boldsymbol{J}^{-1}$$

Now,

$$\begin{aligned} N_1 &= \frac{1}{4}(1-r)(1-s) \\ N_2 &= \frac{1}{4}(1+r)(1-s) \\ N_3 &= \frac{1}{4}(1+r)(1+s) \\ N_4 &= \frac{1}{4}(1-r)(1+s) \end{aligned}$$

Follows from the Q4 element done earlier. Put a=b=1.

$$\epsilon_{xx} = u_{,x} = \Gamma_{11} u_{,r} + \Gamma_{12} u_{,s} = \Gamma_{11}(\sum_i N_{i,r} u_i) + \Gamma_{12}(\sum_i N_{i,s} u_i)$$

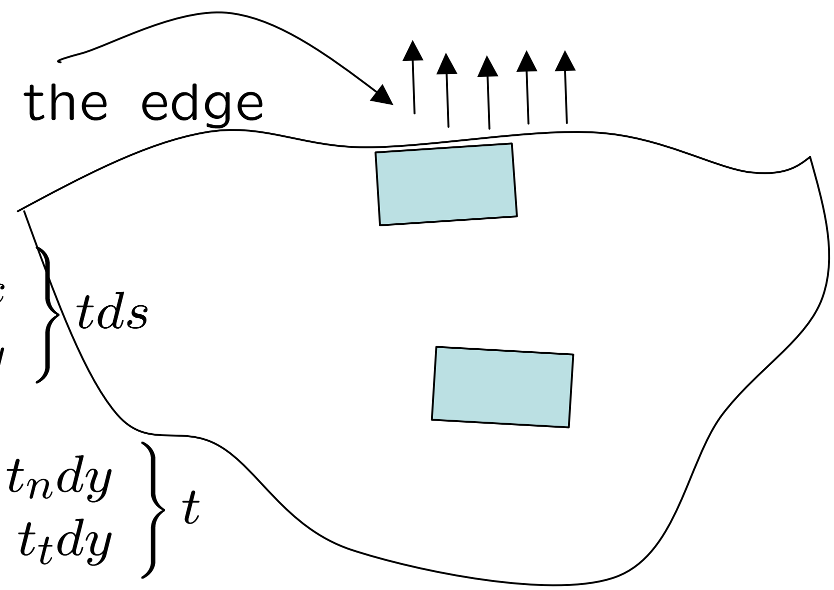
Thus

$$\boldsymbol{B} = \begin{pmatrix} \Gamma_{11} N_{1,r} + \Gamma_{12} N_{1,s} & 0 & \Gamma_{11} N_{2,r} + \Gamma_{12} N_{2,s} & 0 & \dots \\ 0 & \Gamma_{21} N_{1,r} + \Gamma_{22} N_{1,s} & 0 & \Gamma_{21} N_{2,r} + \Gamma_{22} N_{2,s} & \dots \\ \Gamma_{21} N_{1,r} + \Gamma_{22} N_{1,s} & \Gamma_{11} N_{1,r} + \Gamma_{12} N_{1,s} & \Gamma_{21} N_{2,r} + \Gamma_{22} N_{2,s} & \Gamma_{11} N_{2,r} + \Gamma_{12} N_{2,s} & \dots \end{pmatrix}$$

$$\boldsymbol{K}_L = \int_{V_e} \boldsymbol{B}^T \boldsymbol{C} \boldsymbol{B} t dx dy = \int_{-1}^{1} \int_{-1}^{1} \boldsymbol{B}^T \boldsymbol{C} \boldsymbol{B} t J dr ds$$

Stiffness matrix needs to be integrated numerically.

$$\boldsymbol{f}_t = \int_{S_e} \boldsymbol{N}^T \boldsymbol{t} t dS$$



Only N_3 and N_4 are non-zero and the edge maps to $t = 1$.

$$\boldsymbol{f}_t = \int_{S_e} \begin{pmatrix} N_3 & 0 & N_4 & 0 \\ 0 & N_3 & 0 & N_4 \end{pmatrix} \begin{Bmatrix} t_x \\ t_y \end{Bmatrix} t ds$$

$$= \int_{S_e} \begin{pmatrix} N_3 & 0 & N_4 & 0 \\ 0 & N_3 & 0 & N_4 \end{pmatrix} \begin{Bmatrix} t_t dx - t_n dy \\ t_n dx - t_t dy \end{Bmatrix} t$$

Now, on $s = 1$,

$$dx = \frac{\partial x}{\partial r} dr = J_{11} dr, dy = J_{12} dr$$

$$\boldsymbol{f}_t = \int_{S_e} \begin{pmatrix} N_3 & 0 & N_4 & 0 \\ 0 & N_3 & 0 & N_4 \end{pmatrix} \begin{Bmatrix} t_x \\ t_y \end{Bmatrix} t ds$$

$$= \int_{-1}^{1} \begin{pmatrix} N_3 & 0 & N_4 & 0 \\ 0 & N_3 & 0 & N_4 \end{pmatrix} \begin{Bmatrix} t_t J_{11} - t_n J_{12} \\ t_n J_{11} - t_t J_{12} \end{Bmatrix} t dr$$

Assignment 11: Add body force contribution to the force vector

Step 1: Modify data input, create bforce vector bforce(1:nelm)

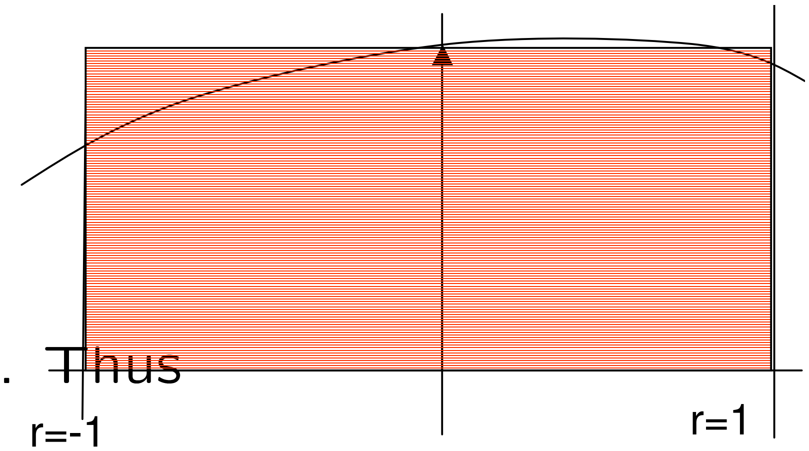
conn

elemnum,matnum,n1,n2,n3,….,bodyforc(1),bodyforce(2)

Step 2: Modify main programme

```
[X,icon,nelm,nnode,nperelem,neltype,nmat,ndof,idisp,specdisp,iforce,specforce,bforce] =
ass1_groupn (fname);
destination = ass2_groupn (X,icon,nelm,nnode,nperelem,neltype,ndof);
for e=1:nelm
    stiff_loc=form local stiffness matrix;
    stiffness_dummy = ass4_groupn(icon,destination,stiff_loc_truss,e);
    stiffness_global=stiffness_global+stiffness_dummy;
    form nodal bosy force vector N'*bodyforce(2) for e
    assemble bodyforce
end
```

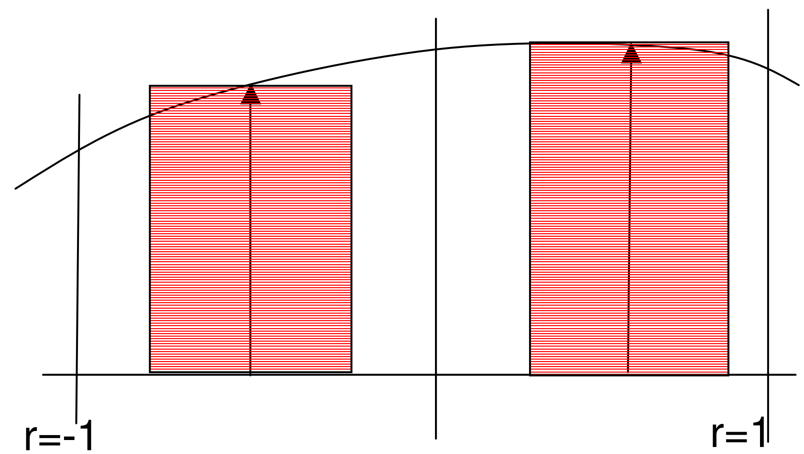
Numerical integration: Gauss quadrature-1d



$$I = \int_{x_1}^{x_2} f(x)dx$$

Substitute $x = 0.5(1-r)x_1 + 0.5(1+r)x_2$. Thus

$$I = \int_{-1}^{1} \phi(r)dr$$



Simplest approx (one point)

$I = \phi(0) * 2$

Exact for a straight line

In general

$$I = \int_{-1}^{1} \phi dr \simeq \sum_{i=1}^{N_{gauss}} W_i \phi(r_i)$$

Weights

Sampling points between -1 and 1

The idea behind a quadrature scheme....

A polynomial passing through n points $\phi(r_i)$

$$\psi(r) = l_1\phi(r_1) + l_2\phi(r_2) + \dots$$

where $l_j(r_i) = \delta_{ij}$ Since at r_i, $\phi(r_i) = \psi(r_i)$,

$$\phi(r) = \psi(r) + P(r)(\beta_0 + \beta_1 r + \beta_2 r^2 + \dots)$$

where

$$P(r) = (r - r_1)(r - r_2)\dots$$

Therefore

$$\int_{-1}^{1} \phi(r)dr = \sum \phi_j \int_{-1}^{1} l_j dr + \sum \beta_j \int_{-1}^{1} r^j P(r)dr$$

Now for $k = 0, 1, 2, 3, \dots (n-1)$

$$\int_{-1}^{1} P(r) r^k dr = 0$$

Thus for $n = 2$

$$\begin{aligned}
\int_{-1}^{1} (r - r_1)(r - r_2) dr &= 0 \\
\int_{-1}^{1} (r - r_1)(r - r_2) r dr &= 0
\end{aligned}$$

Solving we get $r_1 = -1/\sqrt{3}$ and $r_2 = 1/\sqrt{3}$

Also,

$$W_j = \int_{-1}^{1} l_j dr \quad j = 1, 2, \ldots n$$

For $n = 2$,

$$\begin{aligned}
W_1 &= \int_{-1}^{1} \frac{r - r_2}{r_1 - r_2} dr = 1 \\
W_2 &= \int_{-1}^{1} \frac{r - r_1}{r_2 - r_1} dr = 1
\end{aligned}$$

Gauss quadrature: sample points and weights

n	Sampling points	weights
1	0	2
2	0.57735, -0.57735	1 1
3	0.77459, -0.77459 0	0.5555 0.5555 0.8888
4	0.86114 -0.86114 0.33998 -0.33998	0.34785 0.34785 0.65214 0.65214

Example:

$$\phi = a_0 + a_1 r + a_2 r^2 + a_3 r^3$$

Exact integral

$$I = \int_{-1}^{1} \phi dr = 2a_0 + \frac{2}{3} a_2$$

One point quadrature gives

$$I = 2a_0$$

Two point quadrature gives (with $p = 1/\sqrt{3}$)

$$I = 1.0(a_0 - a_1 p + a_2 p^2 - a_3 p^3) + 1.0(a_0 + a_1 p + a_2 p^2 + a_3 p^3) = \text{exact value}$$

Important rule: n point Gauss quadrature integrates a polynomial of order 2n-1 exactly

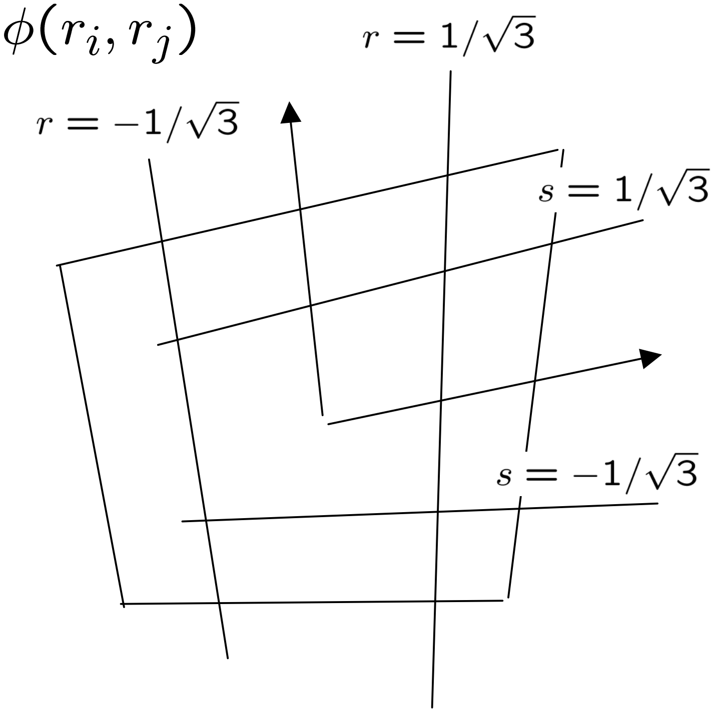
GI in 2-d

We have

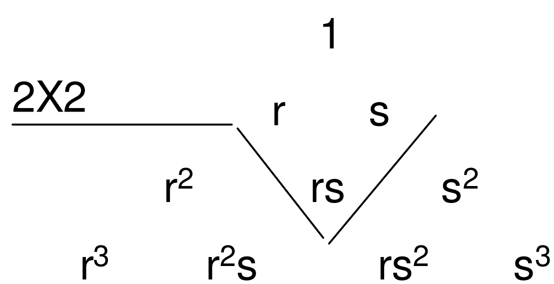
$$\boldsymbol{K}_L = \int_{V_e} \boldsymbol{B}^T \boldsymbol{C} \boldsymbol{B} t dx dy = \int_{-1}^{1}\int_{-1}^{1} \boldsymbol{B}^T \boldsymbol{C} \boldsymbol{B} t J dr ds$$

Now in 2-d rectangular domains

$$I = \int_{-1}^{1}\int_{-1}^{1} \phi(r,s) dr ds \simeq \sum_i \sum_j W_i W_j \phi(r_i, r_j)$$



For the Q4 element, 2X2 quadrature is required



Assignment 12: Formulate the stiffness matrix for an iso-parametric 4 noded quadrilateral element

```
E=1000; nu=0.3

norder=order of gauss quadrature

w(1:norder)=weights

samp(1:norder)=sampling points

locstiff[8,8]=0;

Loop over the number of gauss points lint= 1,ngauss(=norderXnorder)

  get weight(lint), r(lint), s(lint) i.e the weights and sampling point coords

  form shape function matrix at the sampling point lint

  form shape function derivatives wrt r and s at r(lint), s(lint)

  formation of the jacobian matrix at r(lint), s(lint)

  form jacobian inverse Gamma

  form det(jacob) at r(lint), s(lint) [Note 2-d jacobian is 2X2]

  form B matrix in physical space using shape function derivatives and Gamma

  locstiff=locstiff + weight(lint) B'*C*B*det(jacob)

end
```

Term paper topics

Group 1: Model the problem of a line load on a half space. Compare with the theoretical solution of stresses near the load point. Refine the mesh to see if you converge to the exact solution.

Group 2: Model a hard elliptical inclusion in a softer matrix. The matrix is infinitely large compared to the inclusion and is subjected to uniaxial stresses applied at infinity. Find out the theoretical solution to this problem and check how the strains inside the inclusion vary with its ellipticity. Change the Poisson's ratio of the matrix and the inclusion and see how the stress fields in the matrix and inclusion change.

Group 3: Model a sharp crack in a infinite body. Find out the theoretical solution to this problem. The stresses at the crack tip should be infinite. Compare with the theoretical solution and see whether with mesh refinement you can get close to the theoretical solution.

Group 4: Model the problem of a series of periodically placed holes in a thin film. This problem has already been discussed. Discuss further with Dr Ghatak.

Group 5: Model the bending of a functionally graded beam where the gradation is exponential in the depth direction. Compare with the solution for a homogeneous beam to see the differences due to the property variation.

Group 6: Model an internally pressurised hollow cylinder in plane strain. Start with a thick cylinder and using theoretical results show how, as you move towards a thin cylinder the solution changes.

Group 7: Analyse using you own code a deep beam using a structured mesh composed of linear strain and constant strain triangles. Check all stresses with theoretical estimates. Check convergence with mesh refinement.

Group 8: Verify the Euler buckling load of a slender beam using Finite Elements. Your solutions may break down at the point of buckling. How good are your estimates compared to the theoretical solutions?

Group 9: Analyse using your own code a internally pressurised plane strain thick cylinder and check the results with theoretical estimates.

Group 10: Model a infinite wedge with tractions on one of the boundaries. Verify the stress solutions near the tip of the wedge with theoretical estimates. What happens in the case of a right angled wedge subject to shear tractions on one edge?

Group 11: A beam is rotating at a fixed angular velocity about its geometric center. Find the stress distribution in the beam and compare with theoretical results.

Group 12: Solve the stresses for a 90 degree curved beam of inner radius a and outer radius b subjected to a normal sinusoidal loading $A\sin(\lambda\ \eta)$. Find solutions to the problem as λ varies from 0 to 2. Comment on the solution at $\lambda=1$;

Group 13: Solve the problem of a heavy beam on a elastic foundation. Compare your results with theoretical solutions for different beam aspect ratios. Investigate when the theoretical solution breaks down.

Group 14: An infinite plate subjected to a remote tensile load contains an elliptical hole. Using a FEM simulation determine the stress concentration at the tip of the ellipse as a function of the ellipticity of the hole. Compare with theoretical results.

Step by step guide to formulating a problem in FEA

Eg. Heat conduction in a 2-d domain

Q1. What are the governing equations and boundary conditions for the problem?

$$k\nabla^2 T(x,y,t) + Q(x,y,t) - c\rho\dot{T} = 0$$

Assumptions: Steady state $\Rightarrow \dot{T} = 0$

Boundary conditions on the surfaces:

$$\begin{aligned} T|_{S_1} &= T_e \\ k\left.\frac{\partial T}{\partial n}\right|_{S_2} &= k\left(\frac{\partial T}{\partial x}n_x + \frac{\partial T}{\partial y}n_y\right)_{S_2} = q_s \end{aligned}$$

q_s is either specified or convective b.c

$$q_s = h(T_0 - T_s)$$

Q2. What is the variable to be solved for?

T(x,y) → a scalar quantity → 1 dof per node.

Q3. Does a variational statement exist?

Check for self-adjointness → if self adjoint write variational principle

In this case self adjointness exists and hence

$$\Pi = \int_V \frac{1}{2} k \left\{ \left(\frac{\partial T}{\partial x} \right)^2 + \left(\frac{\partial T}{\partial y} \right)^2 \right\} dV - \int_{S_2} T_S q_S dS$$

is the variational principle. Thus the correct T(x,y) makes the above a minimum and also satisfies all the boundary conditions.

If variational principle does not exist, use a weighted residual method (we will learn about it later)

Q.4 Variational principle exists. Now what?

Use Virtual work principle

$$\delta\Pi = 0$$

or Rayleigh-Ritz method of 'assumed displacements'

In this case, if we take the virtual work route

$$\delta\Pi = \int_V k \left\{ T,_x \frac{\partial \delta T}{\partial x} + T,_y \frac{\partial \delta T}{\partial y} \right\} dV - \int_{S_2} \delta T q_s dS = 0$$

If we use the Rayleigh Ritz principle, we should start the discretisation right away. Will deal with that route later..

Q5. What element to choose?

Tricky. Eigenvalue analysis may help. Experience may too.

Let us choose a 4 noded iso-p quad in this case….Thus

$$T(r,s) = N_1(r,s)T_1 + N_2(r,s)T_2 + N_3(r,s)T_3 + N_4(r,s)T_4$$

Also,

$$\begin{aligned} x &= N_1 x_1 + N_2 x_2 + \ldots \\ y &= N_1 y_1 + N_2 y_2 + \ldots \end{aligned}$$

Following procedures learnt earlier

$$\begin{aligned} T &= \boldsymbol{N\theta} \\ \left\{ \begin{array}{c} T,_x \\ T,_y \end{array} \right\} &= \boldsymbol{B\theta} \\ \left\{ \begin{array}{c} \delta T,_x \\ \delta T,_y \end{array} \right\} &= \boldsymbol{B}\delta\boldsymbol{\theta} \end{aligned}$$

Now get the local stiffness matrix:

$$T,_x \frac{\partial \delta T}{\partial x} + T,_y \frac{\partial \delta T}{\partial y} = \delta \boldsymbol{\theta}^T \boldsymbol{B}^T \boldsymbol{B} \boldsymbol{\theta}$$

Also,

$$\delta T q_s = \delta \boldsymbol{\theta}^T \boldsymbol{N}^T h \boldsymbol{N} \boldsymbol{\theta} - \delta \boldsymbol{\theta}^T \boldsymbol{N}^T h T_0$$

Thus

$$\boldsymbol{K}_L = k \int_V \boldsymbol{B}^T \boldsymbol{B} dV$$

What if we used the Rayleigh Ritz technique?

Suppose we use a 3-noded triangle. Then with a triangle

$$T(x,y) = a_0 + a_1 x + a_2 y$$

Thus

$$\begin{aligned} \frac{\partial T}{\partial x} &= a_1 \\ \frac{\partial T}{\partial y} &= a_2 \end{aligned}$$

$$\Pi = \sum_{i=1}^{N_e} \left[\int_{V_e} \frac{1}{2} k \left(a_1^2 + a_2^2 \right) dV - \int_{S_2} (a_0 + a_1 x + a_2 y) q_s dS \right]$$

$$\frac{\partial \Pi}{\partial a_i} = 0$$

$$\boldsymbol{\theta} = \boldsymbol{A}\boldsymbol{a}$$

$$T(x,y) = \langle 1 \;\; x \;\; y \rangle \boldsymbol{A}^{-1}\boldsymbol{\theta}$$

$$\left\{ \begin{array}{c} \frac{\partial T}{\partial x} \\ \\ \frac{\partial T}{\partial y} \end{array} \right\} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \boldsymbol{A}^{-1}\boldsymbol{\theta}$$

Thus first term in Π

$$k \int_{V_e} \boldsymbol{A}^{-T} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \boldsymbol{A}^{-1} dV \left\{ \begin{array}{c} T_1 \\ T_2 \\ T_3 \end{array} \right\}$$

So,

$$\frac{\partial \Pi}{\partial T_i} = 0$$

Define geometry

Mesh geometry

Loop over elements

  Form local stiffness

  Form surface force vector if required

  Assemble stiffness

  Assemble forces

end

Apply essential boundary conditions

Solve

Post process results

Method of weighted residuals

pde

$$\mathcal{L}u - f = 0$$

Approximate solution

$$\mathcal{L}\widehat{u} - f = R$$

The best approximation is the one that gives

$$\int_V w_i R = 0 \quad i = 1, \ldots n$$

weights

Example: beam bending

Governing equation

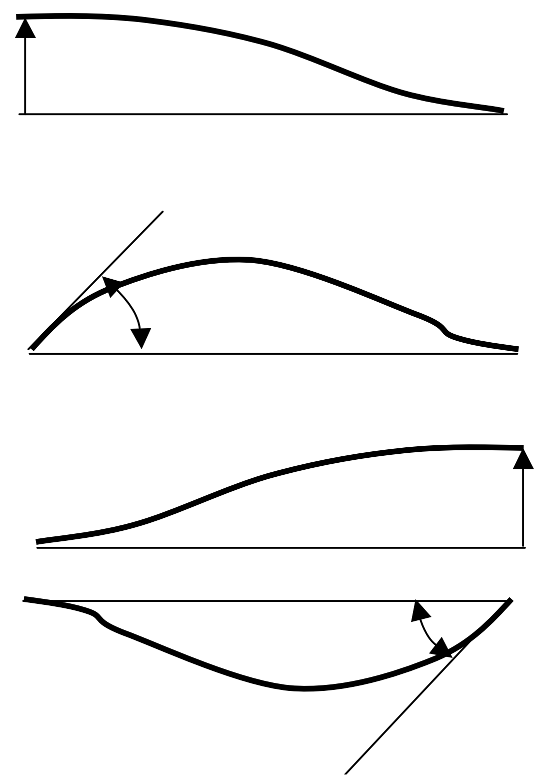
$$EI\frac{d^4w}{dx^4} = 0$$

Boundary conditions at the ends

$$EIw,_{xx} - M_B = 0 \quad \text{and} \quad EIw,_{xxx} - V_B = 0$$

FEM approximation

$$\hat{w}(x) = \boldsymbol{N}\begin{Bmatrix} w_1 \\ \theta_{z1} \\ w_2 \\ \theta_{z2} \end{Bmatrix} = \boldsymbol{NU}$$



$$N_1 = 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3} \quad N_2 = x - \frac{2x^2}{L} + \frac{x^3}{L^2}$$

$$N_3 = \frac{3x^2}{L^2} - \frac{2x^3}{L^3} \quad N_4 = -\frac{x^2}{L} + \frac{x^3}{L^2}$$

Galerkin weighted residual scheme:

$$\int_0^L N_i EI \widehat{w}_{,xxxx} dx = 0 \text{ for } i = 1, \ldots 4$$

$$\begin{aligned} EI \int_0^L N_i \widehat{w}_{,xxxx} dx &= EI \left\{ N_i \widehat{w}_{,xxx} \Big|_0^L - \int_0^L \frac{dN_i}{dx} \widehat{w}_{,xxx} dx \right\} \\ &= EI \left\{ \left[N_i \widehat{w}_{,xxx} - \frac{dN_i}{dx} \widehat{w}_{,xx} \right]_0^L + \int_0^L \frac{d^2 N_i}{dx^2} \widehat{w}_{,xx} dx \right\} \\ &= \left[N_i V_B - \frac{dN_i}{dx} M_B \right]_0^L + EI \int_0^L \frac{d^2 N_i}{dx^2} \widehat{w}_{,xx} dx \end{aligned}$$

$$\boldsymbol{K}_L = \int_0^L \boldsymbol{B}^T EI \boldsymbol{B} dx$$

where

$$\boldsymbol{B} = \boldsymbol{N}_{,xx}$$

Thus after assembly

$$\boldsymbol{KU} = \boldsymbol{R}$$

Example 2: Heat transfer in 2-d

In V

$$\frac{\partial}{\partial x}(k_x T,_x) + \frac{\partial}{\partial y}(k_x T,_y) + Q(x,y) = 0$$

On S

$$n_x k_x T,_x + n_y k_y T,_y - q = 0$$

Discretisation

$$\hat{T} = \langle N_1 \;\; N_2 \;\; \ldots \rangle \left\{ \begin{array}{c} T_1 \\ T_2 \\ \vdots \end{array} \right\} = \boldsymbol{N\theta}$$

For Galerkin scheme consider

$$\int_V N_i \frac{\partial}{\partial x}(k_x \hat{T},_x) dV = -\int_V \frac{\partial N_i}{\partial x} k_x \hat{T},_x dV + \int_S N_i k_x \hat{T},_x n_x dS$$

Also,

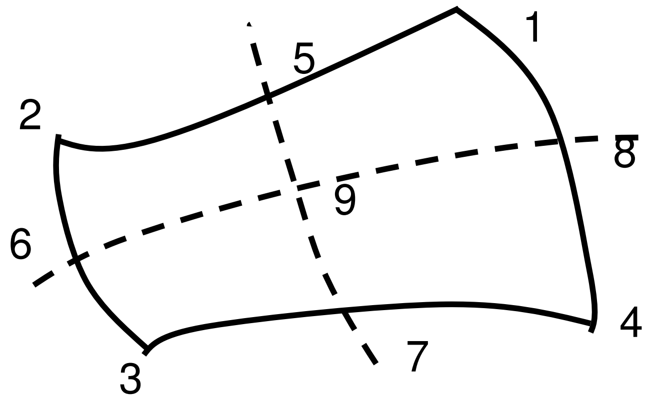
$$\int_V N_i \frac{\partial}{\partial y}(k_y \widehat{T},_y) dV = -\int_V \frac{\partial N_i}{\partial y} k_y \widehat{T},_y dV + \int_S N_i k_y \widehat{T},_y n_y dS$$

Thus we end up with

$$\boldsymbol{K}_L \boldsymbol{\theta} = \boldsymbol{f}_L$$

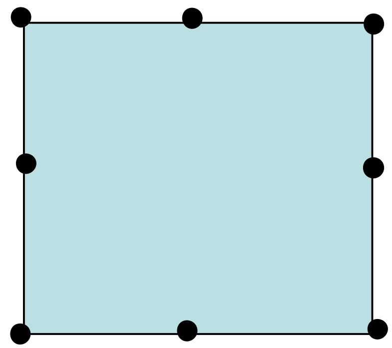
and both $\boldsymbol{K}_L$ and $\boldsymbol{f}_L$ have the usual forms.

Some special elements: singular elements



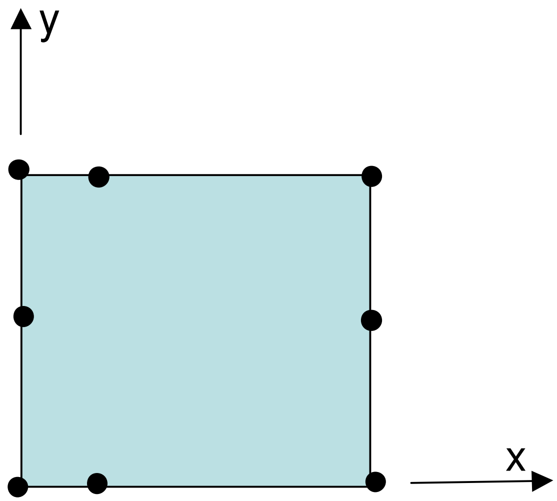
Include if node I is defined

		i=5	i=6	i=7	i=8	i=9
$N_1 =$	$\frac{1}{4}(1+r)(1+s)$	$-\frac{1}{2}N_5$	$\ldots$	$\ldots$	$-\frac{1}{2}N_8$	$-\frac{1}{4}N_9$
$N_2 =$	$\frac{1}{4}(1-r)(1+s)$	$-\frac{1}{2}N_5$	$-\frac{1}{2}N_6$	$\ldots$	$\ldots$	$-\frac{1}{4}N_9$
$N_3 =$	$\frac{1}{4}(1-r)(1-s)$	$\ldots$	$-\frac{1}{2}N_6$	$-\frac{1}{2}N_7$	$\ldots$	$-\frac{1}{4}N_9$
$N_4 =$	$\frac{1}{4}(1+r)(1-s)$	$\ldots$	$\ldots$	$-\frac{1}{2}N_7$	$-\frac{1}{2}N_8$	$-\frac{1}{4}N_9$
$N_5 =$	$\frac{1}{2}(1-r^2)(1+s)$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$-\frac{1}{4}N_9$
$N_6 =$	$\frac{1}{2}(1-s^2)(1-r)$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$-\frac{1}{4}N_9$
$N_7 =$	$\frac{1}{2}(1-s^2)(1+r)$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$-\frac{1}{4}N_9$
$N_8 =$	$\frac{1}{2}(1-s^2)(1+r)$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$-\frac{1}{4}N_9$
$N_9 =$	$\frac{1}{2}(1-r^2)(1-s^2)$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$



On $s = 1$ shape functions for nodes 1,2,5 are

$$
\begin{aligned}
N_1 &= \frac{1}{2}(1+r) - \frac{1}{2}(1-r^2) = -\frac{1}{2}r(1-r) \\
N_2 &= \frac{1}{2}r(1-r) \\
N_5 &= 1 - r^2
\end{aligned}
$$



Thus, for the quarter point element

$$x = -\frac{1}{2}r(1-r)x_1 + \frac{1}{2}(1-r)x_2 + (1-r^2)x_5 = \frac{1}{2}r(1-r)L + (1-r^2)\frac{L}{4}$$

Solving for r

$$r = -1 + 2\sqrt{\frac{x}{L}}$$

Thus

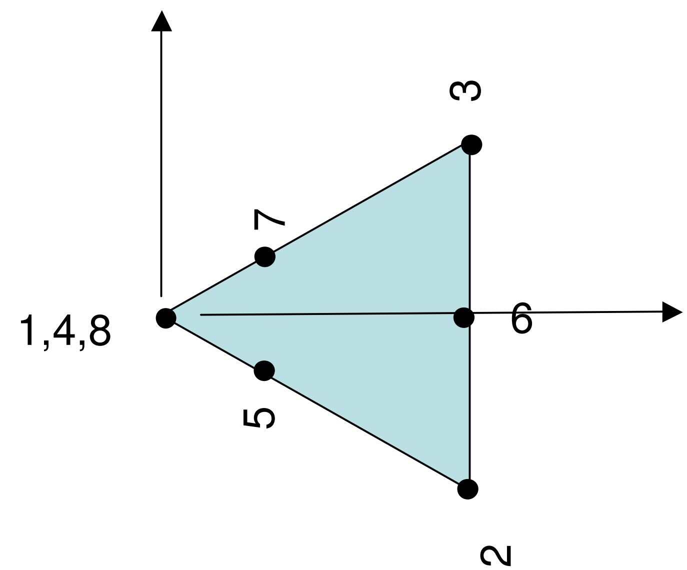
$$\frac{\partial x}{\partial r} = \frac{L}{2}(1+r) = \sqrt{xL}$$

Vanishes at x=0 → Jacobian is singular at x=0

Along the edge 1-2-5

$$u = -\frac{1}{2}\left(-1+2\sqrt{\frac{x}{L}}\right)\left(2-2\sqrt{\frac{x}{L}}\right)u_1 + \left(-1+2\sqrt{\frac{x}{L}}\right)\left(2\sqrt{\frac{x}{L}}\right)u_2 + 4\left(\sqrt{\frac{x}{L}}-\frac{x}{L}\right)u_5$$

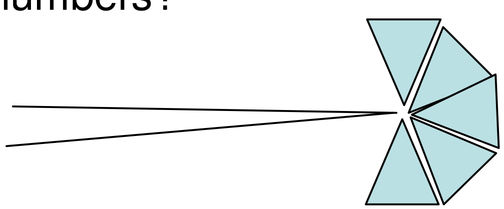
Along this edge $u \sim \sqrt{x}$ and so $\epsilon_x \sim 1/\sqrt{x}$



Singularity exists all over the element.

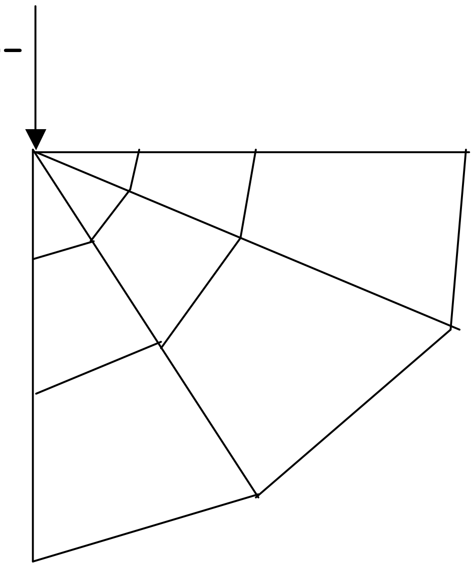
Eg. Show that singularity exists along x axis.

What happens if 1,4,8 are given different node numbers?



Infinite Finite elements

In the 1-d case, $x \to \infty$ as $r \to 1$. Thus mapping functions

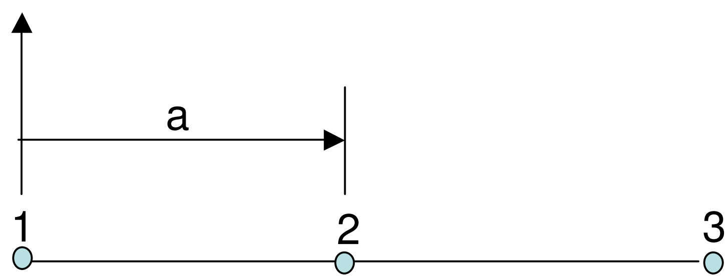


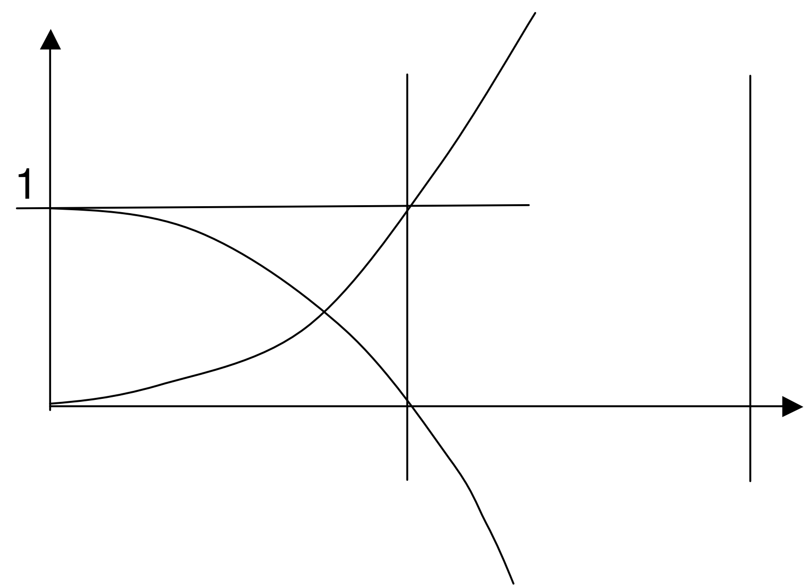
$$x = M_1 x_1 + M_2 x_2 + M_3 x_3$$

$\Rightarrow$

$$M_1 = -\frac{2r}{1-r}$$

$$M_2 = \frac{1+r}{1-r}$$





Also,

$$\phi = N_1\phi_1 + N_2\phi_2 + N_3\phi_3$$

where

$$\begin{aligned} N_1 &= \frac{1}{2}r(r-1) \\ N_2 &= 1 - r^2 \\ N_3 &= \frac{1}{2}r(1+r) \end{aligned}$$

So,

$$\frac{d\Phi}{dx} = \frac{dr}{dx}\frac{d}{dr}\boldsymbol{N}\boldsymbol{\Phi} = \frac{1}{J}\boldsymbol{B}\boldsymbol{\Phi}$$

Again,

$$J = \frac{dM_1}{dr}x_1 + \frac{dM_2}{dr}x_2$$

In the 1-d bar chosen, $x_1 = 0$, $x_2 = a$. Thus

$$x = \frac{1+r}{1-r}a$$

$\Rightarrow$

$$r = \frac{x-a}{x+a}$$

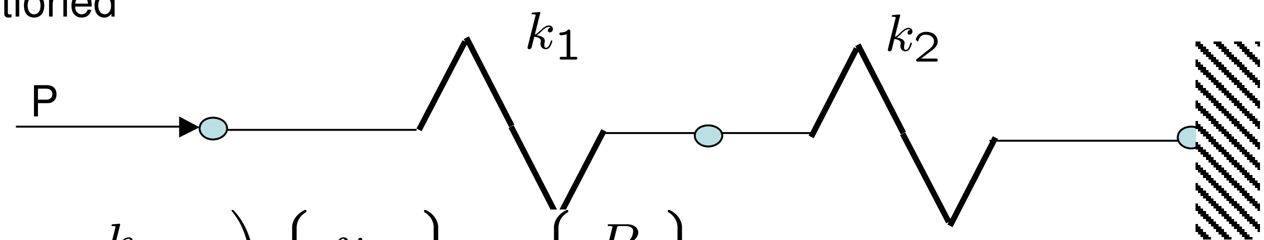
$$\begin{aligned} \lim_{x\to\infty} N_1 &= 0 \\ \lim_{x\to\infty} N_2 &= 0 \\ \lim_{x\to\infty} N_3 &= 1 \end{aligned}$$

Thus, as $x \to \infty$

$$\phi \to \phi_3$$

Errors in FE analysis

Round off errors in ill-conditioned systems



$$\begin{pmatrix} k_1 & -k_1 \\ -k_1 & k_1 + k_2 \end{pmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} P \\ 0 \end{Bmatrix}$$

$$u_2 = \frac{P}{(k_1 + k_2) - k_1}$$

Suppose $k_1 = 1.000000$ and $k_2 = 5.555555 \times 10^{-6}$ $\Rightarrow$ Denominator $\simeq 5 \times 10^{-6}$ if the computer stores upto 7 digits.

Condition number $C = \lambda_{max}/\lambda_{min}$
High condition number $\Rightarrow$ Ill-conditioning

Discretisation error

For the 1-d case:

$$e(x) = u(x) - u^h(x) = u(x) - \left[u_i \left(1 - \frac{x - x_i}{h_i} \right) + u_{i+1} \left(\frac{x - x_i}{h_i} \right) \right]$$

At the nodes: $e(x) = 0$

Displacement error:

For $x_i \leq z \leq x_{i+1}$

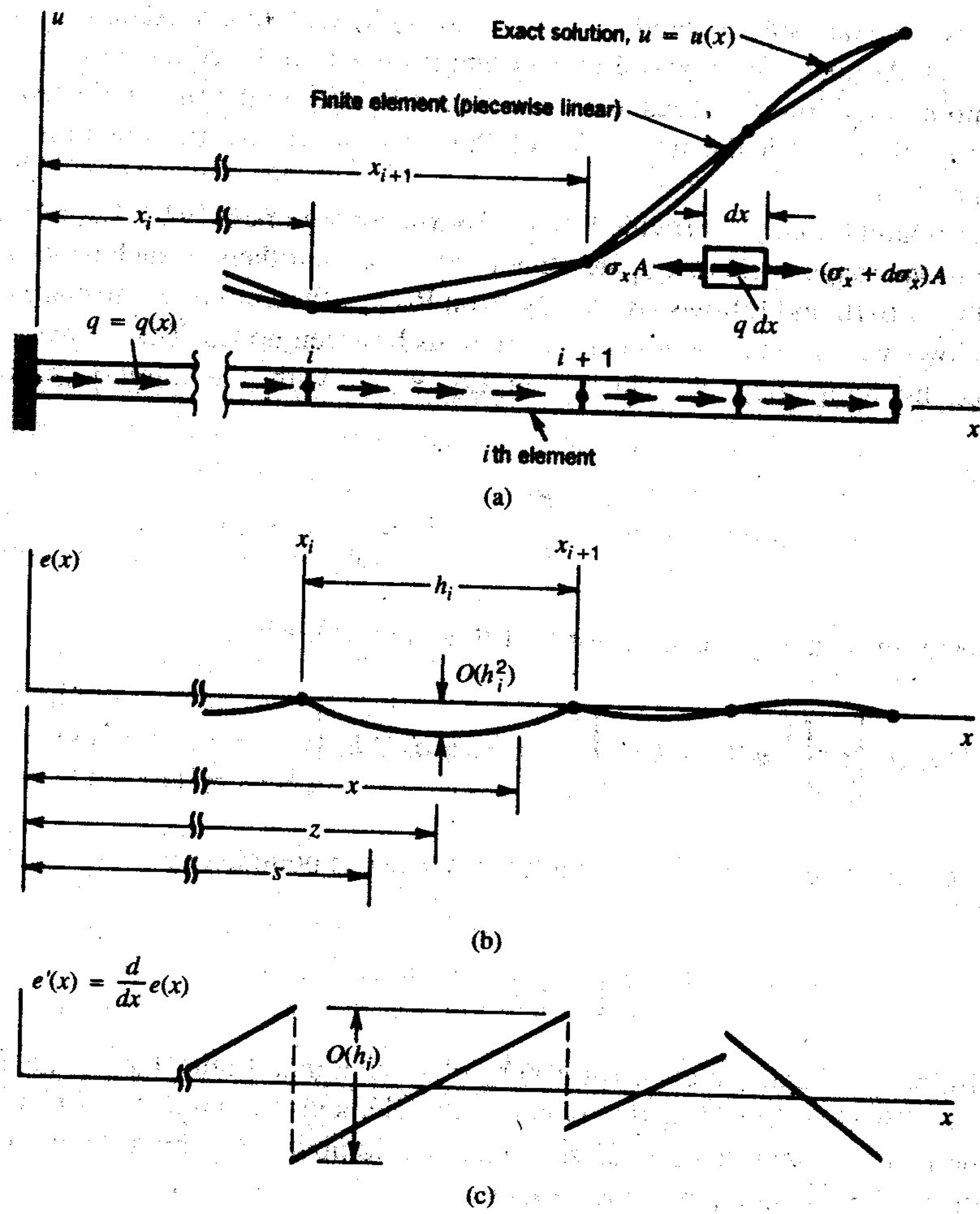
$$e(x_i) = e(z) + (x_i - z)e'(z) + \frac{1}{2}(x_i - z)^2 e''(z)$$

max(e) occurs when $e'(x) = 0$. Thus

$$e(z) = -\frac{1}{2}(x_i - z)^2 u''(z)$$

And in general as $|z - x_i| \leq h_i/2$

$$e(x) \leq \frac{1}{8} h_i^2 \left(\text{max}|u''| \right)$$



From Cook, Malkaus and Plesha (2002)

Strain error:

$$e'(x) - e'(z) = e'(x) = \int_z^x u''(s)ds$$

Thus

$$|e'(x)| \leq h_i \left(\max|u''(x)|\right)$$

In a linear element, displacement error is proportional to the square of element size while strain error is linearly proportional to element size.

Displacements are more accurate near the nodes, strains are accurate inside elements.

If a complete polynomial of order p is used to interpolate,
Error in field quantity: $\mathcal{O}(h^{p+1})$
Error in the rth derivative: $\mathcal{O}(h^{p+1-r})$
Error in energy density: $\mathcal{O}(h^{2(p+1-m)})$
where derivatives of the order m appear in the definition of energy density.

Eg. For a problem meshed with 3 noded triangles, p=1

Thus error in energy O(h^2)

Change to 6 noded triangle: p=2 and energy error O(h^4)

Reduce element size by half: error reduced by a factor of 4.