

Chapter 1

Introduction



Newton's Laws

Newton's three laws:

First law (Law of inertia)[C]: It is always possible to find a coordinate system with respect to which isolated bodies move uniformly. This coordinate system is called an *inertial reference frame*.

The usual statement “A body remains in its state of rest or in uniform linear motion as long as no external force acts on it to change” is not suitable. In an accelerating frame, a force-free body will appear to accelerate.

Inertial reference frame: In which the law of inertia is valid. Any reference frames moving with a constant velocity wrt stationary frame is a inertial reference frame.

Examples: free space, a freely falling elevator.

Second law (C): The change in the linear momentum of a body is proportional to the external force acting on the body and is in the direction of the force.

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}$$

For a point particle: $\mathbf{p} = m\mathbf{v} = m\dot{\mathbf{r}}$

For a rigid body: linear momentum of the CM.

$$\frac{d^2\mathbf{r}}{dt^2} = \mathbf{F}$$

Third law (C): If a given body A exerts a force on another body B, then body B also exerts a force on body A with equal magnitude but opposite in direction.

Not always valid. If the Sun moved by a distance, then the action and reaction with Earth will not take on the new values until the signal reaches the Earth. This is because the Newtonian dynamics assumes instantaneous propagation of signal (action at a distance).

Main points: Second order differential equation.

Other assumptions:

Absolute space: Motion takes place in the background of absolute space. The space is same for all observers.

Consequence: The length of a rod is the same for all observers.

Absolute Time: Evolution of physical system in “time”. Time moves uniformly with equal rate for all observers.

Consequence: Time interval between two events is the same for all observers.

Mass conservation

Infinite speed

Examples:

Oscillator: Equation

$$m\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = F_0 \cos \omega_f t$$

(1) Linear frictionless oscillator with the initial condition is

$$[x(0), v(0)]: x(t) = x(0)\cos \omega t + \frac{v(0)}{\omega} \sin \omega t$$

(2) Solve $\ddot{x} = x$: $x(t) = x(0)\cosh t + v(0)\sinh t$

Exercises:

HW 1.1 (1) Solve for the position and velocity of a damped linear oscillator whose mass $m=1$, the viscous coefficient is γ . The initial condition is $[x(0), v(0)]$.

(2) Do the same for a forced undamped oscillator with initial condition is $[x(0), v(0)]$. Solve for both cases: $\omega_0 \neq \omega_f$ and $\omega_0 = \omega_f$

(3) A spherical ball of mass m falls under gravity in a viscous fluid. Find the position and velocity of the ball as a function of time. Assume that the mass starts at rest from a height h above the ground. Apply the above solution to a raindrop whose radius is 1 mm. Assume the dynamic viscosity of air to be 10^{-5} kg/(m s).

Energy

Work energy theorem:

The work done on a particle by a force \mathbf{F} as it moves from point A to point B :

$$W_{AB} = \int_A^B \mathbf{F} \cdot d\mathbf{r}$$

which can be rewritten as

$$[1] \quad W_{AB} = \int_A^B m \frac{d\mathbf{v}}{dt} \cdot \frac{d\mathbf{r}}{dt} dt = \left[\frac{1}{2} m v^2 \right]_A^B = T_B - T_A$$

where T_A, T_B are the kinetic energy of the particle at the points A and B respectively.

The above statement is *work energy theorem*: The difference in the kinetic energy of a particle between two points is equal to the net work done on the particle by the external forces during the transit.

Conservative force fields and Potentials:

Conservative force fields: Fields for which $\oint \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed path. For such fields, $\int_P^Q \mathbf{F} \cdot d\mathbf{r}$ is only function of the end points, but remain independent of paths. Therefore, we choose P as the reference point and define potential energy $U(\mathbf{r})$ as

$$U(\mathbf{r}) = - \int_P^{\mathbf{r}} \mathbf{F}(\mathbf{r}') \cdot d\mathbf{r}'.$$

Therefore,

$$[2] \quad \int_A^B \mathbf{F} \cdot d\mathbf{r} = \int_P^B \mathbf{F} \cdot d\mathbf{r} - \int_P^A \mathbf{F} \cdot d\mathbf{r} = -U_B + U_A$$

Eqs. [1,2] yield

$$T_A + U_A = T_B + U_B,$$

The sum of kinetic and potential energies is the total energy, and the above statement is the *conservation of total energy* of a mechanical systems that are conservative.

Example:

(1) Spring force $F = -kx$. Therefore

$$U(x) = - \int_{x_0}^x (-kx)dx = \frac{1}{2}kx^2 - \frac{1}{2}kx_0^2$$

Hence, the potential energy is $U(x) = kx^2/2$ apart from a constant, and the total energy

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

(2) For gravitational force $\mathbf{F} = -GMm/r^2\hat{\mathbf{r}}$,

$$U(B) - U(A) = - \int_A^B \mathbf{F} \cdot d\mathbf{r} = - \int_A^B F_r dr = -GMm \left(\frac{1}{r_B} - \frac{1}{r_A} \right)$$

By choosing $r_A = \infty$, we deduce

$$U(r) = - \frac{GMm}{r}$$

Exercises:

HW 1.2

(1) A charged particle $+Q$ is fixed at the origin. A test charged particle $+q$ of mass m is fired head-on towards the charged particle $+Q$ with velocity v_∞ from ∞ . Compute the shortest distance of approach between the particles.

HW 1.3

(2) Four immovable charges of magnitude $+Q$ are placed at the vertices of a square. A test charge $+q$ is moving in the field of

these charges. (a) Find the position(s) of equilibrium point(s). (b) Compute the potential near the equilibrium point. (c) What is the nature of the equilibrium point? Describe the motion of the test particle near this point. (d) Does your analysis depend on the sign of the test charge?

(3) What is the potential of an electric dipole? Compute the electric field induced by it. Sketch the potential.

(4) Philip Morse modelled the potential energy of a diatomic molecule using a potential energy function:

$U(r) = D_e \{1 - \exp[-a(r - r_e)]\}^2$ where r is the distance between the atoms of the dipolar molecule r_e is the equilibrium bond distance and D_e is the depth of the potential well. (a) Sketch the potential. (b) Find the minima of the potential. (c) Expand the potential near its minimum and compute the frequency of oscillation.

System of Particles and Conservation Laws

1. Linear Momentum

Centre of mass:

$$\mathbf{R}_{CM} = \frac{\sum m_a \mathbf{r}_a}{\sum m_a}$$

a : Particle label

CM reference frames: Frame in which $\mathbf{R}_{CM} = 0$.

Linear momentum of a system:

$$\mathbf{P} = \sum m_a \mathbf{v}_a = \sum m_a (\mathbf{V}_{CM} + \mathbf{v}'_a) = M \mathbf{V}_{CM} = \mathbf{P}_{CM}$$

Equation of motion:

$$\dot{\mathbf{P}} = M \frac{d\mathbf{V}_{CM}}{dt} = \sum_a m_a \dot{\mathbf{v}}_a = \sum \mathbf{f}_a$$

\mathbf{f}_a is the net force on the a -th particle.

$$\mathbf{f}_a = \mathbf{f}_{a,\text{ext}} + \mathbf{f}_{a,\text{int}} = \mathbf{f}_{a,\text{ext}} + \sum_b \mathbf{f}_{a,b}$$

where $\mathbf{f}_{a,b}$ is the force on the a -th particle due to the b -th particle.

$$M \mathbf{a}_{CM} = \sum_a \mathbf{f}_a = \sum_a \mathbf{f}_{a,\text{ext}} + \sum_a \sum_b \mathbf{f}_{a,b}$$

From Newton's third law:

$$\mathbf{f}_{a,b} + \mathbf{f}_{b,a} = 0$$

Therefore,

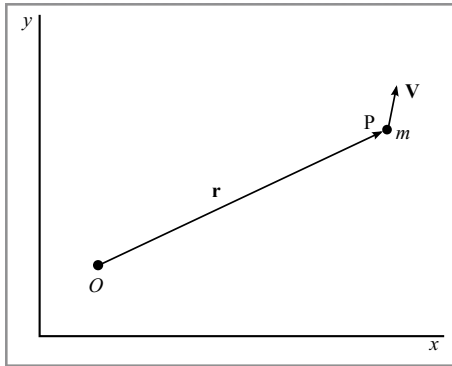
$$\frac{d\mathbf{P}}{dt} = \sum_a \mathbf{f}_{a,\text{ext}} = \mathbf{F}_{\text{ext}}$$

If $\mathbf{F}_{\text{ext}} = 0$, then the total linear momentum of the system \mathbf{P} is a constant. This is the statement of the *conservation of linear momentum*.

Example: Two masses (m) connected by a spring. Example Ch

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2. Angular Momentum of a Single Particle



We define the angular momentum of the particle about a reference point O , where $OP = \mathbf{r}$, as

$$\mathbf{L} = \mathbf{r} \times \mathbf{p},$$

where $\mathbf{p} = m\dot{\mathbf{r}}$ is the linear momentum of the particle measured in the reference frame. The above formulas is valid in any reference frame, inertial or noninertial. Now.

$$\frac{d\mathbf{L}}{dt} = \mathbf{r} \times \frac{d\mathbf{p}}{dt} + \frac{d\mathbf{r}}{dt} \times \mathbf{p} = \mathbf{r} \times \frac{d\mathbf{p}}{dt} = \mathbf{r} \times \mathbf{F} = \mathbf{N}$$

where \mathbf{N} is the torque on the particle about the reference point O .

If the torque, \mathbf{N} , on the particle is zero, then the angular momentum \mathbf{L} of the particle is a constant.

This law is called *conservation of angular momentum*.

3. Angular Momentum of a System of Particles

The Angular momentum of a system of particles is a sum of angular momentum of individual particles. The angular momentum of such system about a reference point is

$$\mathbf{L} = \sum_a \mathbf{r}_a \times \mathbf{p}_a,$$

where \mathbf{r}_a denotes the position vector of the a th particle from the reference point, and \mathbf{p}_a its linear momentum. Using the CM coordinates, $\mathbf{r} = \mathbf{R}_{\text{CM}} + \mathbf{r}'_a$ and $\mathbf{v} = \mathbf{V}_{\text{CM}} + \mathbf{v}'_a$:

$$\begin{aligned} \mathbf{L} &= \sum_a \mathbf{r}_a \times \mathbf{p}_a \\ &= \sum_a m_a (\mathbf{R}_{\text{CM}} + \mathbf{r}'_a) \times (\mathbf{V}_{\text{CM}} + \mathbf{v}'_a) \\ &= \mathbf{R}_{\text{CM}} \times \mathbf{P}_{\text{CM}} + \sum_a \mathbf{r}'_a \times \mathbf{p}'_a \end{aligned}$$

Therefore,

$$\mathbf{L} = \mathbf{R}_{\text{CM}} \times \mathbf{P}_{\text{CM}} + \mathbf{L}_{\text{CM}} = \mathbf{L}_{\text{orbit}} + \mathbf{L}_{\text{CM}},$$

where

$$\mathbf{L}_{\text{CM}} = \sum \mathbf{r}'_a \times \mathbf{p}'_a$$

is the angular momentum of the body about the CM,

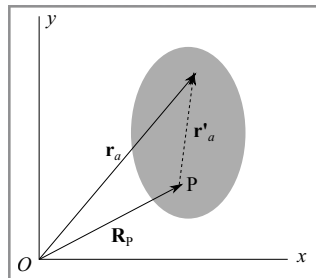
4. Equation of Motion Using Angular Momentum

We assume that xy coordinate system of the figure is an inertial reference frame, and the reference point O belongs to it. The angular momentum of the body about O is

$$\mathbf{L}_O = \sum_a m_a \mathbf{r}_a \times \mathbf{p}_a.$$

We also compute angular momentum \mathbf{L}_P about another point P shown in the following figure:

$$\mathbf{L}_P = \sum_a m_a \mathbf{r}'_a \times \mathbf{p}'_a = \sum_a m_a (\mathbf{r}_a - \mathbf{R}_P) \times (\dot{\mathbf{r}}_a - \dot{\mathbf{R}}_P),$$



The time derivative of \mathbf{L}_P is

$$\frac{d\mathbf{L}_P}{dt} = \sum_a m_a (\mathbf{r}_a - \mathbf{R}_P) \times (\ddot{\mathbf{r}}_a - \ddot{\mathbf{R}}_P).$$

Using $m_a \ddot{\mathbf{r}}_a = \mathbf{f}_a$ is the net force acting on the a th particle, we obtain

$$\frac{d\mathbf{L}_P}{dt} = \sum_a (\mathbf{r}_a - \mathbf{R}_P) \times (\mathbf{f}_a - m_a \ddot{\mathbf{R}}_P).$$

The term $\sum_a (\mathbf{r}_a - \mathbf{R}_P) \times (-m_a \ddot{\mathbf{R}}_P)$ can be interpreted as the torque on the rigid body due to the pseudo force $-m_a \ddot{\mathbf{R}}_P$. The force \mathbf{f}_a is a sum of external and internal forces, i.e.,

$$\mathbf{f}_a = \mathbf{f}_{a,\text{ext}} + \sum_b \mathbf{f}_{a,b}$$

where $\mathbf{f}_{a,\text{ext}}$ is the external force on the a th particle, while $\mathbf{f}_{a,b}$ is the internal force on the a th particle due to the b th particle as shown in Fig. 15.4. Therefore,

$$\frac{d\mathbf{L}_P}{dt} = \sum_a (\mathbf{r}_a - \mathbf{R}_P) \times (\mathbf{f}_{a,\text{ext}} - m_a \ddot{\mathbf{R}}_P) + \frac{1}{2} \sum_a \sum_b (\mathbf{r}'_a \times \mathbf{f}_{a,b} + \mathbf{r}'_b \times \mathbf{f}_{b,a})$$

where $\mathbf{r}'_a = \mathbf{r}_a - \mathbf{R}_P$. According to Newton's third law, $\mathbf{f}_{a,b} = -\mathbf{f}_{b,a}$.

Therefore,

$$[3.1] \quad \frac{d\mathbf{L}_P}{dt} = \sum_a (\mathbf{r}_a - \mathbf{R}_P) \times (\mathbf{f}_{a,\text{ext}} - m_a \ddot{\mathbf{R}}_P) + \frac{1}{2} \sum_a \sum_b (\mathbf{r}'_a - \mathbf{r}'_b) \times \mathbf{f}_{a,b}$$

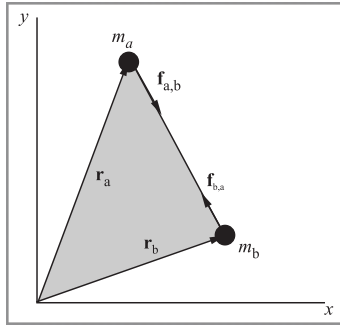


Figure 15.4 Internal forces between particles a and b .

Now we apply the *strong form of Newton's third law*, according to which the action and reaction forces are not only equal and opposite, but they are also along the line joining the two particles. Under this condition, the total internal torque (the second term of Eq. [3.1]) vanishes. Therefore,

$$\frac{d\mathbf{L}_P}{dt} = \sum_a (\mathbf{r}_a - \mathbf{R}_P) \times \mathbf{f}_{a,\text{ext}} - M(\mathbf{R}_{\text{CM}} - \mathbf{R}_P) \times \ddot{\mathbf{R}}_P \quad \text{or}$$

$$\frac{d\mathbf{L}_P}{dt} = \mathbf{N}_{\text{ext}} - M(\mathbf{R}_{\text{CM}} - \mathbf{R}_P) \times \ddot{\mathbf{R}}_P,$$

where \mathbf{N}_{ext} is the external torque acting on the rigid body about the point P , and the second term of the above equation is the torque about the point P induced by the pseudo force $-M\ddot{\mathbf{R}}_{\text{CM}}$. Note that the second term vanishes for one of the following cases:

- (1) $\mathbf{R}_P = \mathbf{R}_{\text{CM}}$, i.e., CM is the reference point P . This result is due to the fact that the torque induced by a pseudo force wrt the CM is zero.
- (2) $\ddot{\mathbf{R}}_P = 0$, or the reference point P is not accelerating. Naturally, the pseudo force is zero for this case.
- (3) $(\mathbf{R}_{\text{CM}} - \mathbf{R}_P) \parallel \ddot{\mathbf{R}}_P$, a case rarely used in solving problems on rigid bodies.

Hence, using the CM or any non-accelerating point as a reference, we obtain

$$\frac{d\mathbf{L}}{dt} = \mathbf{N}_{\text{ext}}. \quad (15.3.10)$$

$\mathbf{N}_{\text{ext}} = 0$, then \mathbf{L} is a constant. That is,

If the net torque on a system of particles is zero, then the total angular momentum of the system is conserved.

Rigid body

Solving For Motion Of A Rigid Body

The equation of motion for the centre of mass of the rigid body in an inertial frame is

$$\frac{d\mathbf{P}_{\text{CM}}}{dt} = \mathbf{F}_{\text{ext}} \quad (1)$$

where \mathbf{P}_{CM} is the linear momentum of the CM, and \mathbf{F}_{ext} is the external force acting on the rigid body.

The equation of motion for the angles of the rigid body is

$$\frac{d\mathbf{L}}{dt} = \mathbf{N}_{\text{ext}}, \quad (2)$$

where \mathbf{L} is the angular momentum of the rigid body about a reference point, and \mathbf{N}_{ext} is the external torque acting on the rigid body about the same reference point.

It is important to note that computation of \mathbf{L} and \mathbf{N}_{ext} requires a reference point. The reference point can be either the centre of mass or a non-accelerating point of the rigid body.

Rotation About a Single Axis

Example 1: A cylinder rolls down an incline without slipping. Describe the motion of the cylinder.

Solution: A cylinder is rolling down the inclined plane without slipping. The forces acting on the rigid body are gravity and friction. The frictional force opposes the tendency to slip, hence the frictional force is along $-\hat{x}$.

For rolling without slipping, it is convenient to use the clockwise rotation as positive. We denote the velocity of the CM by v , and the angular velocity of the cylinder along $-\hat{z}$ by Ω . Note that Ω is the same for all the points on the cylinder. Let a be the acceleration of the cylinder, and α its angular acceleration. We take the CM of the cylinder to be the reference point. We can do so even though the CM is accelerating because the pseudo force passes through the CM, and it causes no additional torque.

Therefore, the equation of motion of the cylinder is

$$ma = mg \sin \theta - f, \quad (1)$$

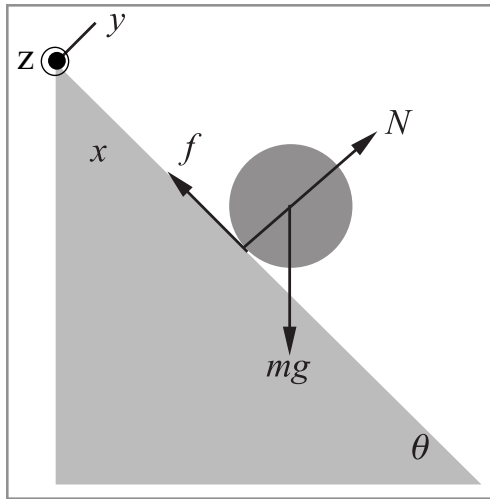
$$I\alpha = fR. \quad (2)$$

The constraint that the cylinder rolls down without slipping yields

$$v = \omega R \quad (3)$$

By taking a time derivative of Eq. (3) we obtain

$$a = \alpha R \quad (4)$$



Now we have three equations (1,2,4), and three unknowns a , α , and f . Solving these equations yields

$$a = \frac{g \sin \theta}{1 + k} \quad (5)$$

and

$$f = \frac{Ia}{R^2} = mg \sin \theta \frac{k}{1 + k}. \quad (6)$$

where $k = I/(mR^2)$. For the cylinder $I = MR^2/2$, hence $k = 1/2$ and $a = (2g/3)\sin \theta$.

$$X_{\text{CM}} = \frac{1}{2}at^2, \quad (7)$$

$$\phi = \frac{1}{2}\alpha t^2, \quad (8)$$

A block slides with an acceleration $a = g \sin \theta$.

Note that $f \leq \mu N$, where μ is the coefficient of friction, and N is the normal force. If the frictional force cannot support rolling, then the ball will slip and roll, and we cannot apply the condition $a = \alpha R$.

For rolling without slipping, the frictional force does not do any work ($\int \mathbf{f} \cdot d\mathbf{r} = 0$) because the contact point does not slip ($d\mathbf{r} = 0$).

Since the frictional force does not dissipate any energy, we can also solve the above problem by applying conservation of total energy:

$$mgx \sin \theta = \frac{1}{2}mV_{\text{CM}}^2 + \frac{1}{2}I\Omega^2, \quad (9)$$

with $V_{\text{CM}} = \Omega R$. The LHS is the loss in potential energy, and the RHS is the gain in KE. The time derivative of the above equation yields the same acceleration as Eq. (5).

In the reference frame xyz , acceleration of any point on the cylinder is $a_p = a - \alpha r = \alpha R - \alpha r$, where r is the distance of the point P from the axis of the cylinder. Clearly the bottom-most point (the contact point) of the cylinder has $a_p = 0$. As a consequence, we can choose the bottom-most point of the cylinder as the reference point since it is a non-accelerating point of the rigid body. The equation for motion for the cylinder about the contact point is

$$I'\alpha = (mg \sin \theta)R \quad (10)$$

with $I' = I + MR^2$. Hence

$$\alpha = \frac{g \sin \theta}{R(1+k)}, \quad (11)$$

$$a = \frac{g \sin \theta}{1+k}, \quad (12)$$

which is consistent with Eq. (5).

We remark that the non-accelerating point of the cylinder at different times are different.

When the ball slips, $f = \mu g \cos \theta$. Hence

$$a = g(\sin \theta - \mu \cos \theta)$$

$$\alpha = k(g/R)\cos \theta.$$

Example 2: The cylinder of Example 16.1 is pushed up the incline in such a way that the cylinder rolls up the incline without slipping. Describe the motion of the cylinder.

Solution: When the cylinder is rolled up, the frictional force is still upward in order to oppose the tendency of the cylinder to slip down the incline. Hence the equations of motion are exactly the same as Eq. (1-4) of the above example, As a result, the cylinder has the same acceleration as Eq. (5); the cylinder slows down while ascending the incline. Note that the motion for this case is the time reversed motion of Exercise 16.1. It is interesting to note that the time-reversal symmetry is respected in this problem even in the presence of frictional force. This is because the rolling frictional force does not dissipate energy, hence it does not break the time-reversal symmetry.

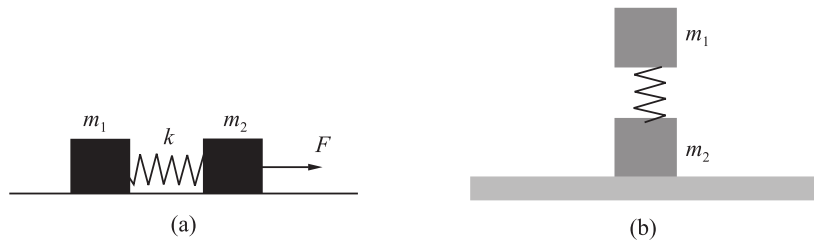
Exercises

- (1) A hoop of mass M and radius R is lying horizontally on a smooth surface. A bug of mass m moves slowly on the hoop.

HW 1.4

Compute the trajectories of the bug and the centre of mass of the hoop.

(2) Two particles rotate around each other in a circular orbit under the influence of gravitational pull of each other. The time period of orbit is T . At a given time, the particles are stopped suddenly, and then let go towards each other. After what interval of time, will the particles meet each other?



HW 1.5

(3) The spring-mass system shown in Fig. (a) is pulled with force F . Compute the position of both the blocks as a function of time.

(4) A spring-mass system shown in Fig. (b) is resting on a horizontal surface. What force should be applied to the upper plate so that the lower one gets lifted after the pressure is removed.

(5) Two masses m_1 and m_2 are hanging on the two sides of a pulley that has moment of inertia I . Assume the string to be massless and inextensible. (a) Compute the acceleration of the masses. (b) Compute the tension of the string.

HW

1.6

(6) A charged ball of mass m , radius R and charge q is resting on a horizontal slab. We apply an electric field E on the ball. The direction of the electric field is parallel to the slab. Describe the motion of the ball.