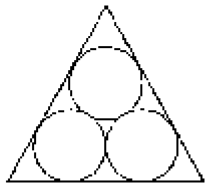
A FILL IN THE BLANKS

1. The area enclosed within the curve $|x| + |y| = 1$ is .... **(IIT 1981; 2M)**
2. The area of the triangle formed by the positive x-axis and the normal and the tangent to the circle $x^2 + y^2 = 4$ at $(1, \sqrt{3})$ is ... **(IIT 1989; 2M)**

C OBJECTIVE QUESTIONS

➡ ***Only one option is correct :***

1. The area bounded by the curves $y = f(x)$, the x-axis and the ordinates $x = 1$ and $x = b$ is $(b-1)\sin(3b+4)$. Then $f(x)$ is : **(IIT 1982; 2M)**
   (a) $(x-1)\cos(3x+4)$
   (b) $\sin(3x-4)$
   (c) $\sin(3x+4) + 3(x-1)\cos(3x+4)$
   (d) none of these
2. The triangle formed by the tangent to the curve $f(x) = x^2 + bx - b$ at the point (1,1) and the coordinate axes, lies in the first quadrant, If its area is 2, then the value of b is : **(IIT 2001)**
   (a) -1 (b) 3
   (c) -3 (d) 1
3. The area bounded by the curves $y = |x| - 1$ and $y = -|x| + 1$ is : **(IIT 2002)**
   (a) 1 (b) 2
   (c) $2\sqrt{2}$ (d) 4
4. The area of the quadrilateral formed by the tangents at the end points of latus rectum to the ellipse $\frac{x^2}{9} + \frac{y^2}{5} = 1$ is : **(IIT 2003)**
   (a) 27/4 sq. units (b) 9 sq. units
   (c) 27/2 sq. units (d) 27 sq. units
5. The area bounded by the curves $y = \sqrt{x}$, $2y + 3 = x$ and x-axis in the Ist quadrant is : **(IIT 2003)**
   (a) 9 (b) 27/4
   (c) 36 (d) 18
6. The area enclosed between the curves $y = ax^2$ and $x = ay^2$ $(a > 0)$ is 1 sq. unit. Then the value of 'a' is : **(IIT 2004)**
   (a) $\frac{1}{\sqrt{3}}$ (b) $\frac{1}{2}$
   (c) 1 (d) $\frac{1}{3}$
7. The area of the equilateral triangle, in which three coins of radius 1 cm are placed, as shown in the figure, is : **(IIT 2005)**



   (a) $6 + 4\sqrt{3}$ (b) $4\sqrt{3} - 6$
   (c) $7 + 4\sqrt{3}$ (d) $4\sqrt{3}$
8. The area bounded by the curves $y = (x-1)^2$, $y = (x+1)^2$ and $y = \frac{1}{4}$ is : **(IIT 2005)**
   (a) $\frac{1}{3}$ (b) $\frac{2}{3}$
   (c) $\frac{1}{4}$ (d) $\frac{1}{5}$

D OBJECTIVE QUESTIONS

➡ ***More than one options are correct :***

1. For which of the following values of m, is the area of the region bounded by the curve $y = x - x^2$ and the line $y = mx$ equals $\frac{9}{2}$? **(IIT 1999; 3M)**
   (a) -4 (b) -2
   (c) 2 (d) 4

E SUBJECTIVE QUESTIONS

1. Find the area bounded by the curve $x^2 = 4y$ and the straight line $x = 4y - 2$ **(IIT 1981; 4M)**

2. Find the area bounded by the x-axis, part of the curve $y = \left(1 + \frac{8}{x^2}\right)$ and the ordinates at $x = 2$ and $x = 4$. If the ordinate at $x = a$ divides the area into two equal parts, find a. **(IIT 1983; 3M)**

3. Find the area of the region bounded by the x-axis and the curves defined by :

$$y = \tan x, \quad -\frac{\pi}{3} \le x \le \frac{\pi}{3}$$

$$y = \cot x, \quad \frac{\pi}{6} \le x \le \frac{\pi}{2}$$ **(IIT 1984; 4M)**

4. Sketch the region bounded by the curves $y = \sqrt{5 - x^2}$ and $y = |x - 1|$ and find its area. **(IIT 1985; 5M)**

5. Find the area bounded by the curves $x^2 + y^2 = 4$, $x^2 = -\sqrt{2}\, y$ and $x = y$. **(IIT 1986; 5M)**

6. Find the area bounded by the curves, $x^2 + y^2 = 25$, $4y = |4 - x^2|$ and $x = 0$ above the x-axis **(IIT 1987; 6M)**

7. Find the area of the region bounded by the curve $C: y = \tan x$, tangent drawn to C at $x = \frac{\pi}{4}$ and the x-axis. **(IIT 1988; 5M)**

8. Find all maxima and minima of the function $y = x(x - 1)^2, 0 \le x \le 2$

Also determine the area bounded by the curve $y = x(x - 1)^2$, the y-axis and the line $x = 2$ **(IIT 1989)**

9. Compute the area of the region bounded by the curves, $y = ex \ln x$ and $y = \frac{\ln x}{ex}$ where $\ln e = 1$. **(IIT 1990; 4M)**

10. Sketch the curves and identify the region bounded by $x = \frac{1}{2}$, $x = 2$, $y = \ln x$ and $y = 2^x$. Find the area of this region. **(IIT 1991; 4M)**

11. Sketch the region bounded by the curves

$$y = x^2 \text{ and } y = 2/(1 + x^2)$$

Find its area. **(IIT 1992; 4M)**

12. In what ratio does the x-axis divide the area of the region bounded by the parabolas $y = 4x - x^2$ and $y = x^2 - x$? **(IIT 1994; 5M)**

13. Consider a square with vertices at (1,1), (−1, 1), (−1, −1) and (1, −1). Let S be the region consisting of all points inside the square which are nearer to the origin than to any edge. Sketch the region S and find its area. **(IIT 1995; 5M)**

14. Let A_n be the area bounded by the curve $y = (\tan x)^n$ and the lines $x = 0$, $y = 0$ and $x = \frac{\pi}{4}$. Prove that for

$$n > 2, A_n + A_{n+2} = \frac{1}{n+1}$$

and deduce $\frac{1}{2n+2} < A_n < \frac{1}{2n-2}$.

(IIT 1996; 3M)

15. Find all the possible values of $b > 0$ so that the area of the bounded region enclosed between the parabolas $y = x - bx^2$ and $y = \frac{x^2}{b}$ is maximum. **(IIT 1997C; 5M)**

16. Let $O(0, 0)$, $A(2, 0)$ and $B\left(1, \frac{1}{\sqrt{3}}\right)$ be the vertices of a triangle. Let R be the region consisting of all those points P inside ΔOAB which satisfy $d(P, OA) \ge \min\{d(P, OB), d(P, AB)\}$, when d denotes the distance from the point to the corresponding line. Sketch the region R and find its area. **(IIT 1997C; 5M)**

17. Let $f(x) =$ maximum

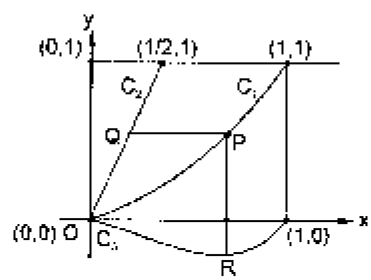
$$\{x^2, (1 - x)^2, 2x(1 - x)\}$$

where $0 \le x \le 1$. Determine the area of the region bounded by the curves $y = f(x)$, x-axis, $x = 0$ and $x = 1$.

(IIT 1997; 5M)

18. Let C_1 and C_2 be the graphs of functions $y = x^2$ and $y = 2x, 0 \le x \le 1$ respectively. Let C_3 be the graph of a function $y = f(x), 0 \le x \le 1, f(0) = 0$. For a point P on C_1, let the lines through P, parallel to the axes, meet C_2 and C_3 at Q and R respectively (see figure). If for every position of P (on C_1) the areas of the shaded regions OPQ and ORP are equal, determine the function $f(x)$.

(IIT 1998; 8M)



19. Let $f(x)$ be a continuous function given by

$$f(x) = \begin{cases} 2x, & |x| \le 1 \\ x^2 + ax + b, & |x| > 1 \end{cases}$$

Find the area of the region in the third quadrant bounded by the curves $x = -2y^2$ and $y = f(x)$ lying on the left on the line $8x + 1 = 0$.

20. Let $b \ne 0$ and for $j = 0, 1, 2, \ldots, n$, let S_j be the area of the region bounded by the y-axis and the curve $xe^{ay} = \sin by$, $\frac{j\pi}{b} \le y \le \frac{(j+1)\pi}{(b)}$. Show that $S_0, S_1, S_2, \ldots, S_n$ are in geometric progression. Also, find their sum for $a = -1$ and $b = \pi$. **(IIT 2001; 5M)**

21. Find the area of the region bounded by the curves $y = x^2$, $y = |2 - x^2|$ and $y = 2$, which lies to the right of the line $x = 1$? **(IIT 2002; 5M)**

22. A curve passes through (2,0) and the slope of tangent at point $P(x, y)$ equals $\frac{(x+1)^2 + y - 3}{(x+1)}$.

Find the equation of the curve and area enclosed by the curve and the x-axis in the fourth quadrant. **(IIT 2004)**

23. The area of the triangle formed by the intersection of a line parallel to x-axis and passing through (h, k) with the lines $y = x$ and $x + y = 2$ is $4h^2$. Find the locus of point P.

(IIT 2005)

24. Find the area bounded by the curves

$$x^2 = y,\ x^2 = -y \text{ and } y^2 = 4x - 3.$$

25. If $\begin{bmatrix} 4a^2 & 4a & 1 \\ 4b^2 & 4b & 1 \\ 4c^2 & 4c & 1 \end{bmatrix} \begin{bmatrix} f(-1) \\ f(1) \\ f(2) \end{bmatrix} = \begin{bmatrix} 3a^2 + 3a \\ 3b^2 + 3b \\ 3c^2 + 3c \end{bmatrix}$, $f(x)$ is a quadratic function and its maximum value occurs at a point V. A is a point of intersection of $y = f(x)$ with x-axis and point B is such that chord AB subtends a right at V. Find the area enclosed by $f(x)$ and chord AB.

(IIT 2005)

ANSWERS

A Fill in the Blanks

1. $\sqrt{2}$ square units **2.** $2\sqrt{3}$ square units

C Objective Questions (Only one option)

1. (c) **2.** (c) **3.** (b) **4.** (d) **5.** (a) **6.** (a) **7.** (a) **8.** (a)

D Objective Questions (More than one option)

1. (b, d)

E Subjective Questions

1. $\frac{9}{8}$ sq. units **2.** $2\sqrt{2}$ **3.** $\frac{1}{2}\log_e 3$ **4.** $\frac{5\pi}{4} - \frac{1}{2}$ **5.** $\frac{1}{3} - \pi$ **6.** $8 + \left(\frac{25\pi}{4}\right)$ **7.** $\left(\frac{\pi^2}{16} - \frac{\pi}{4} + \frac{1}{2}\log 2\right)$ sq. units

8. $y_{max} = \frac{4}{27}$, $y_{min} = 0$, $\frac{2}{3}$ sq. units **9.** $\frac{e^2 - 5}{4e}$ **10.** $\frac{4 - \sqrt{2}}{\log_2} - \frac{5}{2}\log 2 + \frac{3}{2}$ **11.** $\pi - \frac{2}{3}$ **12.** 121 : 4

13. $\frac{1}{3}(16\sqrt{2} - 20)$ **15.** $b - 1$ **16.** $(2 - \sqrt{3})$ sq. units **17.** $\frac{17}{27}$ sq. units **18.** $f(x) = x^3 - x^2, 0 \le x \le +1$

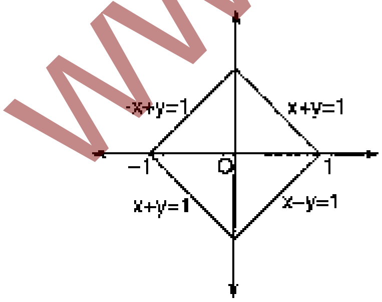
19. $\frac{761}{192}$ **20.** $\frac{\pi(1 + e}{(1 + \pi^2)} \cdot \frac{(e^{\pi+1} - 1}{e - 1}$ **21.** $\left(\frac{20 - 12\sqrt{2}}{3}\right)$ sq. units **22.** $\frac{4}{3}$ sq. units, $y = x^2 - 2x$ **23.** $2x = \pm(y - 1)$

24. $\frac{1}{3}$ sq. units **25.** $\frac{125}{3}$ sq. units

SOLUTIONS

A FILL IN THE BLANKS

1. The area formed by $|x| + |y| = 1$ is square shown as,



whose area $= (\sqrt{2})^2$

$= 2$ square units

2. Equation of tangent to

$$x^2 - y^2 = 4$$

is $$x - \sqrt{3}y = 4$$

whose x-axis intercept (4, 0).

Thus, area of Δ formed by (0, 0) $(1, \sqrt{3})$ and (4, 0) is,

$$\frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ 1 & \sqrt{3} & 1 \\ 4 & 0 & 1 \end{vmatrix} = \frac{1}{2}|(0 - 4\sqrt{3})| = 2\sqrt{3} \text{ square units}$$

C OBJECTIVE (ONLY ONE OPTION)

1. Since, $\int_1^b f(x)\,dx = (b-1)\sin(3b+4)$

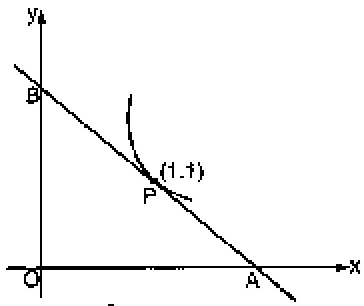
$\therefore$ differentiating both sides with respect to b, we get

$f(b) = 3(b-1)\cdot\cos(3b+4) + \sin(3b+4)$

$\therefore \quad f(x) = \sin(3x+4) + 3(x-1)\cos(3x+4)$

2. $f(x) = x^2 + bx - b$

For equation of the tangent at $P(1, 1)$ is



$y = x^2 + bx - b$

$\Rightarrow \quad 2y = 2x^2 + 2bx - 2b$

$\Rightarrow \quad y + 1 = 2x1 + b(x+1) - 2b \qquad$ (apply $T = 0$)

$\Rightarrow \quad y = (2+b)x - (1+b)$

$\therefore \quad x_A = \dfrac{1+b}{2+b}$ and $y_B = -(1-b)$

Again area of $\Delta OAB = \dfrac{1}{2} OA \times OB = -\dfrac{1}{2} \times \dfrac{(1+b)^2}{(2+b)} = 2$

(given)

$\Rightarrow \quad -(1+b)^2 = 4(2+b)$

$\Rightarrow \quad (1+b)^2 - 4(2+b) = 0$

$\Rightarrow \quad 1 + b^2 + 2b + 8 + 4b = 0$

$\Rightarrow \quad b^2 + 6b + 9 = 0$

$\Rightarrow \quad (b+3)^2 = 0$

$\Rightarrow b = -3$, therefore, (c) is the answer.

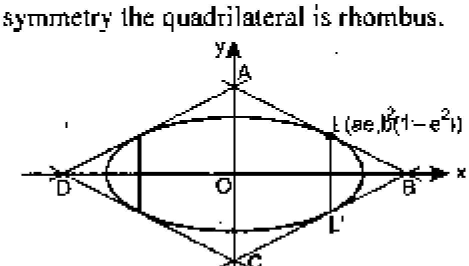
3. The region is clearly square with vertices at the points (1, 0), (0, 1), (−1, 0) and (0, −1). So that its area $= \sqrt{2} \times \sqrt{2} = 2$.

4. Given : $\dfrac{x^2}{9} + \dfrac{y^2}{5} = 1$

To find tangents at the end points of latus rectum, we find ae.

i.e., $\quad ae = \sqrt{a^2 - b^2} = \sqrt{4} = 2$

By symmetry the quadrilateral is rhombus.



So area is four times the area of the right angled Δ formed by the tangent and axes in the I quadrant.

$\Rightarrow$ Equation of tangent at (given)

$(ae, b^2(1-e^2)) = \left(2, \dfrac{5}{3}\right)$ is

$\dfrac{2}{9}x + \dfrac{5}{3}\cdot\dfrac{y}{5} = 1$

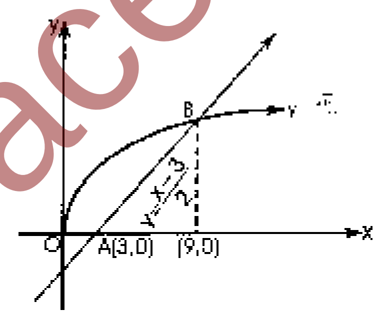
$\Rightarrow \quad \dfrac{x}{9/2} + \dfrac{y}{3} = 1$

$\therefore$ Area of quadrilateral $ABCD$

$= 4(\text{ area of } \Delta AOB)$

$= 4\cdot\left\{\dfrac{1}{2}\cdot\dfrac{9}{2}\cdot 3\right\} = 27$ sq. units

5. To find the area between the curves; $y = \sqrt{x}$ and $2y + 3 = x$ and x-axis in the Ist quadrant (we can plot the above condition as);



To find area of shaded portion $OABO$

$f(x) = \int_0^9 \sqrt{x}\,dx - \int_3^9 \left(\dfrac{x-3}{2}\right)dx$

$= \left(\dfrac{x^{3/2}}{3/2}\right)_0^9 - \dfrac{1}{2}\left(\dfrac{x^2}{2} - 3x\right)_3^9$

$= \left(\dfrac{2}{3}\cdot 27\right) - \dfrac{1}{2}\left\{\left(\dfrac{81}{2} - 27\right) - \left(\dfrac{9}{2} - 9\right)\right\}$

$= 18 - \dfrac{1}{2}\{18\} = 9$ sq. units

6. As from the figure, area enclosed between curves is $OABCO$.

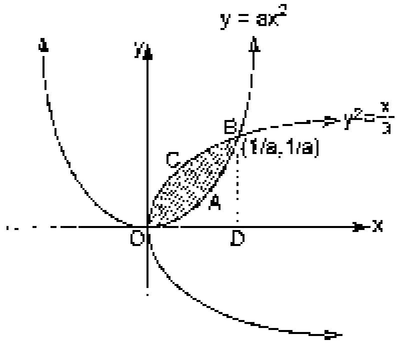
Thus the point of intersection of

$y = ax^2$ and $x = ay^2$

$\Rightarrow \quad x = a(ax^2)$

$\Rightarrow \quad x = 0, \dfrac{1}{a}$

$\therefore$ Point of intersection (0, 0) and $\left(\dfrac{1}{a}, \dfrac{1}{a}\right)$



Thus required area $OABCO$ = Area of $OCBDO$ – area of $OABDO$

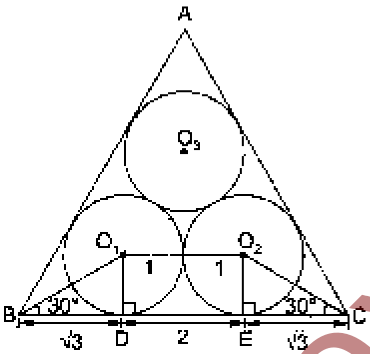
$$\Rightarrow \quad \int_0^{1/a}\left(\sqrt{\frac{x}{a}} - ax^2\right)dx = 1 \qquad \text{(given)}$$

$$\therefore \quad \left(\frac{1}{\sqrt{a}}\cdot\frac{x^{3/2}}{3/2} - \frac{ax^3}{3}\right)_0^{1/a} = 1$$

$$\Rightarrow \quad \frac{2}{3a^2} - \frac{1}{3a^2} = 1$$

$$\Rightarrow \quad a^2 = \frac{1}{3} \text{ or } a = \frac{1}{\sqrt{3}} \qquad (\text{as } a > 0)$$

7. As tangents drawn from external point to the circle subtends equal angle at centre.



$$\therefore \quad \angle O_1BD = 30°$$

In ΔO_1BD $\quad \dfrac{O_1D}{BD} = \tan 30°$

$$\Rightarrow \quad BD = 1\cdot\sqrt{3}$$

Also, $\quad DE = O_1O_2 = 2$ and $EC = \sqrt{3}$

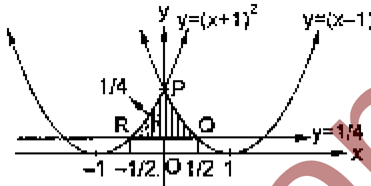
Hence,

$$BC = BD + DE + EC = 2 + 2\sqrt{3}$$

$$\Rightarrow \quad \text{Area} = \frac{\sqrt{3}}{4}(BC)^2 = \frac{\sqrt{3}}{4}\cdot 4(1+\sqrt{3})^2$$

$$= 6 + 4\sqrt{3} \text{ square cm.}$$

8. The curves $y = (x-1)^2$, $y = (x+1)^2$ and $y = 1/4$ are shown as :



where point of intersection are

$$(x-1)^2 = \frac{1}{4}$$

$$\Rightarrow \quad x = \frac{1}{2}, -\frac{1}{2}$$

$$\therefore \quad Q\left(\frac{1}{2}, \frac{1}{4}\right) \text{ and } R\left(\frac{-1}{2}, \frac{1}{4}\right)$$

$$\Rightarrow \quad \text{Area} = 2\int_0^{1/2}\left\{(x-1)^2 - \frac{1}{4}\right\}dx$$

$$= 2\left\{\frac{(x-1)^3}{3} - \frac{1}{4}x\right\}_0^{1/2}$$

$$= 2\left\{\frac{(-1/2)^3}{3} - \frac{1}{8} - \left(-\frac{1}{3} - 0\right)\right\}$$

$$= \frac{8}{24} = \frac{1}{3} \text{ square units}$$

D OBJECTIVE (MORE THAN ONE OPTION)

1. **Case 1** . $m = 0$

In this case $\quad y = x - x^2 \qquad \ldots(1)$

$\qquad y = 0 \qquad \ldots(2)$

are two given curves, $y > 0$ is total region above x-axis.

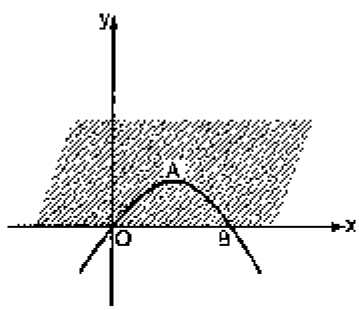
Therefore, area between $y = x - x^2$ and $y = 0$

is area between $y = x - x^2$ and above the x-axis

$$A = \int_0^1 (x - x^2)\,dx = \left(\frac{x^2}{2} - \frac{x^3}{3}\right)_0^1$$

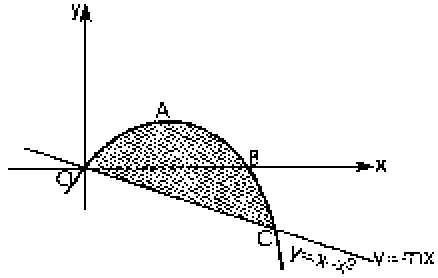
$$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \neq \frac{9}{2}$$

Hence no solution.



Case 2. $m < 0$

In this case area between



$y = x - x^2$ and $y = mx$ is

$OABCO$ and points of intersection are (0,0) and $\{1-m, m(1-m)\}$

Area $OABCO = \int_0^{1-m} [x - x^2 - mx]\, dx.$

Imp. note : Area $OBCO$ considered automatically because m is a parameter

$$= \left[(1-m)\frac{x^2}{2} - \frac{x^3}{3}\right]_0^{1-m}$$

$$= \frac{1}{2}(1-m)^3 - \frac{1}{3}(1-m)^3 = \frac{1}{6}(1-m)^3$$

Put Area $OABCO = 9/2$ (given)

$\therefore \quad \frac{1}{6}(1-m)^3 = \frac{9}{2}$

$\Rightarrow \quad (1-m)^3 = 27$

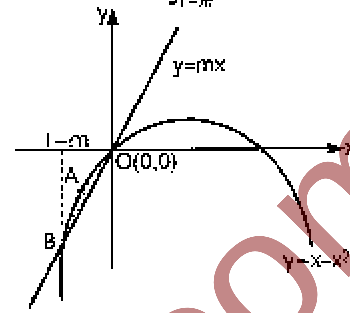
$\Rightarrow \quad 1 - m = 3$

$\Rightarrow \quad m = -2$

Case 3. $m > 0$

In this case $y = mx$ and $y = x - x^2$ intersect in (0,0) and $\{(1-m), m(1-m)\}$ as shown in Fig.

Area of shaded region $= \int_{1-m}^{0} (x - x^2 - mx)\, dx$



$$= \left[(1-m)\frac{x^2}{2} - \frac{x^3}{3}\right]_{1-m}^{0}$$

$$= -\frac{1}{2}(1-m)(1-m)^2 + \frac{1}{3}(1-m)^3$$

$$= -\frac{1}{6}(1-m)^3$$

Area of shaded region = 9/2 sq.unit (given)

$\Rightarrow \quad \frac{9}{2} = -\frac{1}{6}(1-m)^3$

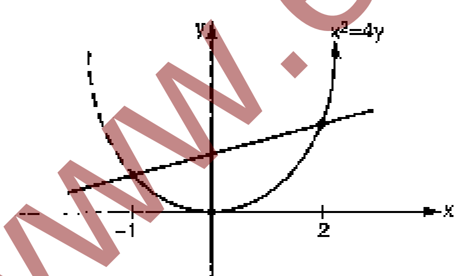
$\Rightarrow \quad (1-m)^3 = -27$

$\Rightarrow \quad (1-m) = -3$

$\Rightarrow m = 3 + 1 = 4$. Therefore (b) and (d) are the answers.

E SUBJECTIVE QUESTIONS

1. The curves $x^2 = 4y$ and $x = 4y - 2$ could be sketched as, whose point of intersection are $x = -1$, and $x = 2$



Thus, area $= \int_{-1}^{2} \left\{\left(\frac{x+2}{4}\right) - \left(\frac{x^2}{4}\right)\right\} dx$

$$= \frac{1}{4}\left\{\frac{x^2}{2} + 2x - \frac{x^3}{3}\right\}_{-1}^{2}$$

$$= \frac{1}{4}\left\{\left(2 + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)\right\}$$

$$= \frac{1}{4} \cdot \frac{9}{2} = \frac{9}{8} \text{ sq. units}$$

2. Here, $\int_2^a \left(1 + \frac{8}{x^2}\right) dx = \int_a^4 \left(1 + \frac{8}{x^2}\right) dx$

$\Rightarrow \quad \left(x - \frac{8}{x}\right)_2^a = \left(x - \frac{8}{x}\right)_a^4$

$\Rightarrow \left(a - \frac{8}{a}\right) - (2 - 4) = (4 - 2) - \left(a - \frac{8}{a}\right)$

$\Rightarrow \quad a - \frac{8}{a} + 2 = 2 - a + \frac{8}{a}$

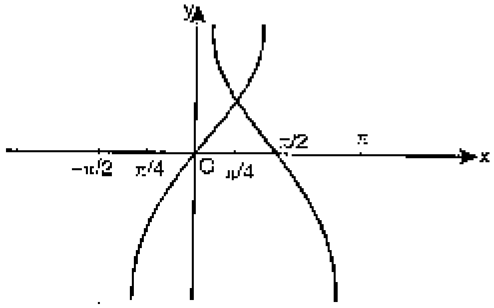
$\Rightarrow \quad 2a - \frac{16}{a} = 0 \text{ or } 2(a^2 - 8) = 0$

$\Rightarrow \quad a = \pm 2\sqrt{2}$, neglecting (–ve)

$a = 2\sqrt{2}.$

3. As $y = \begin{cases} \tan x, & -\frac{\pi}{3} \le x \le \frac{\pi}{3} \\ \cot x, & \frac{\pi}{6} \le x \le \frac{\pi}{2} \end{cases}$

which could be plotted as, y-axis.



$$\text{Required area} = \int_0^{\pi/4} (\tan x)\, dx + \int_{\pi/4}^{\pi/3} (\cot x)\, dx$$

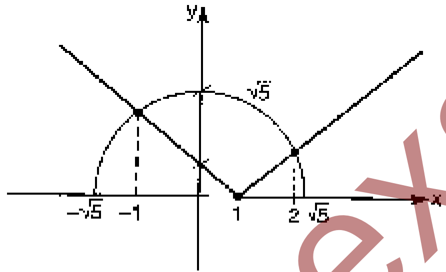
$$= (-\log|\cos x|)_0^{\pi/4} + (\log \sin x)_{\pi/4}^{\pi/3}$$

$$= -\left\{\log \frac{1}{\sqrt{2}} - 0\right\} + \left\{\log \frac{\sqrt{3}}{2} - \log \frac{1}{\sqrt{2}}\right\}$$

$$= \log \frac{\sqrt{3}}{2} - 2\log \frac{1}{\sqrt{2}}$$

$$= \log \frac{\sqrt{3}}{2} - \log \frac{1}{2} = \log \sqrt{3} = \frac{1}{2} \log_e 3$$

4. $y = \sqrt{5 - x^2}$ and $y = |x - 1|$ could be sketched as, whose point of intersection are;



$$5 - x^2 = (x-1)^2$$

or $\quad 5 - x^2 = x^2 - 2x + 1$

or $\quad 2x^2 - 2x - 4 = 0$

$\Rightarrow \quad x = 2, -1$

$\therefore$ Required area

$$= \int_{-1}^{2} \sqrt{5 - x^2}\, dx - \int_{-1}^{1} (-x + 1)\, dx - \int_1^2 (x - 1)\, dx$$

$$= \left[\frac{x}{2}\sqrt{5 - x^2} + \frac{5}{2}\sin^{-1}\left(\frac{x}{\sqrt{5}}\right)\right]_{-1}^{2} - \left\{\frac{-x^2}{2} + x\right\}_{-1}^{1} - \left\{\frac{x^2}{2} - x\right\}_1^2$$

$$= \left(1 + \frac{5}{2}\sin^{-1}\frac{2}{\sqrt{5}}\right) - \left(-1 + \frac{5}{2}\sin^{-1}\left(\frac{-1}{\sqrt{5}}\right)\right)$$

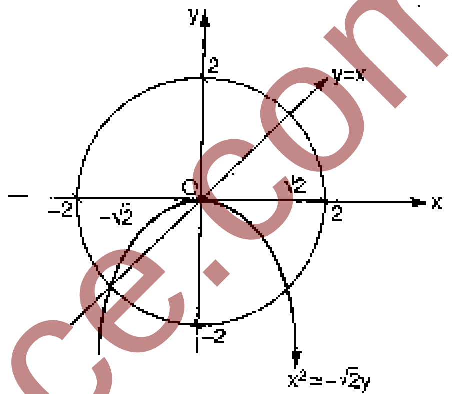
$$-\left(-\frac{1}{2} + 1 + \frac{1}{2} + 1\right) - \left(2 - 2 - \frac{1}{2} + 1\right)$$

$$= \frac{5}{2}\left(\sin^{-1}\frac{2}{\sqrt{5}} + \sin^{-1}\frac{1}{\sqrt{5}}\right) - \frac{1}{2}$$

$$= \frac{5}{2}\sin^{-1}\left(\frac{2}{\sqrt{5}}\sqrt{1 - \frac{1}{5}} + \frac{1}{\sqrt{5}}\sqrt{1 - \frac{4}{5}}\right) - \frac{1}{2}$$

$$= \frac{5}{2}\sin^{-1}(1) - \frac{1}{2} = \frac{5\pi}{4} - \frac{1}{2}$$

5. Curves $x^2 + y^2 = 4, x^2 = -\sqrt{2}y$ and $x = y$ could be shown as.



Thus, the required area

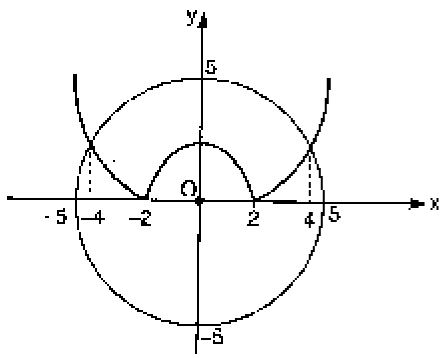
$$= \left|\int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{4 - x^2}\, dx\right| - \left|\int_{-\sqrt{2}}^{0} x\, dx\right| - \left|\int_0^{\sqrt{2}} \frac{-x^2}{\sqrt{2}}\, dx\right|$$

$$= 2\int_0^{\sqrt{2}} \sqrt{4 - x^2}\, dx - \left|\left(\frac{x^2}{2}\right)_{-\sqrt{2}}^{0}\right| - \left|\frac{x^3}{3\sqrt{2}}\right|_0^{\sqrt{2}}$$

$$-2\left\{\frac{x}{2}\sqrt{4 - x^2} - \frac{4}{2}\sin^{-1}\frac{x}{2}\right\}_0^{\sqrt{2}} \quad 1 - \frac{2}{3}$$

$$= (2 - \pi) - \frac{5}{3} = \frac{1}{3} - \pi$$

6. Here, $x^2 + y^2 = 25$, $4y = |4 - x^2|$ could be sketched as, whose point of intersection could be obtained by



$$x^2 + \frac{(4 - x)^2}{16} = 25$$

$\Rightarrow$
$$(x^2 + 24)(x^2 - 16) = 0$$

$\Rightarrow \quad x = \pm 4$

$\therefore$ Required area

$$= 2\left\{\int_0^4 \sqrt{25 - x^2}\,dx - \int_0^2 \left(\frac{4 - x^2}{4}\right)dx - \int_2^4 \left(\frac{x^2 - 4}{4}\right)dx\right\}$$

$$= 2\left\{\left(\frac{x}{2}\sqrt{25 - x^2} + \frac{25}{2}\sin^{-1}\left(\frac{x}{5}\right)\right)_0^4 - \frac{1}{4}\left(4x - \frac{x^3}{3}\right)_0^2 - \frac{1}{4}\left(\frac{x^3}{3} - 4x\right)_2^4\right\}$$

$$= 8 + \left(\frac{25\pi}{4}\right)$$

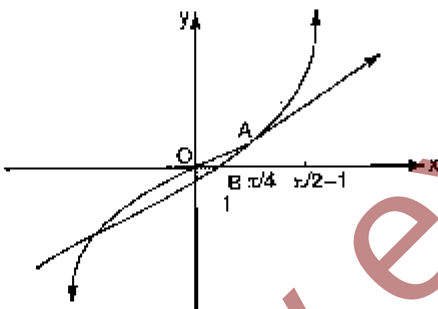
7. $y = \tan x$

$$\Rightarrow \quad \frac{dy}{dx} = \sec^2 x$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{x = \frac{\pi}{4}} = 2,$$

Hence equation of tangent at $\left(\frac{\pi}{4}, 1\right)$ is

$$\frac{y - 1}{x - \pi/4} = 2 \quad \text{or} \quad y - 1 = 2x - \frac{\pi}{2}$$



$$\text{or} \quad (2x - y) = \left(\frac{\pi}{2} - 1\right)$$

$\therefore$ Required area is $OABO$

$$= \int_0^{\pi/4} (\tan x)\,dx - \int_0^{\pi/4} \left(2x + 1 - \frac{\pi}{2}\right)dx$$

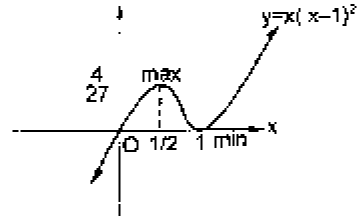
$$= (\log \sec x)_0^{\pi/4} - \left(x^2 + x - \frac{\pi}{2}x\right)_0^{\pi/4}$$

$$= \log\sqrt{2} - \left(\frac{\pi}{4} - \frac{\pi^2}{16}\right) = \left(\frac{\pi^2}{16} - \frac{\pi}{4} + \frac{1}{2}\log 2\right) \text{sq.units}$$

8. $y = x(x - 1)^2$

$$\Rightarrow \quad \frac{dy}{dx} = x \cdot 2(x - 1) + (x - 1)^2$$

$$= (x - 1) \cdot \{2x + x - 1\}$$



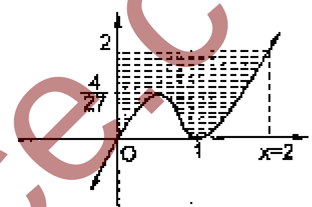
$$= (x - 1)(3x - 1)$$

$+ \quad - \quad +$; $1/3 \quad 1$

$\therefore$ maximum at $x = 1/3 \Rightarrow y_{max} = \frac{1}{3}\left(-\frac{2}{3}\right)^2 = \frac{4}{27}$

minimum at $x = 1 \Rightarrow y_{min} = 0$

Now, to find the area bounded by the curve $y = x(x - 1)^2$, The y-axis and line $x = 2$



$$\Rightarrow \quad \text{Area} = \int_0^2 x(x - 1)^2 \cdot dx$$

$$= \int_0^2 (x^3 - 2x^2 + x)\,dx$$

$$= \left(\frac{x^4}{4} - \frac{2x^3}{3} + \frac{x^2}{2}\right)_0^2$$

$$= \left(4 - \frac{16}{3} + 2\right) = 6 - \frac{16}{3} = \frac{2}{3} \text{ sq. units}$$

9. Both the curves are defined for $x > 0$.

Both are positive when $x > 1$ and negative when $0 < x < 1$.

We know, $\lim_{x \to 0^+} (\log x) \to -\infty$

Hence, $\lim_{x \to 0^+} \frac{\log x}{ex} \to -\infty$. Thus, y-axis is asymptote of second curve.

and $\lim_{x \to 0^+} ex \log x$ $\quad \{(0) \times \infty \text{ form}\}$

$$= \lim_{x \to 0^+} \frac{e \log x}{1/x} \quad \left(-\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x \to 0^+} \frac{e\left(\frac{1}{x}\right)}{\left(-\frac{1}{x^2}\right)} = 0 \quad \{\text{using L-Hospital's rule}\}$$

Thus, the first curve starts from (0, 0) but does not include (0, 0)

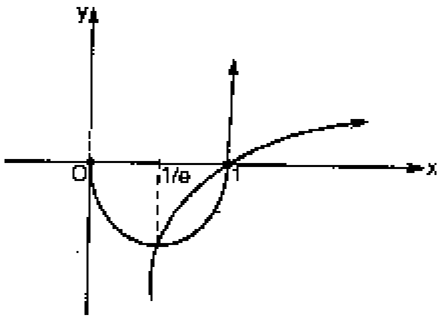
Now, the given curves intersect therefore

$$ex \log x = \frac{\log x}{ex}$$

i.e., $(e^2x^2-1)\log x = 0$

⇒ $x = 1, \frac{1}{e}$ (since $x > 0$)

Hence, using above results figure could be drawn as,



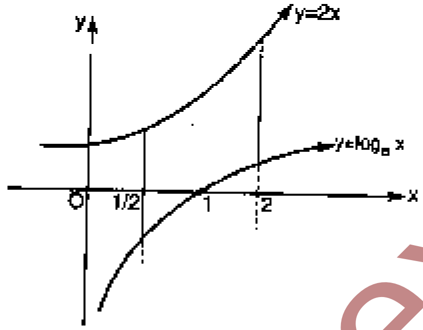
∴ The required area

$$= \int_{1/e}^{1}\left(\frac{(\log x)}{ex} - ex\log x\right)dx$$

$$= \frac{1}{e}\left\{\frac{(\log x)^2}{2}\right\}_{1/e}^{1} - e\left\{\frac{x^2}{4}(2\log x - 1)\right\}_{1/e}^{1}$$

$$= \frac{e^2-5}{4e}$$

10. The required area is the shaded portion in following figure.



In the region $\frac{1}{2} \le x \le 2$ the curve $y = 2^x$ lies above as compared to $y = \log_e x$

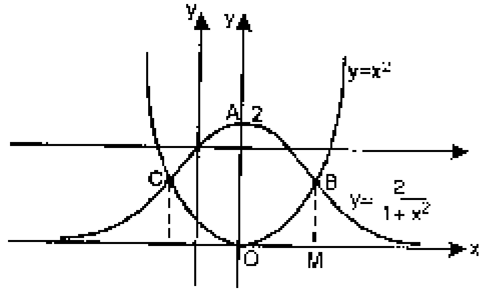
Hence, the required area

$$= \int_{1/2}^{2}(2^x - \log x)\,dx$$

$$= \left(\frac{2^x}{\log 2} - (x\log x - x)\right)_{1/2}^{2}$$

$$= \frac{4-\sqrt{2}}{\log 2} - \frac{5}{2}\log 2 - \frac{3}{2}$$

11. The curve $y = x^2$ is a parabola. It is symmetric about y-axis and has its vertex at (0, 0) and the curve $y = \frac{2}{1+x^2}$ is a bell shaped curve. x-axis is its asymptote and it is symmetric about y-axis and its vertex is (0, 2).



To obtain the area, we need point of intersection of

$$y = x^2 \quad \ldots(1)$$

and $$y = \frac{2}{1+x^2} \quad \ldots(2)$$

⇒ $y = \frac{2}{1+y}$

⇒ $y + y^2 = 2$

⇒ $y^2 + y - 2 = 0$

⇒ $y^2 + 2y - y - 2 = 0$

⇒ $y(y+2) - 1(y+2) = 0$

⇒ $(y-1)(y+2) = 0$

$y = -2, 1$ but $y \ge 0$ so $y = 1 \Rightarrow x = \pm 1$

Therefore coordinates of C are (−1, 1) and coordinates of B are (1, 1).

The required area is $OBACO$ = 2 area of curve $OBAO$

$$= 2\left[\int_0^1 \frac{2}{1+x^2}dx - \int_0^1 x^2\,dx\right]$$

$$= 2[2\tan^{-1}x]_0^1 - \left[\frac{x^3}{3}\right]_0^1$$

$$= 2\left[\frac{2\pi}{4} - \frac{1}{3}\right] = \pi - \frac{2}{3}$$

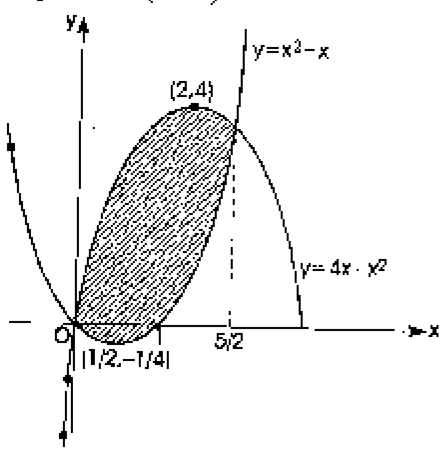
12. $y = 4x - x^2$ (given)

$= -(x^2 - 4x + 4 - 4)$

$= -(x^2 - 4x + 4) + 4$

$y = -(x-2)^2 + 4$

$y - 4 = -(x-2)^2$



Therefore it is a vertically downward parabola with vertex at (2,4) and its axis is $x=2$

and $\quad y=x^2-x \quad$ (given)

$$\Rightarrow \quad y=x^2-x+\frac{1}{4}-\frac{1}{4}$$

$$\Rightarrow \quad y=\left(x-\frac{1}{2}\right)^2-\frac{1}{4}$$

$$\Rightarrow \quad y+\frac{1}{4}=(x-1/2)^2$$

This is a parabola having its vertex at $\left(\frac{1}{2},-\frac{1}{4}\right)$

its axis at $x=\frac{1}{2}$ and opening upwards.

To obtain the x-coordinate of the points of intersection we solve $y=4x-x^2$ and $y=x^2-x$

$$\Rightarrow \quad 4x-x^2=x^2-x$$

$$\Rightarrow \quad 2x^2=5x \Rightarrow 2x^2-5x=0 \Rightarrow x(2-5x)=0$$

$$\Rightarrow \quad x=0,\frac{5}{2}$$

Also $y=x^2-x$, meets x-axis at (0,0) and (1, 0)

Now area, $A_1=\int_0^{5/2}[(4x-x^2)]-[(x^2-x)]\,dx$

$$=\int_0^{5/2}(5x-2x^2)\,dx$$

$$=\left[\left(\frac{5}{2}x^2-\frac{2}{3}x^3\right)\right]_0^{5/2}$$

$$=\frac{5}{2}\left(\frac{5}{2}\right)^2-\frac{2}{3}\cdot\left(\frac{5}{2}\right)^3$$

$$=\frac{5}{2}\cdot\frac{25}{4}-\frac{2}{3}\cdot\frac{125}{8}$$

$$=\frac{125}{8}\left(1-\frac{2}{3}\right)=\frac{125}{24}$$

This area is considering above and below x-axis both. Now for area below x-axis separately. We consider

$$A_2=-\int_0^1(x^2-x)\,dx=\left(\frac{x^2}{2}-\frac{x^3}{3}\right)_0^1$$

$$=\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$$

Therefore net area above the x-axis

$$=A_1-A_2=\frac{125-4}{24}=\frac{121}{24}$$

Hence, ratio of area above the x-axis and area below x-axis

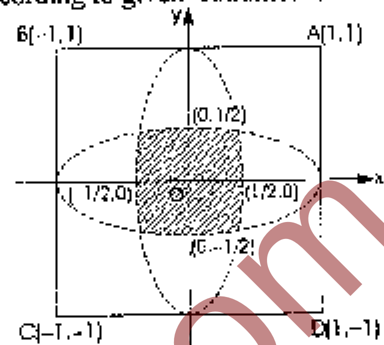
$$=\frac{121}{24}:\frac{1}{6}$$

$$=121:4$$

13. The equations of the sides of the square are as follows :

$AB: y=1,\ BC: x=-1,\ CD: y=-1,\ DA: x=1$

Let the region be S and (x, y) is any point inside it. Then according to given conditions,



$$\sqrt{x^2+y^2}<|1-x|,|1+x|,|1-y|,|1+y|$$

$$\Rightarrow \quad x^2+y^2<(1-x)^2,(1+x)^2,(1-y)^2,(1+y)^2$$

$$\Rightarrow \quad x^2+y^2<x^2-2x+1, x^2+2x+1, y^2-2y+1, y^2+2y+1$$

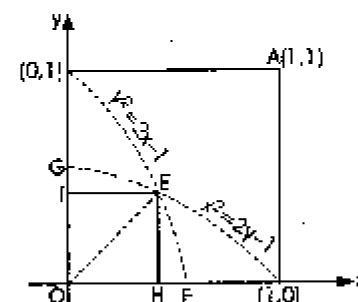
$$\Rightarrow \quad y^2<1-2x,\ y^2<1+2x,\ x^2<1-2y \text{ and } x^2<2y+1$$

Now in $y^2=1-2x$ and $y^2=1+2x$, the first equation represents a parabola with vertex at (1/2,0) and second equation represents a parabola with vertex (−1/2, 0).

And in $x^2=1-2y$ and $x^2=1+2y$, the first equation represents a parabola with vertex at (0, 1/2) and second equation represents a parabola with vertex at (0, −1/2).

Therefore, the region S is the region lying inside the four parabolas

$$y^2=1-2x,\ y^2=1+2x,\ x^2=1+2y,\ x^2=1-2y$$



where S is the shaded region.

Now, S is symmetrical in all four quadrants, therefore, $S=4\times$ area lying in the first quadrant.

Now, $y^2=1-2x$ and $x^2=1-2y$ intersect on the line $y=x$. The point of intersection is $E(\sqrt{2}-1,\ \sqrt{2}-1)$

Area of the region $OEFO$

= area of $\Delta\, OEH$ + area of $HEFH$

$$=\frac{1}{2}(\sqrt{2}-1)^2+\int_{\sqrt{2}-1}^{1/2}\sqrt{1-2x}\,dx$$

$$= \frac{1}{2}(\sqrt{2}-1)^2 + \left[(1-2x)^{3/2}\cdot\frac{2}{3}\cdot\frac{1}{2}(-1)\right]_{-\sqrt{2}-1}^{1/2}$$

$$= \frac{1}{2}(2+1-2\sqrt{2}) + \frac{1}{3}(1+2-2\sqrt{2})^{3/2}$$

$$= \frac{1}{2}(3-2\sqrt{2}) + \frac{1}{3}(3-2\sqrt{2})^{3/2}$$

$$= \frac{1}{2}(3-2\sqrt{2}) + \frac{1}{3}[(\sqrt{2}-1)^2]^{3/2}$$

$$= \frac{1}{2}(3-2\sqrt{2}) + \frac{1}{3}(\sqrt{2}-1)^3$$

$$= \frac{1}{2}(3-\sqrt{2}) + \frac{1}{3}[2\sqrt{2}-1-3\sqrt{2}(\sqrt{2}-1)]$$

$$= \frac{1}{2}(3-2\sqrt{2}) + \frac{1}{3}[5\sqrt{2}-7]$$

$$= \frac{1}{6}[9-6\sqrt{2}+10\sqrt{2}-14] = \frac{1}{6}[4\sqrt{2}-5]$$

Similarly, area $OEGO = \frac{1}{6}(4\sqrt{2}-5)$

Therefore, area of S lying in first quadrant

$$= \frac{2}{6}(4\sqrt{2}-5)$$

$$= \frac{1}{3}(4\sqrt{2}-5)$$

Hence, $S = \frac{4}{3}(4\sqrt{2}-5) = \frac{1}{3}(16\sqrt{2}-20)$

14. We have $A_n = \int_0^{\pi/4} (\tan x)^n \, dx$

Since $0 < \tan x < 1$, when $0 < x < \pi/4$

we have

$0 < (\tan x)^{n+1} < (\tan x)^n$ for each $n \in N$

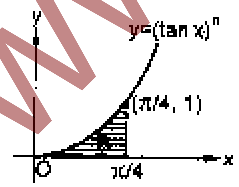
$\Rightarrow \int_0^{\pi/4} (\tan x)^{n+1} \, dx < \int_0^{\pi/4} (\tan x)^n \, dx$

$\Rightarrow A_{n+1} < A_n$

Now, for $n > 2$

$$A_n + A_{n+2} = \int_0^{\pi/4} [(\tan x)^n + (\tan x)^{n+2}] \, dx$$

$$= \int_0^{\pi/4} (\tan x)^n (1+\tan^2 x) \, dx$$



$$= \int_0^{\pi/4} (\tan x)^n \sec^2 x \, dx$$

$$= \left[\frac{1}{(n+1)}(\tan x)^{n+1}\right]_0^{\pi/4}$$

$$\left[\because \int f(x)^n f'(x)\, dx = \frac{f(x)^{n+1}}{n+1}\right]$$

$$= \frac{1}{(n+1)}(1-0) = \frac{1}{n+1}$$

Since $A_{n+2} < A_{n+1} < A_n$, we get

$$A_n + A_{n+2} < 2A_n$$

$\Rightarrow \frac{1}{n+1} < 2A_n$

$\Rightarrow \frac{1}{2n+2} < A_n$ ...(1)

Also for $n > 2$

$$A_n + A_n < A_n + A_{n-2} = \frac{1}{n-1}$$

$\Rightarrow 2A_n < \frac{1}{n-1}$

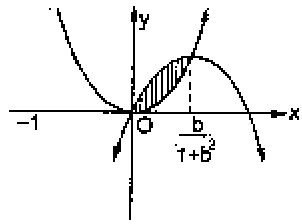
$\Rightarrow A_n < \frac{1}{2n-2}$ ...(2)

from (1) and (2) $\frac{1}{2n+2} < A_n < \frac{1}{2n-2}$.

15. Eliminating y from $y = \frac{x^2}{b}$ and $y = x - bx^2$, we get

$$x^2 = bx - b^2x^2$$

$\Rightarrow x = 0, \frac{b}{1+b^2}$



Thus, the area enclosed between the parabolas.

$$A = \int_0^{b/1+b^2} \left(x - bx^2 - \frac{x^2}{b}\right) dx.$$

$$= \left\{\frac{x^2}{2} - \frac{x^3}{3}\cdot\frac{1+b^2}{b}\right\}_0^{b/1+b^2}$$

$$= \frac{1}{6}\cdot\frac{b^3}{(1+b^2)^2}$$

for maximum, value of A, $\frac{dA}{db} = 0$

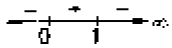
But $\frac{dA}{db} = \frac{1}{6}\cdot\frac{(1+b^2)^2\cdot 2b - 2b^2\cdot(1+b^2)\cdot 2b}{(1+b^2)^4}$

$$= \frac{1}{3}\cdot\frac{b(1-b^2)}{(1+b^2)^3}$$

Hence, $\frac{dA}{db} = 0$ gives $b = -1, 0, 1$ since $b > 0$

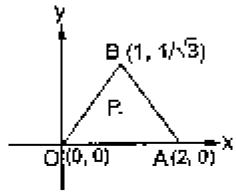
$\therefore$ we consider only $b = 1$.

Sign scheme for $\frac{dA}{db}$ around $b=1$ is as below



from sign scheme it is clear A is maximum at $b=1$.

16. Let the coordinate of P be (x, y)



Equation of line OA be $y=0$

Equation of line OB be $\sqrt{3}\ y=x$

Equation of line AB be $\sqrt{3}\ y=2-x$

$d(P, OA)=$ distance of P from line $OA=y$

$d(P, OB)=$ distance of P from line $OB=\frac{|\sqrt{3}y-x|}{2}$

$d(P, AB)=$ distance of P from line $AB=\frac{|\sqrt{3}y+x-2|}{2}$

Given : $d(P, OA)\le \min.\{d(P, OB), d(P, AB)\}$

$$y\le \min.\left\{\frac{|\sqrt{3}y-x|}{2}, \frac{|\sqrt{3}\ y+x-2|}{2}\right\}$$

$$\Rightarrow \quad y\le \frac{|\sqrt{3}y-x|}{2} \quad \text{and} \quad y\le \frac{|\sqrt{3}y+x-2|}{2}$$

Case I : $y\le \frac{|\sqrt{3}y-x|}{2}$, {since $\sqrt{3}y-x<0$}

$$\Rightarrow \quad y\le \frac{x-\sqrt{3}y}{2}$$

$$\Rightarrow \quad (2+\sqrt{3})y\le x.$$

$$\Rightarrow \quad y\le x\tan 15°$$

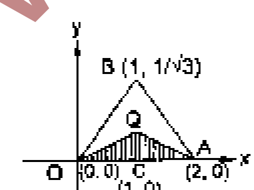
Case II : If $y\le \frac{|\sqrt{3}y+x-2|}{2}$

$$\Rightarrow \quad 2y\le 2-x-\sqrt{3}y \quad [i.e., \sqrt{3}y+x-2<0]$$

$$\Rightarrow \quad (2+\sqrt{3})y\le 2-x$$

$$\Rightarrow \quad y\le \tan 15°.(2-x)$$

from above discussion P moves in side the Δ as shown.



$\Rightarrow$ Area of shaded region

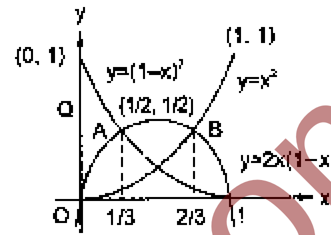
$=$ area of ΔOQA

$=\frac{1}{2}$ (base) $\times$ height

$=\frac{1}{2}(2)(\tan 15°)=\tan 15°$

$=(2-\sqrt{3})$ sq. units

17. We can draw the graph of $y=x^2$, $y=(1-x^2)$ and $y=2x(1-x)$ in following fig. Now, to get the point of intersection of $y=x^2$ and $y=2x(1-x)$. We solve both the equations.



we get $x^2=2x(1-x)$

$$\Rightarrow \quad x^2=2x-2x^2$$

$$\Rightarrow \quad 3x^2=2x$$

$$\Rightarrow \quad 3x^2-2x=0$$

$$\Rightarrow \quad x(3x-2)=0$$

$$\Rightarrow \quad x=0, 2/3$$

Similarly, we can find the coordinate of the points of intersection of

$y=(1-x^2)$ and $y=2x(1-x)$ are $x=1/3$ and $x=1$

From the figure it is clear that

$$f(x)=\begin{cases}(1-x)^2, & 0\le x\le 1/3\\ 2x(1-x), & 1/3\le x\le 2/3\\ x^2, & 2/3\le x\le 1\end{cases}$$

The required area A is given by

$$A=\int_0^1 f(x)\,dx$$

$$=\int_0^{1/3}(1-x)^2\,dx+\int_{1/3}^{2/3}2x(1-x)\,dx+\int_{2/3}^{1}x^2\,dx$$

$$=\left[-\frac{1}{3}(1-x)^3\right]_0^{1/3}+\left[x^2-\frac{2x^3}{3}\right]_{1/3}^{2/3}+\left[\frac{1}{3}x^3\right]_{2/3}^{1}$$

$$=-\frac{1}{3}\left(\frac{2}{3}\right)^3+\frac{1}{3}+\left(\frac{2}{3}\right)^2-\frac{2}{3}\left(\frac{2}{3}\right)^3$$

$$-\left(\frac{1}{3}\right)^2+\frac{2}{3}\left(\frac{1}{3}\right)^3+\frac{1}{3}(1)-\frac{1}{3}\left(\frac{2}{3}\right)^3$$

$$=\frac{17}{27}\text{ sq. units}$$

18. Refer to the Fig. in the question. Let the coordinates of P be (x, x^2), where $0\le x\le 1$.

For the area $(OPRO)$, upper boundary $y=x^2$

lower boundary : $y - f(x)$

lower limit of x : 0

upper limit of x : x

$\therefore$ area $(OPRO) = \int_0^x t^2\, dt - \int_0^x f(t)\, dt$

$$= \left[\frac{t^3}{3}\right]_0^x - \int_0^x f(t)\, dt$$

$$= \frac{x^3}{3} - \int_0^x f(t)\, dt - \int_0^{x^2} \frac{t}{2}\, dt$$

For the area $(OPQO)$ the upper curve : $x = \sqrt{y}$

the lower curve : $x - y/2$

lower limit of y : 0 and upper limit of y : x^2

$\therefore$ area $(OPQO) = \int_0^{x^2} \sqrt{t}\, dt \cdot \int_0^{x^2} \frac{t}{2}\, dt$

$$= \frac{2}{3}[t^{3/2}]_0^{x^2} - \frac{1}{4}[t^2]_0^{x^2}$$

$$= \frac{2}{3}x^3 - \frac{1}{4}x^4$$

according to the given condition,

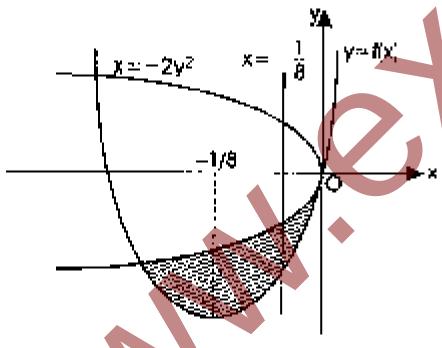
$$\frac{1}{3}x^3 - \int_0^x f(t)\, dt = \frac{2}{3}x^3 \quad \frac{x^4}{4}$$

Differentiating both sides w.r.t. x, we get

$$x^2 - f(x) \cdot 1 = 2x^2 - x^3$$

$$\Rightarrow \quad f(x) = x^3 - x^2,\ 0 \le x \le 1$$

19. $f(x) = \begin{cases} 2x, & |x| \le 1 \\ x^2 + ax + b, & |x| > 1 \end{cases}$



$$\Rightarrow \quad f(x) = \begin{cases} x^2 + ax + b, \text{ if } x < -1 \\ 2x, \text{ if } -1 \le x < 1 \\ x^2 + ax + b \text{ if } 1 \le x \end{cases}$$

f is continuous on R so f is continuous at -1 and 1.

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} f(x) = f(-1)$$

and $\lim_{x \to 1} f(x) = \lim_{x \to 1^-} f(x) = f(1)$

$\Rightarrow \quad 1 - a + b = -2$ and $2 = 1 + a + b$

Thus, $a - b = 3$ and $a + b = 1$

$\therefore \quad a = 2, \quad b = -1$

Hence, $f(x) = \begin{cases} x^2 + 2x - 1 & \text{if } x < -1 \\ 2x & \text{if } -1 \le x < 1 \\ x^2 + 2x - 1 & \text{if } x \ge 1 \end{cases}$

Next, we have to find the point $x = -2y^2$ and $y = f(x)$

The point of intersection is $(-2, -1)$

$\therefore$ Required Area

$$= \int_{-2}^{-1/8} \left[\sqrt{\frac{-x}{2}} - f(x)\right] dx$$

$$= \int_{-2}^{-1/8} \sqrt{\frac{-x}{2}}\, dx - \int_{-2}^{-1} (x^2 + 2x - 1)\, dx - \int_{-1}^{-1/8} 2x\, dx$$

$$= \frac{-2}{3\sqrt{2}}[(-x)^{-1/8}]_{-2}^{-1/8} - \left[\left(\frac{x^3}{3} + x^2 - x\right)\right]_{-2}^{-1} - [x^2]_{-1}^{-1/8}$$

$$= \frac{-2}{3\sqrt{2}}\left[\left(\frac{1}{8}\right)^{3/2} - 2^{3/2}\right] - \left(-\frac{1}{3} + 1 + 1\right) + \left(-\frac{8}{3} + 4 + 2\right) - \left[\frac{1}{64} - 1\right]$$

$$= \frac{\sqrt{2}}{3}[2\sqrt{2} \cdot 2^{-9/2}] + \frac{5}{3} + \frac{63}{64}$$

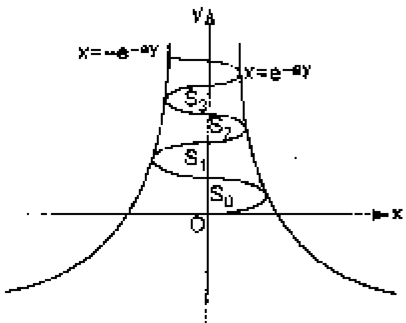
$$= \frac{761}{192}$$

20. Given $x = (\sin by)\, e^{-ay}$

Now, $\quad -1 \le \sin by \le 1$

$\Rightarrow \quad -e^{-ay} \le -e^{-ay} \sin by \le e^{-ay}$

$\Rightarrow \quad -e^{-ay} \le x \le e^{-ay}$



In this case if we take a and b positive, the values $-e^{-ay}$ and e^{-ay} become left bond and right bond of the curve and due to oscillating nature of $\sin by$, it will oscillate between $x = e^{-ay}$ and $x = -e^{-ay}$

Now, $\quad S_j = \int_{j\pi/b}^{(j+1)\pi/b} \sin by . e^{-ay}\, dy$

$\Rightarrow \quad I = \int \sin by . e^{-ay}\, dy$

Integrating by parts, we get

$$I = \frac{-e^{-ay}}{a^2 + b^2} \quad (a \sin by + b \cos by)$$

[by Euler's Integral]

So, $S_j = \left|\frac{-1}{a^2+b^2}\left[e^{\frac{-a(j+1)\pi}{b}}\{a\sin(j+1)\pi + b\cos(j+1)\pi\} - e^{\frac{-aj\pi}{b}}\{a\sin j\pi + b\cos j\pi\}\right]\right|$

$$S_j = \left|-\frac{1}{a^2+b^2}\left[e^{-\frac{a}{b}(j+1)\pi}(0+b(-1)^{j+1}) - e^{-aj\pi/b}(0-b(-1)^j)\right]\right|$$

$$= \left|\frac{b.(-1)^j e^{-\frac{a}{b}j\pi}}{a^2+b^2}\left(e^{-\frac{a}{b}\pi}+1\right)\right|$$

$$= \frac{b.e^{-\frac{a}{b}j\pi}}{a^2+b^2}\left(e^{-\frac{a}{b}\pi}+1\right)$$

$[\because (-1)^{j+2} = (-1)^2(-1)^j = (-1)^j]$

Now, $$\frac{S_j}{S_{j-1}} = \frac{\dfrac{be^{-\frac{a}{b}j\pi}\left(e^{\frac{-a\pi}{b}}+1\right)}{a^2+b^2}}{\dfrac{be^{-\frac{a}{b}(j-1)\pi}\left(e^{\frac{-a\pi}{b}}+1\right)}{a^2+b^2}}$$

$$\Rightarrow \frac{e^{-\frac{a}{b}j\pi}}{e^{-\frac{a}{b}(j-1)\pi}} = e^{-\frac{a}{b}\pi} = \text{constant}$$

$\Rightarrow \frac{S_j}{S_{j-1}} = \text{constant} \Rightarrow S_0, S_1, S_2, \ldots, S_j$ form a G.P.

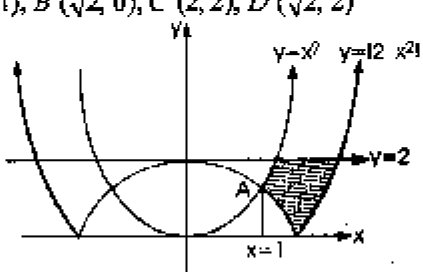
For $a = -1$ and $b = \pi$

$$S_j = \frac{\pi.e^{\frac{1}{\pi}\pi j}}{(1+\pi^2)}\left(e^{\frac{1}{\pi}\pi}+1\right) = \frac{\pi.e^j}{(1+\pi^2)}(1+e)$$

$$\Rightarrow \sum_{j=0}^{n} S_j = \frac{\pi(1+e)}{(1+\pi^2)}\sum_{j=0}^{n} e^j$$

$$= \frac{\pi(1+e)}{(1+\pi^2)}(e^0+e^1+\ldots+e^n) = \frac{\pi(1+e)}{(1+\pi^2)}\cdot\frac{(e^{n+1}-1)}{e-1}$$

21. The points in the graph are :
$A(1,1)$, $B(\sqrt{2},0)$, $C(2,2)$, $D(\sqrt{2},2)$



Required area

$$= \int_1^{\sqrt{2}}(x^2-(2-x^2))\,dx + \int_{\sqrt{2}}^{2}(2-(x^2-2))\,dx$$

$$= \int_1^{\sqrt{2}}(2x^2-2)\,dx + \int_{\sqrt{2}}^{2}(4-x^2)\,dx$$

$$= \left(\frac{2x^3}{3}-2x\right)_1^{\sqrt{2}} + \left(4x-\frac{x^3}{3}\right)_{\sqrt{2}}^{2}$$

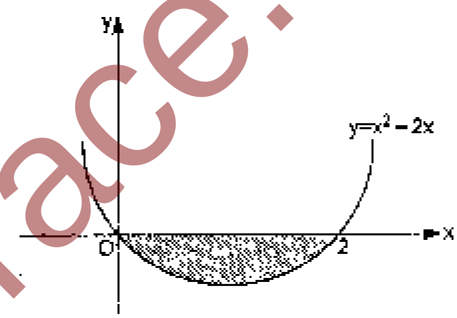
$$= \left(\frac{20-12\sqrt{2}}{3}\right) \text{ sq. units.}$$

22. Here, slope of tangent

$$\Rightarrow \frac{dy}{dx} = \frac{(x+1)^2+y-3}{(x+1)}$$

$$\Rightarrow \frac{dy}{dx} = (x+1) + \frac{(y-3)}{(x+1)}$$

put $x+1 = X$ and $y-3 = Y$



$\Rightarrow \frac{dy}{dx} = \frac{dY}{dX}, \therefore \frac{dY}{dX} = X + \frac{Y}{X}$ or $\frac{dY}{dX} - \frac{1}{X}Y = X$

where integrating factor

$$= e^{\int -\frac{1}{X}dX} = e^{-\log X} = \frac{1}{X}$$

∴ Solution is,

$$Y\cdot\frac{1}{X} = \int X\cdot\frac{1}{X}\,dX + c \text{ or } \frac{Y}{X} = X + c$$

$y-3 = (x+1)^2 + c(x+1)$, which passes through $(2, 0)$

$$-3 = 1 + c$$

$$\Rightarrow \quad c = -4$$

∴ Required curve

$$y = (x+1)^2 - 4(x+1) + 3 \text{ or } y = x^2 - 2x$$

Shown as,

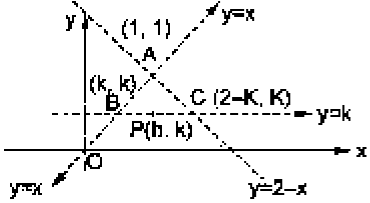
Thus, required area

$$= \left|\int_0^2 (x^2-2x)dx\right| = \left|\left(\frac{x^3}{3}-x^2\right)_0^2\right| = \frac{4}{3} \text{ sq. units}$$

23. Here, the triangle formed by a line parallel to x-axis passing through $P(h,k)$ and the straight line $y = x$ and

$y = 2 - x$ could be shown as,

Since, area of $\Delta\, ABC = 4h^2$



∴ $\frac{1}{2}\cdot AB\cdot AC=4h^2$,

where $AB=\sqrt{2}\,|k-1|$ and $AC=\sqrt{2}(|k-1|)$

⇒ $\frac{1}{2}\cdot 2(k-1) = 4h^2$

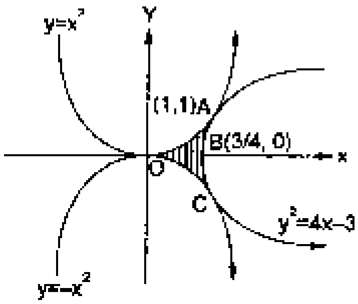
∴ $4h^2=(k-1)^2$

or $2h=\pm(k-1)$ is required locus

i.e., $2x=\pm(y-1)$

24. The region bounded by the curves $y=x^2$, $y=\ x^2$ and $y^2=4x-3$ is symmetrical about x-axis.

where $y=4x-3$ meets at (1, 1)



Hence, area ($OABCO$)

$$=2\left\{\int_0^1 x^2\,dx-\int_{3/4}^1(\sqrt{4x-3})\,dx\right\}$$

$$=2\left\{\left[\left(\frac{x^3}{3}\right)\right]_0^1-\left[\frac{(4x-3)^{3/2}}{32/4}\right]_{3/4}^1\right\}$$

$$=2\left\{\frac{1}{3}-\frac{1}{6}\right\}=1\cdot\frac{1}{6}=\frac{1}{3}\text{ square units}$$

25. Given, $\begin{bmatrix}4a^2 & 4a & 1\\ 4b^2 & 4b & 1\\ 4c^2 & 4c & 1\end{bmatrix}\begin{bmatrix}f(-1)\\ f(1)\\ f(2)\end{bmatrix}=\begin{bmatrix}3a^2+a\\ 3b^2+b\\ 3c^2+c\end{bmatrix}$

⇒ $4a^2 f(-1)+4a f(1)+f(2)=3a^2+a$ ...(1)

$4b^2 f(-1)+4b f(1)+f(2)=3b^2+b$ ...(2)

$4c^2 f(-1)+4c f(1)+f(2)=3c^2+c$ ...(3)

where $f(x)$ is quadratic expression given by,

$f(x)=ax^2+bc+c$ and (1), (2) and (3)

⇒ $4x^2 f(-1)+4x f(1)+f(2)=3x^2+3x$

or $(4f(-1)-3)x^2+(4f(1)-3)x+f(2)=0$ ...(4)

As above equation has 3 roots a, b and c.

∴ above equation is identity in x.

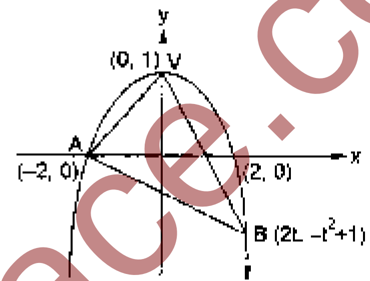
i.e., Coefficients must be zero

⇒ $f(-1)=3/4,\ f(1)=3/4,\ f(2)=0$ ...(5)

∴ $f(x)=ax^2+bx+c$

⇒ $a=-1/4$, $b=0$ and $c=1$, using (5)

Thus, $f(x)=\frac{4-x^2}{4}$ shown as,



Let $A(-2,0)$, $B=(2t,-t^2+1)$

Since AB subtends right angle at vertex V (0, 1)

⇒ $\frac{1}{2}\cdot\frac{-t^2}{2t}=-1$

⇒ $t=4$

∴ $B(8,-15)$

Thus, Area $=\int_2^8\left(\frac{4-x^2}{4}+\frac{3x+6}{2}\right)dx$

$$=\left(x-\frac{x^3}{12}+\frac{3x^2}{4}+3x\right)_{-2}^{8}$$

$$=\frac{125}{3}\text{ square units}$$

□