Partial differential equations

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Introduction

Our objective is to solve traditionally important PDEs by the method of *separation of variables*. This leads to simultaneous ODEs in each variable and the ODEs which generally occur (depending on the geometry of the space or the geometry of the pre-assigned boundary values) are the Legendre equation, the Bessel equation and the equation of a simple harmonic oscillator.

We begin by solving the Legendre equation, and the Bessel equation. (The solution of the equation of the simple harmonic oscillator is assumed to be known.) Next we take up various PDEs one by one and solve some illustrative problems.

(1) Heat equation (also called evolution equation or diffusion equation).
(2) Wave equation.
(3) Laplace equation (also called steady state equation).

Wikipedia gives good information on any topic and is often a good place to start. Some books that you may want to look at for additional information are [2], [3], [7] and [10]. Two useful books specifically written for engineers are [6] and [14]. More references are provided in the text.

Acknowledgement. I am grateful to Akhil Ranjan for many discussions on this subject. These notes are expanded from a detailed draft prepared by him.

Pictures in the text have been borrowed from wikipedia or from similar sources on the internet.

CHAPTER 1

Power series

1.1. Convergence criterion for series

We assume basic familiarity with the notion of convergence of series. A series is said to diverge if it does not converge.

Let $\sum_n a_n$ be a series of (real or complex) numbers. The following are some standard tests for determining its convergence or divergence.

(1) *Weierstrass M-test:*
If $\sum_n M_n$ is a convergent series of nonnegative constants, and $|a_n| \leq M_n$ for all n, then $\sum_n a_n$ is also (absolutely) convergent.

(2) *Ratio test:*
If $a_n \neq 0$ for all n, and if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$, then $\sum_n a_n$
- converges (absolutely) if $L < 1$,
- diverges if $L > 1$.

The test is inconclusive if $L = 1$.

(3) *Root test:*
If $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = L$, then $\sum_n a_n$
- converges (absolutely) if $L < 1$,
- diverges if $L > 1$.

The test is inconclusive if $L = 1$.

(4) *Refined root test:*
If $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = L$, then $\sum_n a_n$
- converges (absolutely) if $L < 1$,
- diverges if $L > 1$.

The test is inconclusive if $L = 1$.

Remark 1.1. In the ratio or root test, existence of the limit L is a pre-requisite. A definition of $\liminf$ (short for limit infimum) and $\limsup$ (short for limit supremum) can be found in [8, Definition 3.16] (or see wikipedia). The advantage of working with $\liminf$ and $\limsup$ is that it always exists. The latter is used in the refined root test. There is also a refined ratio test in which $\limsup < 1$ is used to conclude convergence and $\liminf > 1$ is used to conclude divergence. The Weierstrass M-test is the most basic criterion and is, in fact, used to establish the remaining tests by comparing with a suitable geometric series of positive terms.

1.2. Power series

For a real number x_0 and a sequence (a_n) of real numbers, consider the expression

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + \dots. \tag{1.1}$$

This is called a *power series in the real variable* x. The number a_n is called the n-th coefficient of the series and x_0 is called its *center*.

For instance,

$$\sum_{n=0}^{\infty} \frac{1}{n+1}(x-1)^n = 1 + \frac{1}{2}(x-1) + \frac{1}{3}(x-1)^3 + \dots$$

is a power series in x centered at 1 and with n-th coefficient equal to $\frac{1}{n+1}$.

What can we do with a power series? Note that by substituting a value for x in a power series, we get a series of real numbers. We say that a power series converges (absolutely) at x_1 if substituting x_1 for x in (1.1) yields a (absolutely) convergent series. A power series always converges absolutely at its center x_0.

We would like to know the set of values of x where a power series converges. The following can be shown by applying the refined root test.

Lemma 1.2. *Suppose the power series (1.1) converges for some real number $x_1 \neq x_0$. Let $|x_1 - x_0| = r$. Then the power series is (absolutely) convergent for all x such that $|x - x_0| < r$, that is, in the open interval (or disc) of radius r centered at x_0.*

Definition 1.3. The *radius of convergence* of the power series (1.1) is the *largest* number R, including ∞, such that the power series converges in the open interval (or disc) $\{|x - x_0| < R\}$. The latter is called the *interval of convergence* of the power series.

It is possible that there is no $x_1 \neq x_0$ for which the power series (1.1) converges. An example is $\sum_n n!x^n$. In this case, $R = 0$. The geometric series $\sum_n x^n$ has radius of convergence $R = 1$. It does not converge at $x = 1$ or $x = -1$.

We can calculate the radius of convergence of a power series in terms of its coefficients by the following methods:

(1) *Ratio test:*
If $a_n \neq 0$ for all n, and if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$, then the radius of convergence of the power series is $\frac{1}{L}$.

(2) *Root test:*
If $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = L$, then the radius of convergence of the power series is $\frac{1}{L}$.

(3) *Refined root test:*
If $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = L$, then the radius of convergence of the power series is $\frac{1}{L}$.

These tests are deduced from the corresponding tests for series.

A power series determines a function in its interval of convergence. Denoting this function by f, we may write

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n, \qquad |x - x_0| < R. \tag{1.2}$$

Let us assume $R > 0$. It turns out that f is infinitely differentiable on the interval of convergence, and the successive derivatives of f can be computed by differentiating the power series on the right termwise. From here one can deduce that

$$a_n = \frac{f^{(n)}(x_0)}{n!}. \tag{1.3}$$

Thus, two power series both centered at x_0 take the same values in some open interval around x_0 iff all the corresponding coefficients of the two power series are equal. Explicitly, if

$$a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots = b_0 + b_1(x - x_0) + b_2(x - x_0)^2 + \ldots$$

holds in some interval around x_0, then $a_n = b_n$ for all n. (The converse is obvious.)

The above discussion is also valid if we take x_0 and the a_n to be complex numbers, and let x be a complex variable.

1.3. Real analytic functions

We saw that a power series determines a function in its interval of convergence. Let us now invert the situation. Namely, we start with a function f, and ask whether there is a power series whose associated function is f.

1.3.1. Real analytic functions. Let $\mathbb{R}$ denote the set of real numbers. A subset U of $\mathbb{R}$ is said to be *open* if for each $x_0 \in U$, there is a $r > 0$ such that the open interval $|x - x_0| < r$ is contained in U.

Definition 1.4. Let $f : U \to \mathbb{R}$ be a real-valued function on an open set U. We say f is real analytic at a point $x_0 \in U$ if (1.2) holds in some open interval around x_0. We say f is *real analytic* on U if it is real analytic at all points of U.

In general, we can always consider the set of all points in the domain where f is real analytic. This is called the *domain of analyticity* of f.

Just like continuity or differentiability, real analyticity is a local property.

Suppose f is real analytic on U. Then f is infinitely differentiable on U, and its power series representation around x_0 is necessarily the Taylor series of f around x_0 (that is, the coefficients a_n are given by (1.3)).

If f and g are real analytic on U, then so is cf, $f + g$, fg, and f/g (provided $g \neq 0$ on U).

A power series is real analytic in its interval of convergence. Thus, if a function is real analytic at a point x_0, then it is real analytic in some open interval around x_0. Thus, the domain of analyticity of a function is an open set.

1.3.2. Examples. Polynomials such as $x^3 - 2x + 1$ are real analytic on all of $\mathbb{R}$: A polynomial is a truncated power series (so there is no issue of convergence). By writing $x = x_0 + (x - x_0)$, we can rewrite any polynomial using powers of $x - x_0$. This will be a truncated power series centered at x_0.

The sine, cosine and exponential functions are real analytic on all of $\mathbb{R}$. The Taylor series written using (1.3) are

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots.$$

Using Taylor's remainder formula, one can show that these identities are valid for all x. However this may not always be the case. Consider the function

$$f(x) = \frac{1}{1+x^2}.$$

It is real analytic on all of $\mathbb{R}$. The Taylor series around 0 is

$$\frac{1}{1+x^2} = (1 - x^2 + x^4 - x^6 + \dots).$$

This series has a radius of convergence 1 and the identity only holds for $|x| < 1$. At $x = 1$, note that $f(1) = 1/2$ while the series oscillates between 1 and 0. Thus, even though a function is real analytic, it may not be representable by a single power series. From the analysis done so far, we can only conclude that f is real analytic in $(-1, 1)$. If we want to show that f is real analytic at 1, then we need to need to find another power series centered at 1 which converges to f in an interval around 1. This is possible. Alternatively, one can use a result stated earlier to directly deduce that f is real analytic everywhere. Since 1 and $1+x^2$ are polynomials, they are real analytic analytic, and hence so is their quotient. Draw picture.

The function $f(x) = x^{1/3}$ is defined for all x. It is not differentiable at 0 and hence not real analytic at 0. However it is real analytic at all other points. For instance,

$$x^{1/3} = (1 + (x-1))^{1/3} = 1 + \frac{1}{3}(x-1) + \frac{1}{3}(\frac{1}{3} - 1)\frac{(x-1)^2}{2!} + \dots$$

is valid for $|x - 1| < 1$, showing analyticity in the interval $(0, 2)$. (This is the binomial theorem which we will prove below.) Analyticity at other nonzero points can be established similarly. Thus, the domain of analyticity of $f(x) = x^{1/3}$ is $\mathbb{R} \setminus \{0\}$.

The function

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases} \tag{1.4}$$

is infinitely differentiable. But it is not real analytic at 0 since $f^{(n)}(0) = 0$ for all n and the Taylor series around 0 is identically 0. The domain of analyticity of f is $\mathbb{R} \setminus \{0\}$. Draw picture.

(This kind of behaviour does not happen for functions of a complex variable. If a function of a complex variable is differentiable in a region, then it is complex analytic in that region. If x is a complex variable, then (1.4) has an essential singularity at 0, in particular, it is not differentiable at 0. Also note that if x is allowed to be complex, the function $\frac{1}{1+x^2}$ is not defined at $x = \pm i$ explaining why we got the corresponding power series to have a radius of convergence of 1.)

Remark 1.5. For a function f, never say "convergence of f", instead say "convergence of the Taylor series of f".

1.4. Solving a linear ODE by the power series method

Consider a particle moving on the real line. Laws of classical physics say: If we know its position at a particular time and velocity at all times, then its position at all times is uniquely determined. If we know its position and velocity at a particular time and acceleration at all times, then its position at all times is uniquely determined.

Mathematically, this suggests that a first order ODE with value specified at a point, or a second order ODE with value and derivative specified at a point has a unique solution. It is too much to expect such statements to hold in complete generality. One needs to impose some conditions on the ODE, and further the solutions will only exist in some interval around the point. We now discuss a result of this nature.

Consider the initial value problem

$$p(x)y'' + q(x)y' + r(x)y = g(x), \quad y(a) = y_0, \; y'(a) = y_1, \tag{1.5}$$

where p, q, r and g are real analytic functions in an interval containing the point a, and y_0 and y_1 are specified constants.

Theorem 1.6. *Let $r > 0$ be less than the minimum of the radii of convergence of the functions p, q, r and g expanded in power series around a. Assume that $p(x) \neq 0$ for all $x \in (x_0 - r, x_0 + r)$. Then there is a unique solution to the initial value problem (1.5) in the interval $(x_0 - r, x_0 + r)$, and moreover it can be represented by a power series*

$$y(x) = \sum_{n \geq 0} a_n (x - x_0)^n \tag{1.6}$$

whose radius of convergence is at least r.

Proof. There is an algorithm to compute the power series representation of y: Plug (1.6) into (1.5), take derivatives of the power series formally, and equate the coefficients of $(x - x_0)^n$ for each n to obtain a recursive definition of the coefficients a_n. (In most examples, $x_0 = 0$ and the functions p, q, r and g are polynomials.) The a_n's are uniquely determined and we obtain a formal power series solution. But we are not done here. We need to show that the obtained formal solution has a positive radius of convergence. This argument is given in [10, Chapter 5, Appendix A].

To see why the last step is essential: Suppose in trying to solve an ODE using this power series method, we obtain the recursion $a_{n+1} = (n+1)a_n$ and $a_1 = 1$. Then $a_n = n!$, so the formal solution is $\sum_n n! x^n$. But this does not converge at any point except $x = 0$, so the power series is not really giving a solution. The crux of the last step is to show that such "bad" recursions do not arise. ∎

This result generalizes to any n-th order linear ODE with the first $n-1$ derivatives at x_0 specified. In particular, it applies to the first order linear ODE for which the initial condition is simply the value at x_0. Suppose the first order ODE is

$$p(x)y' + q(x)y = 0.$$

Then, by separation of variables, we see that the general solution is

$$ce^{-\int \frac{q(x)}{p(x)}dx}, \tag{1.7}$$

provided $p(x) \neq 0$ else the integral may not be well-defined. This is an indicator why such a condition is required in the hypothesis of Theorem 1.6.

Example 1.7. Consider the first order linear ODE

$$y' - y = 0, \quad y(0) = 1.$$

The coefficients are constants, so Theorem 1.6 applies and it will yield a solution which is real analytic on all of $\mathbb{R}$. Write $y = \sum_n a_n x^n$. The initial condition $y(0) = 1$ implies $a_0 = 1$. Comparing the coefficient of x^n on both sides of $y' = y$ yields

$$(n+1)a_{n+1} = a_n \text{ for } n \geq 0.$$

This is a 2-step recursion. It can be solved explicitly as $a_n = 1/n!$. Thus

$$y(x) = \sum_n \frac{1}{n!}x^n$$

which we know is e^x. Thus, the function e^x is characterized by the property that its derivative is itself, and its value at 0 is 1.

Remark 1.8. In calculus textbooks, we first define the logarithm function by

$$\log x = \int_1^x \frac{1}{t}dt.$$

The exponential function e^x is then defined as its inverse. The properties $(e^x)' = e^x$ and $e^0 = 1$ are deduced from properties of log. Thus, e^x satisfies the above initial value problem, and so by uniqueness of the solution, we see that the Taylor series of e^x is valid for all x. In calculus textbooks, this is established using Taylor's remainder formula, here we see it as a consequence of Theorem 1.6.

Example 1.9. Consider the first order linear ODE

$$y' - 2xy = 0, \quad y(0) = 1.$$

We proceed as in the previous example. The initial condition $y(0) = 1$ implies $a_0 = 1$. This time we get a 3-step recursion:

$$(n+1)a_{n+1} = 2a_{n-1} \quad \text{for } n \geq 1,$$

and $a_1 = 0$. So all odd coefficients are zero. For the even coefficients, the recursion can be rewritten as $na_{2n} = a_{2n-2}$, so $a_{2n} = 1/n!$. Thus

$$y(x) = \sum_n \frac{1}{n!}x^{2n}$$

(which we know is e^{x^2}).

Example 1.10. Consider the function $f(x) = (1+x)^p$ where $|x| < 1$ and p is any real number. Note that it satisfies the linear ODE.

$$(1+x)y' = py, \quad y(0) = 1.$$

Let us solve this using the power series method around $x = 0$. Since $1 + x$ is zero at $x = -1$, we are guaranteed a solution only for $|x| < 1$. The initial condition

$y(0) = 1$ implies $a_0 = 1$. To calculate the recursion, express each term as a power series:

$$\begin{aligned} y' &= a_1 + 2a_2 x + 3a_3 x^2 + \cdots + (n+1)a_{n+1}x^n + \ldots \\ xy' &= \qquad\quad a_1 x + 2a_2 x^2 + \cdots + n a_n x^n + \ldots \\ py &= pa_0 + pa_1 x + pa_2 x^2 + \cdots + pa_n x^n + \ldots . \end{aligned}$$

Comparing coefficients yields a 2-step recursion:

$$a_{n+1} = \frac{p-n}{n+1}\, a_n \quad \text{for } n \geq 0.$$

This shows that

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2}x^2 + \ldots .$$

This is the *binomial theorem* and we just proved it. The most well-known case is when p is a positive integer (in which case the power series terminates to a polynomial of degree p). Also check by the ratio test that the power series has radius of convergence 1 if p is not a positive integer.

All the above examples were of first order ODEs. Check that the solutions obtained by the power series method agree with (1.7).

Example 1.11. Consider the second order linear ODE

$$y'' + y' - 2y = 0.$$

(The initial conditions are left unspecified.) Proceeding as before, we get a 3-step recursion:

$$(n+2)(n+1)a_{n+2} + (n+1)a_{n+1} - 2a_n = 0 \quad \text{for } n \geq 0,$$

and a_0 and a_1 are arbitrary. There is a general method to solve recursions of this kind. First substitute $b_n = n!a_n$, so we obtain

$$b_{n+2} + b_{n+1} - 2b_n = 0 \quad \text{for } n \geq 0,$$

We now guess that $b_n = \lambda^n$ is a solution. This yields the quadratic $\lambda^2 + \lambda - 2 = 0$ (which can be written directly from the constant-coefficient ODE). Its roots are 1 and -2. Thus the general solution is

$$b_n = \alpha + \beta\,(-2)^n,$$

where α and β are arbitrary. Alternatively, since the sum of the coefficients in the recursion is 0, by add the first $n+1$ recursion, we get $b_{n+2} + 2b_{n+1} = 2b_0 + b_1$ which says that the b_n are in shifted geometric progression. The general solution to the original recursion is

$$a_n = \alpha\,\frac{1}{n!} + \beta\,\frac{(-2)^n}{n!},$$

So the general solution to the ODE is

$$y(x) = \alpha e^x + \beta e^{-2x}.$$

(How would you solve the recursion if the quadratic has repeated roots? Recall how you solve a constant-coefficient ODE whose auxiliary equation has a repeated root?)

Remark 1.12. The Hemachandra-Fibonacci numbers are defined by the recursion $b_{n+2} = b_{n+1} + b_n$ with $b_0 = b_1 = 1$. The above method yields a formula for these numbers.

In all the above examples, we were able to solve the recursion in closed form. But this may not always be possible.

Example 1.13. Consider the second order linear ODE

$$y'' + y = 0.$$

(The initial conditions are left unspecified.) Proceeding as before, we get

$$(n+2)(n+1)a_{n+2} + a_n = 0 \quad \text{for } n \geq 0,$$

and a_0 and a_1 are arbitrary. Observe that the general solution to the ODE is

$$y(x) = a_0 \cos(x) + a_1 \sin(x).$$

Remark 1.14. How should one define the sine and cosine functions? In school, the sine is defined as the ratio of the opposite side and the hypotenuse. But this assumes that we know what angle is. We can take the unit circle and define angle as the length of the corresponding arc, and we know how to define length of the arc. Thus, for $x^2 + y^2 = 1$, if we let

$$z = \int_0^y \sqrt{1 + (\frac{dx}{dy})^2 dy},$$

then $\sin z = y$. So what we do in school amounts to defining $\sin^{-1}$ by the formula

$$z = \sin^{-1}(y) = \int_0^y \frac{1}{\sqrt{1-t^2}} dt.$$

Note the similarity with the approach to exp via log.

Question 1.15. Given a linear ODE with polynomial coefficients, how can you determine by inspection what step recursion you are going to get? (If the coefficients are not polynomials, then the recursion is not finite-step. As n increases, one has to look further and further back to determine a_n.) Give formula.

Remark 1.16. While computing the recursion for a linear ODE, it is useful to keep in mind that: Taking derivative lowers degree by 1 while multiplying by x raises degree by 1. The two operators do not commute. In fact

$$\frac{d}{dx}x - x\frac{d}{dx} = 1.$$

This is the famous Heisenberg relation.

CHAPTER 2

Legendre equation

We study the Legendre equation. There is one for each real number p. We solve it using the power series method. When p is a nonnegative integer n, we get a polynomial solution. This is the n-th Legendre polynomial. We consider orthogonality properties of these polynomials, and also write down their generating function.

2.1. Legendre equation

Consider the second order linear ODE:

$$(1-x^2)y'' - 2xy' + p(p+1)y = 0. \tag{2.1}$$

This is known as the *Legendre equation*. Here p denotes a fixed real number. (It suffices to assume $p \geq -1/2$. Why?) The equation can also be written in the form

$$((1-x^2)y')' + p(p+1)y = 0. \tag{2.2}$$

This is called the Sturm-Liouville or self-adjoint form of the ODE.

This ODE is defined for all real numbers. The coefficients $(1-x^2)$, $-2x$ and $p(p+1)$ are polynomials (and in particular real analytic). However $1-x^2=0$ for $x=\pm 1$. These are the *singular points* of the ODE. So by Theorem 1.6, we are only guaranteed a power series solution around $x=0$ in the interval $(-1,1)$. (The solution is unique when $y(0)$ and $y'(0)$ are specified.) Accordingly, let $y=\sum a_n x^n$. Equating the coefficients of x^n from each term yields a 3-step recursion:

$$a_{n+2} = \frac{(n-p)(n+p+1)}{(n+2)(n+1)} a_n, \quad n \geq 0, \tag{2.3}$$

with a_0 and a_1 arbitrary. To see this, we rewrite the ODE as $y'' = x^2y'' + 2xy' - p(p+1)y$. The lhs contributes the term $(n+2)(n+1)a_{n+2}$ while the rhs contributes $[n(n-1)+2n-p(p+1)]a_n$ which is the same as $[n(n+1)-p(p+1)]a_n$. Observe that something interesting is going to happen when p is a nonnegative integer. We will be discussing this case in detail.

Since the a_{n+1} term is absent in the recursion, it breaks into two 2-step recursions relating the odd and even terms respectively. (Note the similarity with the analysis done in Example 1.13.) Explicitly,

$$a_2 = -\frac{p(p+1)}{2!}a_0, \; a_4 = +\frac{(p(p-2)(p+1)(p+3)}{4!}a_0,$$
$$a_6 = -\frac{p(p-2)(p-4)(p+1)(p+3)(p+5)}{6!}a_0, \ldots$$

and

$$a_3 = -\frac{(p-1)(p+2)}{3!}a_1, \ a_5 = +\frac{(p-1)(p-3)(p+2)(p+4)}{5!}a_1,$$
$$a_7 = -\frac{(p-1)(p-3)(p-5)(p+2)(p+4)(p+6)}{7!}a_1, \ldots.$$

Thus

$$(2.4) \quad y(x) = a_0\left(1 - \frac{p(p+1)}{2!}x^2 + \frac{(p(p-2)(p+1)(p+3)}{4!}x^4 + \dots\right) + a_1\left(x - \frac{(p-1)(p+2)}{3!}x^3 + \frac{(p-1)(p-3)(p+2)(p+4)}{5!}x^5 + \dots\right).$$

The first series is an even function while the second series is an odd function. The above is the general solution of (2.1) in the interval $(-1, 1)$. It is called the *Legendre function*.

Remark 2.1. If p is not an integer, then both series in (2.4) have radius of convergence 1. (You can check this by the ratio test.)

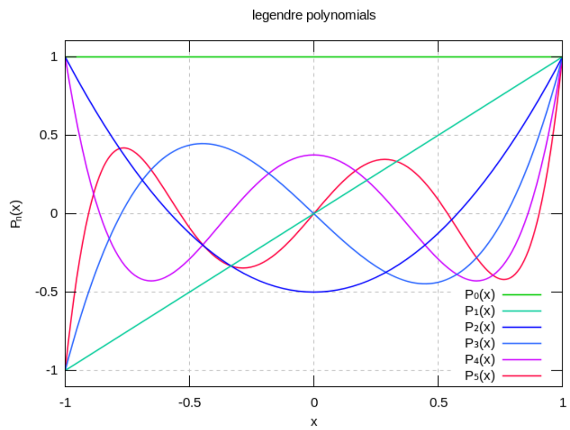
2.2. Legendre polynomials

Now suppose the parameter p in the Legendre equation (2.1) is a nonnegative integer. Then one of the two series in (2.4) terminates: if p is even, then the first series terminates. In this case, if we put $a_1 = 0$, then we get a polynomial solution. Similarly, if p is odd, then the second series terminates, and we get a polynomial solution by putting $a_0 = 0$. Thus we obtain a sequence of polynomials $P_m(x)$ (up to multiplication by a constant) for each nonnegative integer m. These are called the *Legendre polynomials*. The m-th Legendre polynomial $P_m(x)$ solves (2.1) for $p = m$ and for all x, not just for $x \in (-1, 1)$. It is traditional to normalize the constants so that $P_m(1) = 1$. The first few values are as follows.

n	$P_n(x)$
0	1
1	x
2	$\frac{1}{2}(3x^2 - 1)$
3	$\frac{1}{2}(5x^3 - 3x)$
4	$\frac{1}{8}(35x^4 - 30x^2 + 3)$
5	$\frac{1}{8}(63x^5 - 70x^3 + 15x)$

Note that only even or odd powers are present, but other than that by (2.3), there are no "missing" powers.

Many interesting features of these polynomials are visible in the interval $(-1, 1)$. So it is customary to restrict them to this interval, though they are defined over the entire real line. The graphs of the first few Legendre polynomials in this interval are given below.



Remark 2.2. In order to normalize by the condition $P_m(1) = 1$, we need to know that no polynomial solution vanishes at $x = 1$. For this, suppose $f(x)$ is a polynomial solution of (2.1). Write $f(x) = (x-1)^k g(x)$ with $g(1) \neq 0$. Substitute this in (2.1), simplify, remove the common factor of $(x-1)^{k-1}$, and finally put $x = 1$ to obtain $k^2 g(1) = 0$, which implies that $k = 0$. So $f(1) \neq 0$.

Remark 2.3. We do not need to consider negative integral values of p separately, since by our earlier observation it suffices to assume $p \geq -1/2$. Explicitly, $-k(-k+1) = (k-1)k$.

Now let us consider the second independent solution given by (2.4). It is a honest power series. For $p = 0$, it is

$$x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots = \frac{1}{2} \log\big(\frac{1+x}{1-x}\big),$$

while for $p = 1$, it is

$$1 - \frac{x^2}{1} - \frac{x^4}{3} - \frac{x^6}{5} - \cdots = 1 - \frac{1}{2} x \log\big(\frac{1+x}{1-x}\big).$$

These nonpolynomial solutions always have a log factor of the above kind and hence are unbounded at both $+1$ and -1. (Since they are either even or odd, the behavior at $x = 1$ is reflected at $x = -1$.) They are called the *Legendre functions of the second kind* (with the Legendre polynomials being those of the first kind). Most books tend to ignore these. The reason for the log factor will become evident when we apply the Frobenius method at $x = \pm 1$, see Example 3.10. Include pictures.

2.3. Orthogonality and Legendre series

We now look at the orthogonality property of the Legendre polynomials wrt to an inner product on the space of polynomials. This property allows us to expand any square-integrable function on $[-1, 1]$ into a Legendre series.

2.3.1. Orthogonal basis for the vector space of polynomials. Recall that the space of polynomials in one variable is a vector space (that is, one can scalar-multiply and add two polynomials). Its dimension is countably infinite and the set $\{1, x, x^2, \dots\}$ is a canonical basis. Since $P_m(x)$ is a polynomial of degree m, it follows that $\{P_0(x), P_1(x), P_2(x), \dots\}$ is also a basis. Explain.

Further the space of polynomials carries an inner product defined by

$$\langle f, g\rangle := \int_{-1}^{1} f(x)g(x)dx. \tag{2.5}$$

Note that we are integrating only between -1 and 1. This ensures that the integral is always finite. The norm of a polynomial is defined by

$$\|f\| := \left(\int_{-1}^{1} f(x)f(x)dx\right)^{1/2}. \tag{2.6}$$

In the discussion below, we will repeatedly use the following simple consequence of integration by parts: For differentiable functions f and g, if $(fg)(b) = (fg)(a)$, then

$$\int_a^b fg'dx = -\int_a^b f'gdx. \tag{2.7}$$

(This process transfers the derivative from g to f. This has theoretical significance, it is used to define the notion of weak derivatives.)

The following result says that the Legendre polynomials provide an orthogonal basis (that is, distinct Legendre polynomials are perpendicular to each other wrt the above inner product).

Proposition 2.4. *Suppose m and n are two nonnegative integers with $m \neq n$. Then*

$$\int_{-1}^{1} P_m(x)P_n(x)dx = 0. \tag{2.8}$$

Orthogonality does not depend on the normalization factor, so the specific normalization used to define P_n is not relevant here. To prove (2.8), one can try a brute force method using the formulas (2.4). Since $P_n(x)$ is an odd or even function depending on whether n is odd or even, the result is immediate if m and n are of opposite parity. However, handling the cases when m and n are both even, or both odd, would require some effort. A computation-free proof is given below.

Proof. Since $P_m(x)$ solves the Legendre equation (2.2) for $p = m$, we have

$$((1-x^2)P_m')' + m(m+1)P_m = 0.$$

Multiply by P_n and integrate by parts to get

$$-\int_{-1}^{1}(1-x^2)P_m'P_n' + (1-x^2)P_m'P_n\big|_{-1}^{1} + m(m+1)\int_{-1}^{1} P_m(x)P_n(x)dx = 0.$$

The boundary term vanishes because $1 - x^2 = 0$ at both $x = 1$ and $x = -1$. (We are in the situation of (2.7).) Thus,

$$-\int_{-1}^{1}(1-x^2)P_m'P_n' + m(m+1)\int_{-1}^{1} P_m(x)P_n(x)dx = 0.$$

Interchanging the roles of m and n,

$$-\int_{-1}^{1}(1-x^2)P_m'P_n' + n(n+1)\int_{-1}^{1} P_m(x)P_n(x)dx = 0.$$

Subtracting the two identities, we obtain

$$(m(m+1) - n(n+1)) \int_{-1}^{1} P_m(x)P_n(x)dx = 0.$$

Since $m \neq n$, the scalar in front can be canceled and we get (2.8). □

Remark 2.5. Suppose $\{f_n(x)\}$ is a sequence of polynomials such that $f_n(1) = 1$ and $f_n(x)$ has degree n, and they are orthogonal wrt the inner product (2.5). Then $f_n(x) = P_n(x)$. This is a geometric way to (introduce and) understand the Legendre polynomials.

2.3.2. Rodrigues formula. There is another nice way to obtain an orthogonal basis for the space of polynomials. For this, consider the sequence of polynomials

$$q_n(x) := \left(\frac{d}{dx}\right)^n (x^2-1)^n = D^n(x^2-1)^n, \quad n \geq 0.$$

(We are writing D as a shorthand for the derivative operator.) Observe that $q_n(x)$ has degree n. The first few polynomials are 1, $2x$, $4(3x^2-1)$, ... [Note the similarity with the Legendre polynomials.]

Proposition 2.6. *Suppose m and n are two nonnegative integers with $m \neq n$. Then*

$$\int_{-1}^{1} q_m(x)q_n(x)dx = 0.$$

Proof. Assume wlog that $m < n$. Using (2.7) repeatedly, transfer all the n derivatives in $q_n(x)$ to $q_m(x)$.

$$\begin{aligned}\int_{-1}^{1} D^m(x^2-1)^m D^n(x^2-1)^n dx &= -\int_{-1}^{1} D^{m+1}(x^2-1)^m D^{n-1}(x^2-1)^n dx \\ &= \cdots = (-1)^m \int_{-1}^{1} D^{2m}(x^2-1)^m D^{n-m}(x^2-1)^n dx \\ &= (-1)^{m+1} \int_{-1}^{1} D^{2m+1}(x^2-1)^m D^{n-m-1}(x^2-1)^n dx = 0,\end{aligned}$$

since $D^{2m+1}(x^2-1)^m \equiv 0$.

In order to apply (2.7), we need to check that the boundary terms always vanish: At each step, the boundary term in the integration by parts has a factor $D^r(x^2-1)^n$, $r < n$ which vanishes at ± 1. For instance, the first boundary term is

$$q_m(x)D^{n-1}(x^2-1)^n\Big|_{-1}^{1}.$$

Since $(x^2-1)^n$ has roots $+1$ and -1 of multiplicity n, the second term vanishes at both $+1$ and -1. □

So the sequence $\{q_n(x)\}$ is also an orthogonal basis of the space of polynomials. Further the span of $q_0(x), \ldots, q_n(x)$ consists of all polynomials of degree n or less. This feature is also exhibited by the Legendre polynomials. So it follows that $P_n(x)$ and $q_n(x)$ are scalar multiples of each other (in view of the following general result).

Lemma 2.7. *In any inner product space, suppose $\{u_1, u_2, \ldots, u_n\}$ and $\{v_1, v_2, \ldots, v_n\}$ are two orthogonal systems of vectors such that for each $1 \leq k \leq n$, the span of $u_1, \ldots, u_k$ equals the span of $v_1, \ldots, v_k$. Then $u_i = c_i v_i$ for each i for certain nonzero scalars c_i.*

PROOF. Nice exercise. $\square$

The scalar relating q_n and P_n can be found for instance by computing $q_n(1)$. This is done below.

Proposition 2.8. *We have* $q_n(1) = 2^n n!$.

PROOF. Repeatedly apply the product rule to $(x-1)^n$ and $(x+1)^n$ to obtain

$$D^n(x^2-1)^n = \sum_{r=0}^{n} \binom{n}{r} D^r(x-1)^n \cdot D^{n-r}(x+1)^n.$$

Since

$$D^r(x-1)^n\big|_{x=1} = \begin{cases} 0 & \text{if } r < n, \\ n! & \text{if } r = n. \end{cases}$$

Thus after substuting $x = 1$, only the term where all n derivatives are on $(x-1)^n$ survives. This gives $n!$, while substituting 1 in $(x+1)^n$ gives 2^n. $\square$

Since by convention $P_n(1) = 1$,

$$P_n(x) = \frac{1}{2^n n!}\Big(\frac{d}{dx}\Big)^n (x^2-1)^n. \tag{2.9}$$

This is known as *Rodrigues formula.*

Proposition 2.9. *We have*

$$\|q_n(x)\|^2 = \frac{2}{2n+1}(2^n n!)^2.$$

PROOF. By repeated use of (2.7), as in the proof of Proposition 2.6, we obtain

$$\begin{aligned} \|q_n(x)\|^2 &= \int_{-1}^{1} (x^2-1)^n (-1)^n D^{2n}(x^2-1)^n dx \\ &= \int_{-1}^{1} (1-x^2)^n (2n)! dx = 2(2n)! \int_0^{\pi/2} \sin^{2n+1}\theta d\theta. \end{aligned}$$

The rhs is a standard integral. Instead of the trigonometric substitution, one can continue to exploit (2.7):

$$\begin{aligned} (2n)! \int_{-1}^{1} (1-x)^n (1+x)^n dx &= (2n)! \int_{-1}^{1} n(1-x)^{n-1} \frac{(1+x)^{n+1}}{n+1} dx = \dots \\ &= (2n)! \frac{n}{n+1} \frac{n-1}{n+2} \cdots \frac{1}{2n} \int_{-1}^{1} (1+x)^{2n} dx = \frac{2^{2n+1}(n!)^2}{2n+1}. \end{aligned}$$

$\square$

This along with the Rodrigues formula (2.9) implies that

$$\|P_n(x)\|^2 = \int_{-1}^{1} P_m(x)^2 dx = \frac{2}{2n+1}. \tag{2.10}$$

Using the Rodrigues formula (2.9) one can show that

$$P_n(x) = \sum_{m=0}^{[n/2]} (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!} x^{n-2m} \tag{2.11}$$

where $[n/2]$ denotes the greatest integer less than or equal to $n/2$. (This is a tutorial problem.) Note that (2.4) gives an explicit formula for $P_n(x)$ but only up to a scalar multiple. The point of (2.11) is that this scalar is now determined.

2.3.3. Fourier-Legendre series. We have already noted that a polynomial $f(x)$ of degree m can be uniquely expressed as a linear combination of the first m Legendre polynomials:

$$f(x) = c_0 P_0(x) + c_1 P_1(x) + \cdots + c_m P_m(x).$$

Since the Legendre polynomials are orthogonal, we obtain for $0 \leq n \leq m$,

$$\int_{-1}^{1} f(x) P_n(x) dx = c_n \int_{-1}^{1} P_n(x) P_n(x) dx.$$

Thus

$$c_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x) dx. \tag{2.12}$$

A function $f(x)$ on $[-1, 1]$ is *square-integrable* if

$$\int_{-1}^{1} f(x) f(x) dx < \infty. \tag{2.13}$$

For instance, (piecewise) continuous functions are square-integrable. The set of all square-integrable functions on $[-1, 1]$ is a vector space. Moreover, it an inner product space under (2.5). It contains polynomials as a subspace. Observe from (2.6) that the condition (2.13) simply says that $\|f\|$ is finite. The Legendre polynomials no longer form a basis for this larger space. Nevertheless, one can do the following.

Any square-integrable function $f(x)$ on $[-1, 1]$ can be expanded in a series of Legendre polynomials

$$f(x) \approx \sum_{n \geq 0} c_n P_n(x), \tag{2.14}$$

where c_n is as in (2.12). This is called the *Fourier-Legendre series* (or simply the Legendre series) of $f(x)$. This series converges to $f(x)$ in the sense that

$$\|f(x) - \sum_{n=0}^{m} c_n P_n(x)\| \to 0 \quad \text{as} \quad m \to \infty.$$

This is known as *convergence in norm*. (The reason this works is that the Legendre polynomials form a maximal orthogonal set in the space of square-integrable functions.) We are only claiming here that $f(x)$ can be approximated by a sequence of polynomials of increasing degrees in an average sense. Pointwise convergence is more delicate. There are two issues here: Does the series on the right in (2.14) converge at x? If yes, then does it converge to $f(x)$? A useful result in this direction is given below (see [10, footnote on page 345]).

Theorem 2.10. *If both $f(x)$ and $f'(x)$ have at most a finite number of jump discontinuities in the interval $[-1, 1]$, then the Legendre series converges to*

$$\frac{1}{2}(f(x_-) + f(x_+))$$

for $-1 < x < 1$, to $f(-1_+)$ at $x = -1$, and to $f(1_-)$ at $x = 1$. In particular, the series converges to $f(x)$ at every point of continuity.

This is sometimes referred to as the *Legendre expansion theorem.*

Example 2.11. Consider the function

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1, \\ -1 & \text{if } -1 < x < 0. \end{cases}$$

Theorem 2.10 applies. By explicit calculation using the Rodrigues formula, one can show that the Legendre series of $f(x)$ is

$$\sum_{k\geq 0} \frac{(-1)^k(4k+3)}{2^{2k+1}(k+1)} \binom{2k}{k} P_{2k+1}(x) = \frac{3}{2}P_1(x) - \frac{7}{8}P_3(x) + \frac{11}{16}P_5(x) - \dots.$$

(This is a tutorial problem.) Sketch the first few graphs. They wiggle around the graph of $f(x)$. Include pictures. One can use Stirling's approximation for the factorial, and get an asymptotic expression for the coefficients. The ratio of successive coefficients approaches -1. It comes from the $(-1)^k$ part, the ratio of the remaining part goes to 1.

2.4. Generating function

Consider the function of two variables

$$\varphi(x,t) := \frac{1}{\sqrt{1-2xt+t^2}}.$$

We restrict the domain to those x and t which satisfy $2|xt| + |t|^2 < 1$. (The function may be defined for more values, but we make this assumption to justify the manipulations below.) First by the binomial theorem,

$$\begin{aligned}\varphi(x,t) = (1-2xt+t^2)^{-1/2} = 1 - \frac{1}{2}(-2xt+t^2) + \frac{1}{2!}(-\frac{1}{2})(-\frac{3}{2})(-2xt+t^2)^2 \\ + \frac{1}{3!}(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})(-2xt+t^2)^3 + \dots.\end{aligned}$$

Next by rearranging terms and writing them as a power series in t,

$$\varphi(x,t) = 1 + xt + \frac{1}{2}(3x^2-1)t^2 + \dots.$$

The coefficients are precisely the Legendre polynomials. Above we only computed the first three terms, the fact that they will always be the Legendre polynomials can be shown using (2.11). Thus:

Proposition 2.12. *For any x and t such that $2|xt| + |t|^2 < 1$,*

$$(1-2xt+t^2)^{-1/2} = \sum_{n\geq 0} P_n(x)t^n.$$

Remark 2.13. The binomial expansion only requires $|-2xt+t^2| < 1$. However we then rearranged terms of the resulting series. This is justified whenever the series converges absolutely. The assumption $2|xt| + |t|^2 < 1$ implies absolute convergence.

Consider the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

which converges to $\log 2$. Now rearrange this series as follows

$$(1-\frac{1}{2}) - \frac{1}{4} + (\frac{1}{3} - \frac{1}{6}) - \frac{1}{8} + (\frac{1}{5} - \frac{1}{10}) - \frac{1}{12} + \dots$$

and observe that it converges to $\frac{1}{2}\log 2$. Thus rearranging the terms of a series can alter its sum. Another example and some related theory is given in [8, Example 3.53].

The above result says that φ is the generating function for the Legendre polynomials. We give another argument for this result which does not require knowing (2.11).

The function φ satisfies the PDE:

$$(1-x^2)\frac{\partial^2 \varphi}{\partial x^2} - 2x\frac{\partial \varphi}{\partial x} + t\frac{\partial^2 (t\varphi)}{\partial t^2} = 0. \tag{2.15}$$

This is a direct check. First,

$$\frac{\partial \varphi}{\partial x} = t\varphi^3 \quad \text{and} \quad \frac{\partial \varphi}{\partial t} = (x-t)\varphi^3. \tag{2.16}$$

Hence

$$(1-x^2)\frac{\partial^2 \varphi}{\partial x^2} - 2x\frac{\partial \varphi}{\partial x} = (1-x^2)t.3\varphi^2.t\varphi^3 - 2xt\varphi^3 = t\varphi^5(3t - 2(1+t^2)x + tx^2),$$

and

$$\begin{aligned} t\frac{\partial^2 (t\varphi)}{\partial t^2} = t(2\frac{\partial \varphi}{\partial t} + t\frac{\partial^2 \varphi}{\partial t^2}) &= t(2(x-t)\varphi^3 - t\varphi^3 + t(x-t).3\varphi^2(x-t)\varphi^3) \\ &= t\varphi^5(-3t + 2(1+t^2) - tx^2), \end{aligned}$$

proving (1.5).

Now write

$$\varphi(x,t) = \sum q_n(x)t^n \quad \text{where} \quad q_n(x) = \frac{1}{n!}\frac{\partial^n \varphi}{\partial t^n}\Big|_{t=0}.$$

The $q_n(x)$ are polynomials in x due to (2.16). Substituting this expression in (2.15), we see that $q_n(x)$ satisfies (2.1) for $p = n$. Further,

$$\varphi(1,t) = (1-t)^{-1} = \sum t^n$$

shows that $q_n(1) = 1$ for all n. Hence $q_n = P_n$ for all n.

2.4.1. Gravitational potential of a point mass. Write $x = \cos\varphi$ for $0 \le \varphi \le \pi$. Consider the triangle two of whose sides are a and r and the angle between them is φ. Then the length of the third side (by the cosine rule) is $\sqrt{a^2 - 2ar\cos\varphi + r^2}$. Let the vertex with angle φ be denoted O. Let the remaining two vertices be denoted A and B. Then the gravitational (or electric) potential at B due to a point mass (or charge) at A is (up to normalization) given by

$$\frac{1}{\sqrt{a^2 - 2ar\cos\varphi + r^2}}. \tag{2.17}$$

If $a < r$, then this can expressed as a power series in $t := a/r$ and the coefficients are $P_n(\cos\varphi)$. This is how Legendre discovered these polynomials. (If $a > r$, then we can let $t := r/a$.)

If $r = a$, then this value is the reciprocal of $2a\sin\varphi/2$.

CHAPTER 3

Frobenius method for regular singular points

Some of the most important second order ODEs which occur in physics and engineering problems when written in the form $y'' + p(x)y' + q(x)y = 0$ do not allow power series expansion of $p(x)$ and $q(x)$. For example, this is true for the Cauchy-Euler equation $x^2y'' + xy' + y = 0$. The solution of such an equation may involve negative powers of x or a term such as $\log x$, which are singular at the origin. We now study a general method called the Frobenius method for dealing with an ODE of the above kind. A nice and detailed exposition of these ideas is given by Rainville [7, Chapter 18].

3.1. Ordinary and singular points

Consider the second-order linear ODE

$$p(x)y'' + q(x)y' + r(x)y = 0$$

defined on an interval I. We assume that p, q, and r are real analytic on I, and do not have any common zeroes.

A point $x_0 \in I$ is an *ordinary point* of the ODE if $p(x_0) \neq 0$, and a *singular point* if $p(x_0) = 0$. A singular point x_0 is a *regular singular point* if

$$\lim_{x \to x_0} (x - x_0) \frac{q(x)}{p(x)} \quad \text{and} \quad \lim_{x \to x_0} (x - x_0)^2 \frac{r(x)}{p(x)}$$

exist. In other words, the bad behavior of q/p and r/p at x_0 can be tamed by multiplying them by $x - x_0$ and $(x - x_0)^2$ respectively. Equivalently, (using the real analyticity of p, q, and r,) at a regular singular point x_0, the ODE can be written in the form

$$y'' + \frac{b(x)}{x - x_0}\, y' + \frac{c(x)}{(x - x_0)^2}\, y = 0, \tag{3.1}$$

where $b(x)$ and $c(x)$ are real analytic in a neighborhood of x_0.

A point which is not regular singular is called *irregular singular*. We will not be considering ODEs with irregular singular points.

Example 3.1. Consider the Legendre equation (2.1). The singular points are $x = \pm 1$, the rest are all ordinary points. Rewrite the equation as

$$y'' + \frac{2x}{x^2 - 1}\, y' - \frac{p(p+1)}{x^2 - 1}\, y = 0. \tag{3.2}$$

This can be put in the form (3.1) with $x_0 = 1$ for

$$b(x) = \frac{2x}{x + 1} \quad \text{and} \quad c(x) = -\frac{(x - 1)p(p+1)}{x + 1}. \tag{3.3}$$

Both are real analytic in a neighborhood of $x = 1$. So $x = 1$ is a regular singular point. By similar reasoning, we see that $x = -1$ is also a regular singular point.

Example 3.2. Consider the Cauchy-Euler equation

$$x^2y'' + b_0xy' + c_0y = 0, \tag{3.4}$$

where b_0 and c_0 are constants with $c_0 \neq 0$. Then $x = 0$ is a singular point, the rest are all ordinary points. The ODE can be put in the form (3.1) with $b(x) \equiv b_0$ and $c(x) \equiv c_0$ being constants, hence $x = 0$ is a regular singular point. So Theorem 1.6 does not apply. (Nonetheless, if we try the power series method, then we get a one-step recursion

$$(n(n-1) + b_0n + c_0)a_n = 0, \quad n \geq 0,$$

an indicator that something is not right here. It is very likely that no integer n satisfies $n(n-1) + b_0n + c_0 = 0$, in which case $a_n = 0$ for all n. So we do not get anything beyond the trivial solution.) As a concrete example, consider

$$x^2y'' + 2/9y = 0. \tag{3.5}$$

The power series method leads to the equation $n(n-1)+2/9 = (n-1/3)(n-2/3) = 0$ which has no integer solutions. However, if n is allowed to be a real variable, then the equation does have two roots, and one can readily check that the functions $x^{1/3}$ and $x^{2/3}$ are two independent solutions of (3.5). Note that neither of the two solutions is real analytic in a neighborhood of 0.

The general Cauchy-Euler equation (3.4) can be solved in a similar manner. Assume $x > 0$. Substitute $y = x^r$, and compute the roots r_1 and r_2 of the equation

$$r^2 + (b_0 - 1)r + c_0 = 0. \tag{3.6}$$

Since this is an equation satisfied by the index r (more commonly called exponent), it is called the *indicial equation*.

- If the roots are real and unequal, then x^{r_1} and x^{r_2} are two independent solutions.
- If the roots are complex (written as $a \pm ib$), then $x^a \cos(b \log x)$ and $x^a \sin(b \log x)$ are two independent solutions.
- If the roots are real and equal, then x^{r_1} and $(\log x)x^{r_1}$ are two independent solutions.

Note that unless r_1 or r_2 is a nonnegative integer, the solutions have singularities at 0. For the interval $x < 0$, one can make a substitution $u = -x$, and reduce to the above case. The upshot is that the solutions are again as above with $-x$ instead of x.

The analysis of (3.4) is strikingly similar to the second-order linear ODE with constant coefficients. This is not an accident. After making the substitution $x = e^t$, (3.4) transforms to

$$\frac{d^2y}{dt^2} + (b_0 - 1)\frac{dy}{dt} + c_0y = 0.$$

The auxiliary equation of this constant coefficient ODE is precisely (3.6). In case of distinct real roots, the solutions are e^{r_1t} and e^{r_2t}, which when expressed in the variable x are x^{r_1} and x^{r_2}.

Remark 3.3. The function x^r for r real and $x > 0$ is not trivial to define. One standard way is to first define $\log x$ and e^x, and then define

$$x^r := e^{r \log x}.$$

Note that this makes sense for r real and $x > 0$. This function is always nonnegative. Using properties of log and exp, one can then easily show that for r and s real and $x > 0$,

$$x^r x^s = x^{r+s}, \quad (x^r)^s = x^{rs}, \quad \text{and} \quad \frac{d}{dx}(x^r) = rx^{r-1}.$$

These identities get routinely used. For instance, we used them in Example 3.2 when we substituted x^r in the Cauchy-Euler equation.

3.2. Fuchs-Frobenius theory

There is a satisfactory theory to understand the solutions of a second-order linear ODE at a regular singular point. Wlog assume that this point is the origin, and write the ODE as

$$L[y] := x^2 y'' + xb(x)\, y' + c(x)\, y = 0, \tag{3.7}$$

where $b(x)$ and $c(x)$ are real analytic in a neighborhood of the origin. Here L denotes the differential operator

$$L := x^2 \frac{d^2}{dx^2} + xb(x)\frac{d}{dx} + c(x). \tag{3.8}$$

Let

$$b(x) = \sum_{n \geq 0} b_n x^n \quad \text{and} \quad c(x) = \sum_{n \geq 0} c_n x^n \tag{3.9}$$

be their power series expansions. If we take $b(x)$ and $c(x)$ to be constants, then we recover the Cauchy-Euler equation (3.4). The analysis below generalizes the one given in Example 3.2.

3.2.1. Frobenius method. We restrict to $x > 0$. (This is a standing assumption whenever the Frobenius method is applied.) Assume that (3.7) has a solution of the form

$$y(x) = x^r \sum_{n \geq 0} a_n x^n, \quad a_0 \neq 0, \tag{3.10}$$

where r is a fixed real number. (The point is that r need not be a nonnegative integer. Nothing is gained by allowing $a_0 = 0$. If the first few terms of the series are zero and it starts at say $a_k x^k$ with $a_k \neq 0$, then x^k could be absorbed with x^r.) Substituting (3.10) in (3.7) and equating the coefficient of x^r (the lowest degree term) and canceling off a_0 (which is nonzero), we obtain the *indicial equation*:

$$r(r-1) + b_0 r + c_0 = 0, \quad \text{or equivalently,} \quad r^2 + (b_0 - 1)r + c_0 = 0. \tag{3.11}$$

Let us denote the quadratic in r by $I(r)$. Equating the coefficient of x^{r+1}, we obtain

$$[(r+1)r + (r+1)b_0 + c_0]a_1 + a_0(rb_1 + c_1) = 0.$$

More generally, equating the coefficient of x^{r+n}, we obtain the recursion

$$I(r+n)a_n + \sum_{j=0}^{n-1} a_j((r+j)b_{n-j} + c_{n-j}) = 0, \quad n \geq 1. \tag{3.12}$$

This shows that a_n depends on all the preceding coefficients $a_0, \dots, a_{n-1}$.

Remark 3.4. Note that (3.12) is not a finite-step recursion, since the number of terms depends on n. This happens because $b(x)$ and $c(x)$ are not assumed to be polynomials. Also observe that b_0 and c_0 appear in the first term as a part of $I(r+n)$, while the higher b_i and c_i appear in the sum. In most examples we look at, $b(x)$ and $c(x)$ are polynomials of degree 1, so the sum in (3.12) has only one term which involves a_{n-1}, and we get a two-step recursion.

Let r_1 and r_2 be the two roots of $I(r) = 0$. If both are real, we follow the convention that $r_1 \geq r_2$. Note that the recursion (3.12) can be solved uniquely (starting with $a_0 = 1$ for definiteness) provided $I(r+n)$ is never 0 for any $n \geq 1$. Now $I(x) = 0$ only for $x = r_1$ or $x = r_2$. Hence it is clear that for the larger root r_1, $I(r_1 + n)$ is never 0 for any $n \geq 1$. The same is true for r_2 provided $r_1 - r_2$ is not a positive integer. This leads to the following result.

Theorem 3.5. *With notation as above, the ODE (3.7) has as a solution for $x > 0$*

$$y_1(x) = x^{r_1}(1 + \sum_{n \geq 1} a_n x^n) \tag{3.13}$$

where the a_n solve the recursion (3.12) for $r = r_1$ and $a_0 = 1$.

If, in addition, $r_1 - r_2$ is not zero or a positive integer, then a second independent solution for $x > 0$ is given by

$$y_2(x) = x^{r_2}(1 + \sum_{n \geq 1} A_n x^n) \tag{3.14}$$

where the A_n solve the recursion (3.12) for $r = r_2$ and $a_0 = 1$.

The power series in (3.13) and (3.14) converge in the interval in which both power series in (3.9) converge.

The above solutions are called the *fractional power series solutions*. Note that we have not proved this theorem. The important step is the last part which claims that these power series have a nonzero radius of convergence (similar to what is needed to prove Theorem 1.6). This argument is given in [10, Chapter 5, Appendix A] or [4, Chapter 4, Section 5].

Remark 3.6. Suppose the indicial equation has complex roots. The above solutions are still valid except that they are now complex-valued. One can obtain real-valued solutions by taking the real and imaginary parts of $y_1(x)$ (or of $y_2(x)$). Since r_1 and r_2 are complex conjugates, the real and imaginary parts of $y_1(x)$ and $y_2(x)$ are closely related, so we are only going to get two independent real-valued solutions.

Example 3.7. Consider the ODE

$$2x^2y'' - xy' + (1+x)y = 0.$$

Observe that $x = 0$ is a regular singular point. (The functions $b(x) = -1/2$ and $c(x) = (1+x)/2$ are in fact polynomials, so they converge everywhere.) Let us apply the Frobenius method. The indicial equation is

$$r(r-1) - \frac{r}{2} + \frac{1}{2} = 0.$$

The roots are $r_1 = 1$ and $r_2 = 1/2$. Their difference is not an integer. Hence by Theorem 3.5, we will get two fractional power series solutions converging everywhere.

Let us write them down explicitly. The general recursion is

$$(2(r+n)(r+n-1)-(r+n)+1)a_n + a_{n-1} = 0, \quad n \geq 1.$$

This is an instance of (3.12) (multiplied by 2 for convenience). Instead of specializing (3.12), one may also directly substitute (3.10) into the ODE and obtain the above recursion by equating the coefficient of x^{n+r}. In the present case, we got a simple two-step recursion since among the higher b_i and c_i, only c_1 is nonzero.

For the root $r = 1$, the recursion simplies to

$$a_n = \frac{-1}{(2n+1)n} a_{n-1}, \quad n \geq 1$$

leading to the solution for $x > 0$

$$y_1(x) = x\Big(1 + \sum_{n \geq 1} \frac{(-1)^n x^n}{(2n+1)n(2n-1)(n-1)\dots(5 \cdot 2)(3 \cdot 1)}\Big).$$

Similarly, for the root $r = 1/2$, the recursion simplies to

$$a_n = \frac{-1}{2n(n-1/2)} a_{n-1}, \quad n \geq 1$$

leading to the second solution for $x > 0$

$$y_2(x) = x^{1/2}\Big(1 + \sum_{n \geq 1} \frac{(-1)^n x^n}{2n(n-1/2)(2n-2)(n-3/2)\dots(4 \cdot 3/2)(2 \cdot 1/2)}\Big).$$

One can also directly check using the ratio test that the power series in both solutions converge everywhere, in accordance with Theorem 3.5.

3.2.2. Indicial equation with repeated roots. Suppose the indicial equation has repeated roots $r_1 = r_2$. Then the Frobenius method yields only one solution. To obtain the other, let us go through that procedure carefully again. Let us leave aside (3.11). Given any r, we can uniquely solve (3.12) for the unknowns a_n (given a_0). These unknowns depend on r, so to show this dependence, let us write $a_n(r)$. These are rational functions in r, that is, a quotient of two polynomials in r. Now consider

$$\varphi(r,x) := x^r \sum_{n \geq 0} a_n(r) x^n = x^r \big(a_0 + \sum_{n \geq 1} a_n(r) x^n\big). \tag{3.15}$$

This is a function of two variables r and x. By our choice of the $a_n(r)$, we have

$$L[\varphi(r,x)] = a_0 I(r) x^r \tag{3.16}$$

(at least formally). Here L is the differential operator (3.8). If we now put $r = r_1$, then the rhs of (3.16) is zero and this is the first fractional power series solution (3.13). If instead we differentiate the rhs wrt r and put $r = r_1$, we still get 0 (since r is a repeated root). So

$$\frac{\partial}{\partial r} L[\varphi(r,x)](r_1, x) = 0.$$

Interchanging the order of the partial derivatives wrt r and wrt x suggests that the second solution is

$$\begin{aligned} y_2(x) = \frac{\partial \varphi(r,x)}{\partial r}\Big|_{r=r_1} &= \frac{\partial}{\partial r}\Big(x^r \sum_{n\geq 0} a_n(r)x^n\Big)\Big|_{r=r_1} \\ &= x^{r_1}\log x \sum_{n\geq 0} a_n(r_1)x^n + x^{r_1}\sum_{n\geq 1} a_n'(r_1)x^n \\ &= y_1(x)\log x + x^{r_1}\sum_{n\geq 1} a_n'(r_1)x^n. \end{aligned}$$

It is possible to justify these steps leading to the following result.

Theorem 3.8. *If the indicial equation has repeated roots, then there is a second independent solution of (3.7) of the form*

$$y_2(x) = y_1(x)\log x + x^{r_1}\sum_{n\geq 1} A_n x^n \tag{3.17}$$

with $y_1(x)$ as in (3.13). The power series converges in the interval in which both power series in (3.9) converge.

In fact, the preceding discussion shows that $A_n = a_n'(r_1)$. Recall that the $a_n(r)$ are rational functions in r. There is a nice procedure to compute their derivatives: Suppose $f(r) = f_1(r)\dots f_n(r)$. Then by the product rule

$$f'(r) = f(r)\big(\frac{f_1'(r)}{f_1(r)} + \dots + \frac{f_n'(r)}{f_n(r)}\big). \tag{3.18}$$

Further if $f(r) = (\alpha r + \beta)^k$, then

$$\frac{f'(r)}{f(r)} = \frac{k\alpha}{\alpha r + \beta}.$$

For instance, if

$$f(r) = \frac{(r-1)^2}{(r+1)(r-3)^4}$$

then

$$f'(r) = f(r)\big(\frac{2}{r-1} - \frac{1}{r+1} - \frac{4}{r-3}\big).$$

Note that $y_2(x)$ always has a singularity at 0 (that is, it is unbounded near 0). This is because of the presence of $\log x$. In addition, $y_1(x)$ also has a singularity at 0 whenever r_1 is not a nonnegative integer, and its product with $\log x$ only makes matters worse.

Given an ODE, it is often possible to calculate the derivatives of $a_n(r)$ by the above procedure. Another alternative is to substitute (3.17) in (3.7) and solve for the A_n. (Yet another alternative is to use the method of variation of parameters, and compute the second solution from $y_1(x)$. This would express the solution as an integral rather than a power series.)

Example 3.9. Consider the ODE

$$x^2y'' + 3xy' + (1-2x)y = 0.$$

This has a regular singularity at 0. Written in the form (3.1), $b(x) = 3$ and $c(x) = 1 - 2x$. The indicial equation is $r^2 + 2r + 1 = 0$ which has a repeated root at -1. One may check that the recursion is

$$(n + r + 1)^2 a_n = 2a_{n-1}, \quad n \geq 1.$$

(Why did we get a two-step recursion?) Hence

$$a_n(r) = \frac{2^n}{[(r+2)(r+3)\dots(r+n+1)]^2}\, a_0.$$

(Since we had a two-step recursion, this rational function comes factorized into linear factors.) Setting $r = -1$ (and $a_0 = 1$) yields the fractional power series solution

$$y_1(x) = \frac{1}{x} \sum_{n \geq 0} \frac{2^n}{(n!)^2}\, x^n.$$

The power series converges everywhere. The formula for $a_n(r)$ in simple enough that we can compute the second solution explicitly. First, from (3.18) and using the method explained there,

$$a_n'(r) = -2a_n(r)\Big(\frac{1}{r+2} + \frac{1}{r+3} + \dots + \frac{1}{r+n+1}\Big), \quad n \geq 1.$$

Evaluating at $r = -1$,

$$a_n'(-1) = -\frac{2^{n+1} H_n}{(n!)^2},$$

where

$$H_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}. \tag{3.19}$$

(These are the partial sums of the harmonic series.) So the second solution is

$$y_2(x) = y_1(x) \log x - \frac{1}{x} \sum_{n \geq 1} \frac{2^{n+1} H_n}{(n!)^2}\, x^n.$$

The power series converges everywhere.

Example 3.10. Let us go back to the Legendre equation (3.2). We have seen that it has regular singular points at $x = \pm 1$. Let us look at the point $x = 1$ in more detail. We are guaranteed one fractional power series solution

$$(x-1)^r \sum_{n \geq 0} a_n (x-1)^n.$$

We should expand $b(x)$ and $c(x)$ in powers of $x - 1$. The constant coefficients b_0 and c_0 can be obtained by putting $x = 1$ in (3.3). Hence $b_0 = 1$ and $c_0 = 0$. So the indicial equation is $r^2 = 0$ which has a repeated root at 0. The fractional power series solution can be computed to be

$$\begin{aligned} y_1(x) = 1 + \\ \sum_{n \geq 1} (-1)^{n+1} \frac{p(p+1)[1 \cdot 2 - p(p+1)] \dots [n(n-1) - p(p+1)]}{2^n (n!)^2}\, (x-1)^n. \end{aligned} \tag{3.20}$$

Since $r = 0$, this is in fact a power series. Observe that if p is a nonnegative integer, then $y_1(x)$ is a polynomial of degree p. Up to a scalar multiple, this is the p-th Legendre polynomial expanded in powers of $x - 1$. If p is not an integer, then the above power series has a radius of convergence of 2 with a singularity at $x = -1$. (This should be checkable by the ratio test.) Now let us compare (3.20) with the general solution (2.4). A suitable choice of a_0 and a_1 in (2.4) must give (3.20). If p is an integer, then one of a_0 or a_1 is 1 and the other is 0. However in the noninteger case, both a_0 and a_1 are nonzero. In this case, both power series in (2.4) have singularities at $x = 1$, but this particular choice of a_0 and a_1 gets rid of the singularity at $x = 1$. (The singularity at $x = -1$ stays.)

By Theorem 3.8, the second solution is given by (3.17). This has a logarithmic singularity at $x = 1$. It should be possible to write down an explicit formula for the A_n.

3.2.3. Roots differing by an integer. Suppose the indicial equation has roots differing by a positive integer. Write $r_1 - r_2 = N$. The Frobenius method yields a solution for r_1. However, there may be a problem for r_2. So we proceed as follows. With φ as in (3.15), consider

$$\psi(r,x) := (r - r_2)\varphi(r,x) = x^r(r - r_2)\sum_{n\geq 0} a_n(r)x^n.$$

(The recursion (3.12) is linear in the a_i. So if we start with $(r - r_2)a_0$ as the 0-th term, then the n-th term will be $(r - r_2)a_n(r)$.) It follows from (3.16) that

$$L[\psi(r,x)] = a_0 x^r(r - r_2)I(r).$$

If we differentiate the rhs wrt r and put $r = r_2$, we get zero. (Wlog, let us assume from now on that $a_0 = 1$.) This suggests that the second solution is

$$\begin{aligned} y_2(x) = \frac{\partial\psi(r,x)}{\partial r}\Big|_{r=r_2} &= \frac{\partial}{\partial r}\left(x^r\sum_{n\geq 0}(r - r_2)a_n(r)x^n\right)\Big|_{r=r_1} \\ &= x^{r_2}\log x\left[\sum_{n\geq 0}(r - r_2)a_n(r)x^n\right]\Big|_{r=r_2} + x^{r_2}(1 + \sum_{n\geq 1} A_n x^n) \\ &= Ky_1(x)\log x + x^{r_2}(1 + \sum_{n\geq 1} A_n x^n), \end{aligned}$$

where

$$A_n = \frac{d}{dr}((r - r_2)a_n(r))\big|_{r=r_2}, \qquad n \geq 1, \tag{3.21}$$

and

$$K = \lim_{r\to r_2}(r - r_2)a_N(r). \tag{3.22}$$

The last step requires some explanation. Since the $a_n(r)$ have no singularity at r_2 till $n < N$, for all such n

$$\lim_{r\to r_2}(r - r_2)a_n(r) = 0.$$

So the first contributing term will be for $n = N$, and this term is Kx^N multiplied by $x^{r_2}\log x$, that is, $Kx^{r_1}\log x$. Note that this is the first term in $Ky_1(x)\log x$. Can you see why terms for $n > N$ will match the terms coming from the power series of $y_1(x)$?

Theorem 3.11. *If $r_1 - r_2$ is a positive integer, then there is a second solution of (3.7) of the form*

$$y_2(x) = Ky_1(x)\log x + x^{r_2}(1 + \sum_{n\geq 1} A_n x^n), \tag{3.23}$$

with K and A_n as defined above. The power series converges in the interval in which both power series in (3.9) converge.

More precisely, we can write the solution as

$$y_2(x) = Ky_1(x)\log x + x^{r_2}(1 + \sum_{n=1}^{N-1} a_n(r_2)x^n) + x^{r_2}\sum_{n\geq N} A_n x^n. \tag{3.24}$$

This is becuase $a_n(r)$ has no singularity for $n < N$, and for all such n, $A_n = a_n(r_2)$. (Use: if $f(r)$ is differentiable at r_2, then $\frac{d}{dr}(r-r_2)f(r)\big|_{r=r_2} = f(r_2)$.)

In general, $a_N(r)$ (and the subsequent $a_n(r)$ will have a singularity at $r = r_2$ and K will be nonzero. The only way this does not happen is if the sum in (3.12) is zero for $n = N$, that is, if the rational function given by the sum has a factor of $r - r_2$. In such a situation, $K = 0$ and there is no log term. Further, none of the $a_n(r)$ have a singularity at $r = r_2$, and $A_n = a_n(r_2)$. So the second solution is given more simply by

$$y_2(x) = x^{r_2}(1 + \sum_{n\geq 1} a_n(r_2)x^n). \tag{3.25}$$

Remark 3.12. If the log term is absent, then the smaller root r_2 can be used to get the general solution directly. Start with arbitrary a_0, and compute till $a_{N-1}(r_2)$. Then instead of taking $a_N(r_2)$, take an arbitrary b as the value for a_N, and solve the rest of the recursion uniquely. Then this solution is

$$a_0y_2(x) + (b - a_N(r_2))y_1(x).$$

Here a_0 and b are both arbitrary. The point is that one does not have to do the $y_1(x)$ calculation separately.

Example 3.13. Consider the ODE

$$x^2y'' + x(1-x)y' - (1+3x)y = 0.$$

The indicial equation is $(r+1)(r-1) = 0$, with the roots differing by a positive integer. The recursion is

$$(n+r+1)(n+r-1)a_n = (n+r+2)a_{n-1}, \quad n \geq 1.$$

Hence

$$a_n(r) = \frac{r+n+2}{(r+2)[r(r+1)\dots(r+n-1)]}\,a_0.$$

Setting $r = 1$ (and $a_0 = 1$) yields the fractional power series solution

$$y_1(x) = \frac{1}{3}\sum_{n\geq 0}\frac{n+3}{n!}\,x^{n+1}.$$

Note that setting $r = -1$ in $a_n(r)$ is problematic since there is a $r+1$ in the denominator. Using (3.21) and (3.22), we calculate $K = -3$, $A_1 = -2$, $A_2 = -1$, and

$$A_n = -\frac{n+1}{(n-2)!}\Big(\frac{1}{n+1} - H_{n-2}\Big), \quad n \geq 3,$$

where H_{n-2} is the harmonic sum (3.19). This yields the second solution

$$y_2(x) = -3y_1(x)\log x + \frac{1}{x}\big(1 - 2x - x^2 + \sum_{n\geq 3} \frac{1-(n+1)H_{n-2}}{(n-2)!}x^n\big).$$

Example 3.14. Now let us consider an ODE where the log term is absent.

$$xy'' - (4+x)y' + 2y = 0$$

The indicial equation is $r(r-5) = 0$, with the roots differing by a positive integer. The recursion is

$$(n+r)(n+r-5)a_n = (n+r-3)a_{n-1}, \quad n \geq 1.$$

Hence

$$a_n(r) = \frac{(n+r-3)\dots(r-2)}{(n+r)\dots(1+r)(n+r-5)\dots(r-4)}a_0.$$

Setting $r = 5$ (and $a_0 = 1$) yields the fractional power series solution

$$y_1(x) = \sum_{n\geq 0} \frac{60}{n!(n+5)(n+4)(n+3)}\, x^{n+5}.$$

For the second solution, let us look at the 'critical' function

$$a_5(r) = \frac{(r+2)(r+1)r(r-1)(r-2)}{(r+5)\dots(r+1)r(r-1)\dots(r-4)}.$$

Note that there is a factor of r in the numerator also. So this function does not have a singularity at $r = 0$, and $K = 0$, and $a_5(0) = 1/720$.

$$y_2(x) = (1 + \frac{1}{2}x + \frac{1}{12}x^2) + a_5(0)\big(x^5 + \sum_{n\geq 6} \frac{60}{(n-5)!n(n-1)(n-2)}\, x^n\big).$$

Here the first solution $y_1(x)$ is clearly visible. This happens because following Remark 3.12, if we take $b = 0$ for a_5, then all higher a_i are zero (since the recursion is two-step). Check that

$$1 + \frac{1}{2}x + \frac{1}{12}x^2$$

solves the ODE, this is a combination of $y_1(x)$ and $y_2(x)$, the two independent solutions given by the method, both of which were complicated!

As a simple illuminating exercise, apply the Frobenius method to the Cauchy-Euler equation (3.4). In the case when the roots differ by a positive integer, we always have $K = 0$ and the log term is absent. Why?

CHAPTER 4

Bessel equation and Bessel functions

We study the Bessel equation. There is one for each real number p. We solve it using the Frobenius method. The fractional power series solution for the larger root of the indicial equation is the Bessel function (of the first kind) of order p. It is denoted by $J_p(x)$. We study some important identities of the Bessel functions, their orthogonality properties, and also write down their generating function. In our analysis, it is a standing assumption that $x > 0$.

There is a Bessel-pedia by Watson [12]. There is also a much shorter book by Bowman [1]. Another source is [3, Chapter 7]. Most books cited in the bibliography contain some material on Bessel functions.

4.1. Bessel functions

Consider the second-order linear ODE

$$x^2y'' + xy' + (x^2 - p^2)y = 0. \tag{4.1}$$

This is known as the *Bessel equation*. Here p denotes a fixed real number. We may assume $p \geq 0$. There is a regular singularity at $x = 0$. All other points are ordinary. We apply the Frobenius method to find the solutions. All the different nuances of the method are visible in the analysis.

4.1.1. Bessel function of the first kind. Writing (4.1) in the form (3.1), $b(x) = 1$ and $c(x) = x^2 - p^2$. The indicial equation is

$$r^2 - p^2 = 0.$$

The roots are $r_1 = p$ and $r_2 = -p$. The recursion is

$$(r + n + p)(r + n - p)a_n + a_{n-2} = 0, \quad n \geq 1.$$

By convention, $a_{-1} = 0$. Let us solve this recursion with r as a variable. For that, first note that $a_1(r) = 0$ and hence $a_n(r) = 0$ for all odd n. The even terms are

$$a_2(r) = -\frac{1}{(r+2)^2 - p^2}\, a_0, \quad a_4(r) = \frac{1}{((r+2)^2 - p^2)((r+4)^2 - p^2)}\, a_0, \dots$$

and in general

$$a_{2n}(r) = \frac{(-1)^n}{((r+2)^2 - p^2)((r+4)^2 - p^2)\dots((r+2n)^2 - p^2)}\, a_0. \tag{4.2}$$

The fractional power series solution for the larger root $r_1 = p$ obtained by setting $a_0 = 1$ and $r = p$ is

$$y_1(x) = x^p \sum_{n\geq 0} \frac{(-1)^n}{((p+2)^2 - p^2)((p+4)^2 - p^2)\dots((p+2n)^2 - p^2)} x^{2n} \tag{4.3}$$
$$= x^p \sum_{n\geq 0} \frac{(-1)^n}{2^{2n} n!(1+p)\dots(n+p)} x^{2n}.$$

The power series converges everywhere. (We follow the convention that

$$\prod_{j=1}^{n}(j+p) = 1 \quad \text{if} \quad n = 0.$$

As a general convention, a product over an empty set is taken to be 1, while a sum over an empty set is taken to be 0.) The solution $y_1(x)$ itself is bounded at $x = 0$: the value at $x = 0$ is 0 if $p > 0$, and 1 if $p = 0$. The solution is analytic at $x = 0$ if p is a nonnegative integer. Define

$$J_p(x) := \left(\frac{x}{2}\right)^p \sum_{n\geq 0} \frac{(-1)^n}{n!(n+p)!} \left(\frac{x}{2}\right)^{2n}, \quad x > 0. \tag{4.4}$$

This is called the Bessel function of the first kind of order p. It is obtained by multiplying $y_1(x)$ by $\frac{1}{2^p p!}$. (The choice of this constant is traditional.) So $J_p(x)$ is a solution of (4.1). Since p may not be an integer, a more correct way to say things is to use the gamma function (A.1): Normalize $y_1(x)$ by $\frac{1}{2^p \Gamma(1+p)}$ and define

$$J_p(x) := \left(\frac{x}{2}\right)^p \sum_{n\geq 0} \frac{(-1)^n}{n!\,\Gamma(n+p+1)} \left(\frac{x}{2}\right)^{2n}, \quad x > 0. \tag{4.5}$$

Explicitly, the Bessel functions of order 0 and 1 are as follows.

$$J_0(x) := \sum_{n\geq 0} \frac{(-1)^n}{(n!)^2} \left(\frac{x}{2}\right)^{2n} \tag{4.6}$$
$$= 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} - \frac{x^6}{2^2 4^2 6^2} + \dots$$

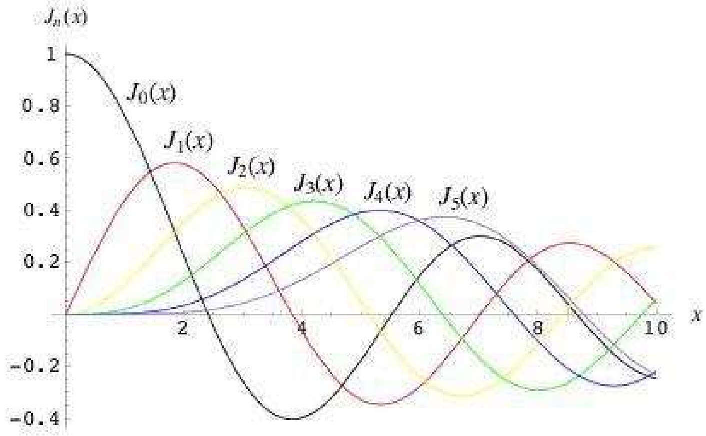
$$J_1(x) := \sum_{n\geq 0} \frac{(-1)^n}{n!(n+1)!} \left(\frac{x}{2}\right)^{2n+1} \tag{4.7}$$
$$= \frac{x}{2} - \frac{1}{1!2!}\left(\frac{x}{2}\right)^3 + \frac{1}{2!3!}\left(\frac{x}{2}\right)^5 + \dots.$$

Both $J_0(x)$ and $J_1(x)$ have a damped oscillatory behavior having an infinite number of zeroes, and these zeroes occur alternately much like the functions $\cos x$ and $\sin x$. Further, they also satisfy similar derivative identities

$$J_0'(x) = -J_1(x) \quad \text{and} \quad [xJ_1(x)]' = xJ_0(x). \tag{4.8}$$

More general identities of this kind are discussed later.

If p is a nonnegative integer, then $J_p(x)$ is a power series convergent everywhere, so it is real analytic everywhere. In the remaining cases, we restrict to $x > 0$.



4.1.2. Second Frobenius solution. Observe that $r_1 - r_2 = 2p$. Suppose $2p$ is not an integer. Then by Theorem 3.5, there is a second fractional power series solution (3.14) with coefficients $a_n(r_2)$:

$$y_2(x) = x^{-p} \sum_{n \geq 0} \frac{(-1)^n}{2^{2n} n!(1-p)\dots(n-p)} x^{2n}. \tag{4.9}$$

(Note that replacing p by $-p$ in $y_1(x)$ results in $y_2(x)$.) Normalizing by $\frac{1}{2^{-p}\Gamma(1-p)}$, define

$$J_{-p}(x) := \left(\frac{x}{2}\right)^{-p} \sum_{n \geq 0} \frac{(-1)^n}{n!\,\Gamma(n-p+1)} \left(\frac{x}{2}\right)^{2n}, \quad x > 0. \tag{4.10}$$

This is a second solution of (4.1) linearly independent of $J_p(x)$. It is clearly unbounded at $x = 0$, behaving like x^{-p} as x approaches 0.

4.1.3. Half-integer case. Now consider the case when p is a positive half-integer such as 1/2, 3/2, 5/2, etc. Then the difference of the roots $N = 2p$ is an odd integer. Since we have seen that $a_n(r) = 0$ for all odd n, in particular, $a_N(r)$ has no singularity at $r = -p$, and in fact, is identically zero. So we get a second fractional power series solution (3.25) which coincides with $y_2(x)$ above. Hence $J_{-p}(x)$ is a second solution of (4.1) in the half-integer case as well.

Remark 4.1. Recall from Remark 3.12 that in the above situation, we can get the general solution directly by choosing a_N arbitrarily. However note that the solution separates into the even and odd parts, and we get $a_0 J_{-p}(x) + a_N J_p(x)$, so there is no particular advantage doing things this way.

Explicitly, for $p = 1/2$, the two solutions are

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x \quad \text{and} \quad J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x. \tag{4.11}$$

(The first function is bounded at 0 but does not have a derivative at 0, while the second function is not even bounded.) This can be seen by a direct calculation. For instance, substituting $p = 1/2$ in (4.3)

$$x^{1/2} \sum_{n \geq 0} \frac{(-1)^n}{2^{2n} n!\, 3/2 \dots (2n+1)/2} x^{2n} = x^{1/2} \sum_{n \geq 0} \frac{(-1)^n}{2^n n!\, 3 \dots (2n+1)} x^{2n}$$

$$= x^{1/2} \sum_{n \geq 0} \frac{(-1)^n}{(2n+1)!} x^{2n} = x^{-1/2} \sum_{n \geq 0} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x^{-1/2} \sin x.$$

Now normalize by the reciprocal of $\sqrt{2}\,\Gamma(3/2)$. Alternatively, one can also substitute $p = 1/2$ in (4.5), and repeatedly use the identity (A.2).

Formulas (4.11) can be derived more simply as follows. (The method is suggested by the formulas themselves.) The substitution $u(x) = \sqrt{x}\, y(x)$ transforms the Bessel equation into

$$u'' + \big(1 + \frac{1 - 4p^2}{4x^2}\big) u = 0. \tag{4.12}$$

For $p = 1/2$, this is exactly $u'' + u' = 0$, whose solutions are $\sin x$ and $\cos x$.

4.1.4. Case $p = 0$. In this case, the indicial equation has a repeated root $r_1 = r_2 = 0$. Substituting $p = 0$ in (4.2),

$$a_{2n}(r) = \frac{(-1)^n}{(r+2)^2 (r+4)^2 \dots (r+2n)^2} a_0.$$

The $a_n(r)$ for n odd are the zero functions. The fractional power series solution is a power series with coefficients $a_n(0)$:

$$y_1(x) = J_0(x) = \sum_{n \geq 0} \frac{(-1)^n}{2^{2n} (n!)^2} x^{2n}, \quad x > 0.$$

(In this case, note that the normalization factor is 1.)

To find the second solution, we must differentiate the $a_{2n}(r)$ wrt r, and then put $r = 0$. Using the method outlined earlier to calculate the derivatives of rational functions,

$$a'_{2n}(r) = -2 a_{2n}(r) \big(\frac{1}{r+2} + \frac{1}{r+4} + \dots + \frac{1}{r+2n}\big).$$

Now setting $r = 0$, we obtain

$$a'_{2n}(0) = \frac{(-1)^{n-1} H_n}{2^{2n} (n!)^2} a_0,$$

where H_n is the harmonic sum (3.19). Thus, the second solution is

$$y_2(x) = J_0(x) \log x - \sum_{n \geq 1} \frac{(-1)^n H_n}{2^{2n} (n!)^2} x^{2n}, \quad x > 0.$$

It is customary to take the following linear combination of $y_2(x)$ and $J_0(x)$ as the second solution.

$$\begin{aligned} Y_0(x) &= \frac{2}{\pi} [y_2(x) + (\gamma - \log 2) J_0(x)], \\ &= \frac{2}{\pi} \big[\big(\gamma + \log \frac{x}{2}\big) J_0(x) - \sum_{n \geq 1} \frac{(-1)^n H_n}{2^{2n} (n!)^2} x^{2n}\big] \end{aligned} \tag{4.13}$$

where γ is the Euler constant defined by

$$\gamma := \lim_{m\to\infty}(H_m - \log m). \tag{4.14}$$

Its value is $\gamma \approx 0.57721566490\ldots$. *It is an open problem in number theory to determine whether γ is rational or not.*

4.1.5. Case when p is a positive integer. Suppose p is a positive integer. Recall from (3.24) that the second solution is of the form

$$y_2(x) = Ky_1(x)\log x + x^{r_2}(1 + \sum_{n=1}^{N-1} a_n(r_2)x^n) + x^{r_2}\sum_{n\geq N} A_n x^n.$$

(We assume $a_0 = 1$.) In the present situation $N = 2p$. Since only the even powers show up, the solution can be written as

$$y_2(x) = Ky_1(x)\log x + x^{-p}(1 + \sum_{n=1}^{p-1} a_{2n}(-p)x^{2n}) + x^{-p}\sum_{n\geq p} A_{2n} x^{2n}.$$

Rewriting (4.2),

$$a_{2n}(r) = \frac{(-1)^n}{(r+2+p)\ldots(r+2n+p)(r+2-p)\ldots(r+2n-p)}. \tag{4.15}$$

For $n < p$,

$$a_{2n}(-p) = \frac{(-1)^n}{2\ldots 2n.2(1-p)\ldots 2(n-p)} = \frac{(-1)^n}{2^{2n}n!(1-p)\ldots(n-p)}.$$

Note that if $n \geq p$, then $p-p$ would appear in the denominator. So we indeed have a singularity and there will be a log term. The constant K is computed by cancelling the $r+p$ which appears in the denominator of (4.15) and then substituting $r = -p$:

$$K = \frac{(-1)^p}{2\ldots 2p.2(1-p)\ldots 2(-1)} = -\frac{1}{2^{2p-1}p!(p-1)!}.$$

Similarly, to compute A_{2n} for $n \geq p$, we multiply $a_{2n}(r)$ by $r+p$, take derivative wrt r (using the method outlined before) and finally substitute $r = -p$:

$$\frac{d}{dr}(r+p)a_{2n}(r) =$$
$$(r+p)a_{2n}(r)\left[\frac{1}{r+2+p} + \cdots + \frac{1}{r+2n+p} + \frac{1}{r+2-p} + \cdots + \frac{1}{r+2n-p}\right]$$

with the understanding that the troublesome term $\frac{1}{r+p}$ is absent from the sum written inside the brackets. Now substituting $r = -p$ and simplifying, we obtain,

$$A_{2n} = \frac{(-1)^{n+1}(H_{n-p} + H_n - H_{p-1})}{2^{2n}n!(n-p)!(1-p)\ldots(-1)} = \frac{(-1)^{n-p}(H_{n-p} + H_n - H_{p-1})}{2^{2n}n!(n-p)!(p-1)!},$$

where H_n is the harmonic sum (3.19). We follow the convention that $H_0 = 0$.

Putting all these values in, the second solution is

$$y_2(x) = -\frac{1}{2^{2p-1}p!(p-1)!}y_1(x)\log x + x^{-p}(1 + \sum_{n=1}^{p-1} \frac{(-1)^n}{2^{2n}n!(1-p)\dots(n-p)}x^{2n}) + x^{-p}\sum_{n\geq p}\frac{(-1)^{n-p}(H_{n-p}+H_n-H_{p-1})}{2^{2n}n!(n-p)!(p-1)!}x^{2n}.$$

Note that the first term above is

$$-\frac{1}{2^{2p-1}p!(p-1)!}y_1(x)\log x = -\frac{1}{2^{p-1}(p-1)!}J_p(x)\log x.$$

In particular, for $p = 1$, the second solution is

$$-J_1(x)\log x + x^{-1}(1 + \sum_{n\geq 1}\frac{(-1)^{n-1}(H_{n-1}+H_n)}{2^{2n}n!(n-1)!}x^{2n}).$$

The above analysis up to some language translation is given in [7, Section 120].

4.2. Identities

Note that the second solution (4.9) makes no sense if p is a positive integer (since we start getting zeroes in the denominator. However, $J_{-p}(x)$ defined in (4.10) does make sense for all p after imposing the natural condition that the reciprocal of Γ evaluated at a nonpositive integer is 0. (How did this happen?) Further, $J_{-p}(x)$ is a solution of the Bessel equation (4.1) for all p. However, in the integer case, one can see that

$$J_{-p}(x) = (-1)^p J_p(x),$$

so this is not linearly independent of the first solution.

In the following discussion, we prove four important identities involving $J_p(x)$ where p is any real number. The observation (4.8) will end up being a special case.

$$\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x). \tag{4.16}$$

We calculate using (4.4).

$$\begin{aligned}
(x^p J_p(x))' &= \left(2^p \sum_{n\geq 0}\frac{(-1)^n}{n!\,\Gamma(n+p+1)}\left(\frac{x}{2}\right)^{2n+2p}\right)' \\
&= 2^p \sum_{n\geq 0}\frac{(-1)^n(2n+2p)}{n!\,\Gamma(n+p+1)}\frac{1}{2}\left(\frac{x}{2}\right)^{2n+2p-1} \\
&= 2^p \sum_{n\geq 0}\frac{(-1)^n}{n!\,\Gamma(n+p)}\left(\frac{x}{2}\right)^{2n+2p-1} \\
&= x^p \left(\frac{x}{2}\right)^{p-1}\sum_{n\geq 0}\frac{(-1)^n}{n!\,\Gamma(n+p)}\left(\frac{x}{2}\right)^{2n} \\
&= x^p J_{p-1}(x).
\end{aligned}$$

In the third step, we used (A.3). In the second step, we differentiated a series termwise. Can you justify this? We know that a power series can be differentiated termwise, but this is not a power series unless p is a nonnegative integer.

A companion identity is

$$\frac{d}{dx}\left[x^{-p}J_p(x)\right] = -x^{-p}J_{p+1}(x). \tag{4.17}$$

The proof is a similar calculation given below.

$$\begin{aligned}
(x^{-p}J_p(x))' &= \left(2^{-p}\sum_{n\geq 0}\frac{(-1)^n}{n!\,\Gamma(n+p+1)}\left(\frac{x}{2}\right)^{2n}\right)' \\
&= 2^{-p}\sum_{n\geq 0}\frac{(-1)^n(2n)}{n!\,\Gamma(n+p+1)}\frac{1}{2}\left(\frac{x}{2}\right)^{2n-1} \\
&= 2^{-p}\sum_{n\geq 1}\frac{(-1)^n}{(n-1)!\,\Gamma(n+p+1)}\left(\frac{x}{2}\right)^{2n-1} \quad \text{(since 0-th term vanishes)} \\
&= 2^{-p}\sum_{n\geq 0}\frac{(-1)^{n+1}}{n!\,\Gamma(n+p+2)}\left(\frac{x}{2}\right)^{2n+1} \quad \text{(relabelling the terms)} \\
&= -x^{-p}\left(\frac{x}{2}\right)^{p+1}\sum_{n\geq 0}\frac{(-1)^n}{n!\,\Gamma(n+p+2)}\left(\frac{x}{2}\right)^{2n} \\
&= -x^{-p}J_{p+1}(x).
\end{aligned}$$

These lead to two more identities

$$J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x}J_p(x) \tag{4.18}$$

and

$$J_{p-1}(x) - J_{p+1}(x) = 2J_p'(x). \tag{4.19}$$

as follows.

$$\begin{aligned}
J_{p-1}(x) \pm J_{p+1}(x) &= x^{-p}[x^pJ_p(x)]' \mp x^p[x^{-p}J_p(x)]' \\
&= J_p'(x) + \frac{p}{x}J_p(x) \mp [J_p'(x) - \frac{p}{x}J_p(x)] \\
&= \frac{2p}{x}J_p(x),\ 2J_p'(x) \quad \text{respectively.}
\end{aligned}$$

The identity (4.18) can be thought of as a 3-step recursion in p. In other words, $J_{p+n}(x)$ can be computed algorithmically from $J_p(x)$ and $J_{p+1}(x)$ for all integer n. For example, using (4.11):

$$\begin{aligned}
J_{\frac{3}{2}}(x) &= \frac{1}{x}J_{\frac{1}{2}}(x) - J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\Big(\frac{\sin x}{x} - \cos x\Big) \\
J_{-3/2}(x) &= -\frac{1}{x}J_{-\frac{1}{2}}(x) - J_{\frac{1}{2}}(x) = -\sqrt{\frac{2}{\pi x}}\Big(\frac{\cos x}{x} + \sin x\Big) \\
J_{\frac{5}{2}}(x) &= \frac{3}{x}J_{\frac{3}{2}}(x) - J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\Big(\frac{3\sin x}{x^2} - \frac{3\cos x}{x} - \sin x\Big) \\
J_{-\frac{5}{2}}(x) &= -\frac{3}{x}J_{-\frac{3}{2}}(x) - J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\Big(\frac{3\cos x}{x^2} + \frac{3\sin x}{x} - \cos x\Big)
\end{aligned}$$

and so on. Thus we see that all Bessel functions of the form $J_{n+1/2}(x)$ can be expressed using sin, cos, powers and square-roots. These are sometimes called the *spherical Bessel functions*. Liouville showed that these are the only cases for which $J_p(x)$ is an elementary transcendental function. This result is discussed in [12, Section 4.74].

4.3. Orthogonality

We now look at the orthogonality property of the Bessel functions wrt an inner product on square-integrable functions. This property allows us to expand any square-integrable function on $[0, 1]$ into a Bessel series.

4.3.1. Zeroes of the Bessel function. Fix $p \geq 0$. The equation (4.12) suggests that for large values of x, $J_p(x)$ should behave like a sine or cosine function. A precise fact (without proof) is stated below.

$$J_p(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4} - \frac{p\pi}{2}\right) + \frac{\epsilon(x)}{x^{3/2}}, \tag{4.20}$$

where $\epsilon(x)$ is bounded as $x \to \infty$. This fact is stated in [10, Section 34]. Plenty of information on such asymptotic expansions of $J_p(x)$ is given in [12, Chapter VII].

Let $Z^{(p)}$ denote the set of zeroes of $J_p(x)$. It follows from (4.20) that the set of zeroes is a sequence increasing to infinity. (For $p = 1/2$, this is also clear from the formula (4.11). In this case, (4.20) holds with $\epsilon(x) = 0$.) This fact can be established by Sturm-Liouville theory which is not discussed in these notes.

The first five positive zeroes of some Bessel functions are given below.

	$J_0(x)$	$J_1(x)$	$J_2(x)$	$J_3(x)$	$J_4(x)$	$J_5(x)$
1	2.4048	3.8317	5.1356	6.3802	7.5883	8.7715
2	5.5201	7.0156	8.4172	9.7610	11.0647	12.3386
3	8.6537	10.1735	11.6198	13.0152	14.3725	15.7002
4	11.7915	13.3237	14.7960	16.2235	17.6160	18.9801
5	14.9309	16.4706	17.9598	19.4094	20.8269	22.2178

The first row gives the value of the first positive zero, and so on. Compare with the graphs drawn earlier.

Proposition 4.2. *Let x_1 and x_2 be successive positive zeroes of a nontrivial solution $y_p(x)$ of the Bessel equation.*

- *If $0 \leq p < 1/2$, then $x_2 - x_1$ is less than π and approaches π as $x_1 \to \infty$.*
- *If $p = 1/2$, then $x_2 - x_1 = \pi$.*
- *If $p > 1/2$, then $x_2 - x_1$ is greater than π and approaches π as $x_1 \to \infty$.*

The result is clear for $p = 1/2$. The other two cases can be proved using the Sturm comparison theorem. For details, see [10, Section 23].

4.3.2. Orthogonality. Define an inner product on square-integrable functions on $[0, 1]$ by

$$\langle f, g \rangle := \int_0^1 x f(x) g(x) dx. \tag{4.21}$$

This is similar to (2.5) except that $f(x)g(x)$ is now multiplied by x, and the interval of integration is from 0 to 1. The multiplying factor x is called a weight function. We could instead work on the interval $0 \le x \le R$, and the formulas below could be adapted accordingly.

Fix $p \ge 0$. The set of *scaled functions*

$$\{J_p(zx) \mid z \in Z^{(p)}\}$$

indexed by the zero set $Z^{(p)}$ form an orthogonal family wrt the inner product (4.21). More precisely:

Proposition 4.3. *If k and ℓ are any two positive zeroes of the Bessel function $J_p(x)$, then*

$$\int_0^1 xJ_p(kx)J_p(\ell x)dx = \begin{cases} \frac{1}{2}[J_p'(k)]^2 = \frac{1}{2}J_{p\pm1}^2(k) & \text{if } k = \ell, \\ 0 & \text{if } k \ne \ell. \end{cases}$$

The identity

$$J_p'(k)^2 = J_{p\pm1}(k)^2$$

can be shown as follows: The identities (4.16) and (4.17) evaluated at $x = k \in Z^{(p)}$ yield

$$J_p'(k) = J_{p-1}(k) \quad \text{and} \quad J_p'(k) = -J_{p+1}(k).$$

Proof. We begin with the fact that $J_p(x)$ is a solution of the Bessel equation

$$y'' + \frac{1}{x}y' + \big(1 - \frac{p^2}{x^2}\big)y = 0.$$

For any positive constants a and b, the functions $u(x) = J_p(ax)$ and $v(x) = J_p(bx)$ satisfy

$$u'' + \frac{1}{x}u' + \big(a^2 - \frac{p^2}{x^2}\big)u = 0 \quad \text{and} \quad v'' + \frac{1}{x}v' + \big(b^2 - \frac{p^2}{x^2}\big)v = 0$$

respectively. Multiplying the first equation by v, the second by u and subtracting,

$$(u'v - v'u)' + \frac{1}{x}(u'v - v'u) = (b^2 - a^2)uv.$$

Multiplying by x, we obtain

$$[x(u'v - v'u)]' = (b^2 - a^2)xuv.$$

Note that $x(u'v - v'u)$ is bounded on the interval $(0, 1)$: The only problem is near 0 where u' and v' may blow up if $0 < p < 1$, however multiplying by x tempers this, and in fact the value goes to 0 as x approaches 0.

Integrating from 0 to 1, we get

$$(b^2 - a^2)\int_0^1 xuv\,dx = u'(1)v(1) - v'(1)u(1).$$

Suppose $a = k$ and $b = \ell$ are distinct zeroes of $J_p(x)$. Then $u(1) = v(1) = 0$, so the rhs is zero. Further, $b^2 - a^2 \ne 0$, so the integral in the lhs is zero. This proves orthogonality.

Now let $a = k$, and b be close to but not equal to k. Then

$$\int_0^1 xJ_p(kx)J_p(bx)\,dx = \frac{kJ_p'(k)J_p(b)}{b^2 - k^2}.$$

Now take limit as b approaches k. Since the integrand in the lhs is bounded, the limit can be pushed inside the integral to obtain

$$\int_0^1 x J_p(kx) J_p(kx)\, dx.$$

On the other hand, the rhs reads

$$\lim_{b\to k} \frac{kJ_p'(k)J_p(b)}{b^2-k^2} = \lim_{b\to k} \frac{k}{b+k} J_p'(k) \frac{J_p(b)-J_p(k)}{b-k} = \frac{1}{2} J_p'(k)^2.$$

□

4.3.3. Fourier-Bessel series. Fix $p \geq 0$. Any square-integrable function $f(x)$ on $[0,1]$ can be expanded in a series of scaled Bessel functions $J_p(zx)$

$$f(x) \approx \sum_{z\in Z^{(p)}} c_z J_p(zx), \tag{4.22}$$

where

$$c_z = \frac{2}{J^2_{p\pm1}(z)} \int_0^1 x\, f(x)\, J_p(zx)\, dx. \tag{4.23}$$

This is called the *Fourier-Bessel series* of $f(x)$. The formula for the coefficients is derived from Proposition 4.3. The series converges to $f(x)$ in norm (see the parallel discussion for Legendre polynomials.) By general Sturm-Liouville theory, the scaled functions form a maximal orthogonal set.

For pointwise convergence, the following result is useful.

Theorem 4.4. *If both $f(x)$ and $f'(x)$ have at most a finite number of jump discontinuities in the interval $[0,1]$, then the Bessel series converges to*

$$\frac{1}{2}(f(x_-)+f(x_+))$$

for $0 < x < 1$. At $x = 1$, the series always converges to 0 irrespective of f, and at $x = 0$, it converges to $f(0_+)$. In particular, the series converges to $f(x)$ at every interior point of continuity.

This is sometimes called the *Bessel expansion theorem*. This is stated in [10, Section 35, Theorem A]. A more general statement with a proof is given in [12, Section 18.24].

Example 4.5. Let us compute the Fourier-Bessel series (for $p = 0$) of the function $f(x) = 1$ in the interval $0 \leq x \leq 1$. Theorem 4.4 applies. Using (4.8),

$$\int_0^1 x\, J_0(zx)\, dx = \frac{1}{z} x\, J_1(zx)\Big|_0^1 = \frac{J_1(z)}{z},$$

so using (4.23),

$$c_z = \frac{2}{zJ_1(z)}.$$

It follows that

$$1 = \sum_{z\in Z^{(0)}} \frac{2}{zJ_1(z)} J_0(zx), \quad 0 \leq x < 1.$$

4.4. Schlomilch's formula

Consider the function

$$\varphi(x,t) := e^{x/2(t-1/t)}.$$

This is a function of two variables defined for all x and t except for $t = 0$.

$$e^{xt/2} = \sum_{j\geq 0} \frac{1}{j!}\frac{x^j}{2^j}t^j \quad \text{and} \quad e^{-x/2t} = \sum_{k\geq 0} \frac{(-1)^k}{k!}\frac{x^k}{2^k}t^{-k}.$$

Both series are absolutely convergent. If we multiply them, we obtain a double series whose terms are all possible products of a term from the first series with a term in the second series. Absolute convergence allows us to rearrange the terms of the double series in any manner that we like. Let us group terms together according to powers of t. Observe that the coefficient of t^0 is

$$\sum_{n\geq 0} \frac{(-1)^n}{(n!)^2}\left(\frac{x}{2}\right)^{2n}$$

and this is precisely $J_0(x)$ from (4.6). More generally, one can show by a simple explicit calculation that the coefficient of t^n is $J_n(x)$. This yields:

Proposition 4.6. *For any x and t such that $t \neq 0$,*

$$e^{x/2(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n(x)t^n.$$

This is known as *Schlomilch's formula.* One can use this to deduce the addition formula

$$J_n(x+y) = \sum_{k=-\infty}^{\infty} J_{n-k}(x)J_k(y). \tag{4.24}$$

In particular, putting $n = 0$,

$$J_0(x+y) = J_0(x)J_0(y) - 2J_1(x)J_1(y) + 2J_2(x)J_2(y) - \dots.$$

If we replace y by $-x$ and use the fact that $J_n(x)$ is even or odd according as n is even or odd, then we obtain

$$1 = J_0(x)^2 + 2J_1(x)^2 + 2J_2(x)^2 + \dots.$$

We deduce that $|J_0(x)| \leq 1$ and $|J_n(x)| \leq \frac{1}{\sqrt{2}}$ for $n \geq 1$.

4.5. Neumann functions

Let p be any real number. For noninteger values of p, the Neumann function (or Weber function) is defined by

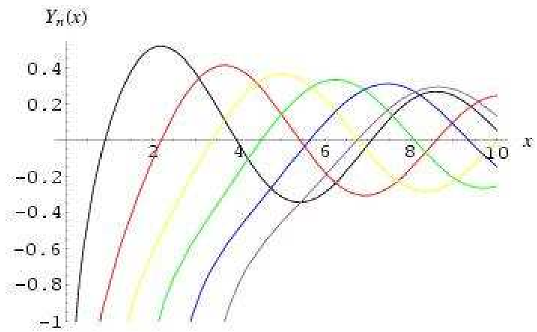
$$Y_p(x) := \frac{\cos p\pi J_p(x) - J_{-p}(x)}{\sin p\pi}. \tag{4.25}$$

This is linearly independent of $J_p(x)$ and provides a second solution to the Bessel equation (since $J_{-p}(x)$ is a linearly independent solution).

If $p = n$ is a nonnegative integer, then the denominator in $Y_n(x)$ is 0, but so is the numerator. In this case, the Neumann function is defined as

$$Y_n(x) := \lim_{p\to n} Y_p(x) = \lim_{p\to n} \frac{\cos p\pi J_p(x) - J_{-p}(x)}{\sin p\pi}. \tag{4.26}$$

The graphs of some Neumann functions are given below. They are all unbounded near 0.



The rhs of (4.26) can *in principle* be evaluated by L'Hôpital's rule. This results in a second solution linearly independent of $J_n(x)$. Actual computation is very hard.

We illustrate by computing $Y_0(x)$. For that, we need to find $\left[\frac{\partial}{\partial p} J_{\pm p}\right]_{p=0}$. So for $0 < p < 1$, write

$$J_p(x) := \left(\frac{x}{2}\right)^p \sum_{n \geq 0} \frac{(-1)^n}{n!\,\Gamma(n+p+1)} \left(\frac{x}{2}\right)^{2n}$$

It is imperative to compute $\left[\frac{\partial \Gamma(n+p+1)}{\partial p}\right]_{p=0}$. By definition,

$$\Gamma(n+p+1) = \int_0^\infty t^{n+p} e^{-t} dt.$$

Therefore,

$$\left[\frac{\partial \Gamma(n+p+1)}{\partial p}\right]_{p=0} = \int_0^\infty t^n (\log t) e^{-t} dt,$$

and hence

$$\left[\frac{\partial [\Gamma(n+p+1)]^{-1}}{\partial p}\right]_{p=0} = -\frac{1}{(n!)^2} \int_0^\infty t^n \log t e^{-t} dt = \frac{\gamma_n}{n!},$$

where

$$\gamma_n := -\frac{1}{n!} \int_0^\infty t^n (\log t) e^{-t} dt. \tag{4.27}$$

Using the above computation, we now differentiate the series for J_p wrt p to obtain

$$\left[\frac{\partial}{\partial p} J_p\right]_{p=0} = \log \frac{x}{2} J_0(x) + \sum_{n \geq 0} \gamma_n \frac{(-1)^n}{(n!)^2} \left(\frac{x}{2}\right)^{2n}.$$

Similarly,

$$\left[\frac{\partial}{\partial p} J_{-p}\right]_{p=0} = -\log \frac{x}{2} J_0(x) - \sum_{n \geq 0} \gamma_n \frac{(-1)^n}{(n!)^2} \left(\frac{x}{2}\right)^{2n}.$$

Therefore applying L'Hôpital's rule, we finally get,

$$Y_0(x) = \frac{2}{\pi} \left[\log \frac{x}{2} J_0(x) + \sum_{n \geq 0} \gamma_n \frac{(-1)^n}{(n!)^2} \left(\frac{x}{2}\right)^{2n}\right]. \tag{4.28}$$

This proves that the solution Y_0 which is independent of J_0 has a free logarithmic term since the power series for J_0 starts with 1. Thus Y_0 is unbounded as $x \to 0$ and it cannot be a power series in x.

We note that integrating (4.27) by parts establishes the recurrence:

$$\gamma_{n+1} = \gamma_n - \frac{1}{n+1}.$$

Hint: First we need to establish the easy indefinite integral

$$\int t^n (\log t) dt = \frac{t^{n+1} \log t}{n+1} - \frac{t^{n+1}}{(n+1)^2}.$$

This implies that

$$\gamma_n = \gamma_0 - \frac{1}{n} - \frac{1}{n-1} - \cdots - \frac{1}{2} - 1 = -\int_0^\infty (\log t) e^{-t} dt - H_n.$$

This allows us to write (4.28) in a more conventional form,

$$Y_0(x) = \frac{2}{\pi} \left[\left(\log \frac{x}{2} + \gamma_0 \right) J_0(x) - \sum_{n \geq 0} H_n \frac{(-1)^n}{(n!)^2} \left(\frac{x}{2} \right)^{2n} \right]. \tag{4.29}$$

The constant $\gamma_0 = -\int_0^\infty (\log t) e^{-t} dt$ can be shown to be the Euler constant (4.14), see (B.1). Hence the above expression equals (4.13).

Remark 4.7. For the fastidious reader, note that the integral $\Gamma(s+1) = \int_0^\infty t^s e^{-t} dt$ is convergent for $s > -1$. Since we restricted p to be in $(0, 1)$, all the integral expressions for $\Gamma(m \pm p + 1)$ that we used above are valid.

CHAPTER 5

Fourier series

We have seen the Fourier-Legendre series and the Fourier-Bessel series for square-integrable functions on a closed interval. We now look at another series of this kind called simply the Fourier series. The role of the Legendre polynomials and the scaled Bessel functions is now played by the trigonometric functions.

The book by Brown and Churchill [3] is well-suited for our purposes. It gives an elementary treatment of Fourier analysis, and explains their role in the theory of differential equations. A nice but slightly advanced book is [11].

5.1. Orthogonality of the trigonometric family

Consider the space of square-integrable functions on $[-\pi, \pi]$. Define an inner product by

$$\langle f, g\rangle := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)g(x)dx. \tag{5.1}$$

Note that we are integrating only between $-\pi$ and π. This ensures that the integral is always finite. The norm of a function is then given by

$$\|f\| = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)f(x)dx\right)^{1/2}. \tag{5.2}$$

(Some books follow a different normalization convention for the inner product.)

The result below shows that the set $\{1, \cos nx, \sin nx\}_{n\geq 1}$ is an orthogonal family wrt the inner product (5.1).

Proposition 5.1. *Let m and n denote any positive integers. Then*

$$\langle 1, 1\rangle = 1.$$

$$\langle \cos mx, \cos nx\rangle = \begin{cases} 0 & \text{if } m \neq n, \\ 1/2 & \text{if } m = n. \end{cases}$$

$$\langle \sin mx, \sin nx\rangle = \begin{cases} 0 & \text{if } m \neq n, \\ 1/2 & \text{if } m = n. \end{cases}$$

$$\langle \sin mx, \cos nx\rangle = \langle 1, \cos nx\rangle = \langle 1, \sin mx\rangle = 0 \quad \text{for all} \;\; m, n \geq 1.$$

Proof. Assuming $m \neq n$,

$$\langle \cos mx, \cos nx\rangle = \frac{1}{4\pi}\int_{-\pi}^{\pi} \cos(m+n)x + \cos(m-n)x\, dx$$

$$= \frac{1}{4\pi}\frac{\sin(m+n)x}{(m+n)} + \frac{\sin(m-n)x}{(m-n)}\Big|_{-\pi}^{\pi} = 0.$$

If $m = n$, then the second term in the integral above is identically 1, so it contributes 1/2 (instead of 0).

The remaining formulas can be proved by similar calculations. $\square$

Alternatively, one may also prove orthogonality using the differential equation satisfied by the sine and cosine functions:

Proposition 5.2. *For each integer $n \geq 1$, let y_n be a nontrivial solution of*

$$y'' + n^2 y = 0.$$

(Explicitly, y_n is a linear combination of $\sin nx$ and $\cos nx$.) Then for any $m \neq n$,

$$\langle y_m, y_n \rangle = 0.$$

PROOF. Multiply the ODE for y_n by y_m, the one for y_m by y_n, and subtract:

$$(y_n' y_m - y_n y_m')' = (m^2 - n^2) y_m y_n.$$

Integrating from $-\pi$ to π yields

$$(m^2 - n^2) \int_{-\pi}^{\pi} y_m y_n dx = y_n' y_m - y_n y_m' \big|_{-\pi}^{\pi}.$$

For any integer k, y_k and y_k' are 2π-periodic, so rhs is zero. Since $m \neq n$, the integral in the lhs must be zero, as required. $\square$

5.2. Fourier series

Any square-integrable function $f(x)$ on $[-\pi, \pi]$ can be expanded in a series of the trigonometric functions

$$f(x) \approx a_0 + \sum_1^\infty (a_n \cos nx + b_n \sin nx) \tag{5.3}$$

where the a_n and b_n are given by

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx, \quad \text{and} \tag{5.4}$$
$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx, \quad n \geq 1.$$

This is called the *Fourier series* of $f(x)$, and the a_n and b_n are called the *Fourier coefficients*. The formulas (5.4) are derived from Proposition 5.1. They are sometimes called the *Euler formulas*.

5.2.1. Convergence in norm and Parseval's identity. The Fourier series of $f(x)$ converges to $f(x)$ in the sense that

$$\| f(x) - a_0 - \sum_{n=1}^m (a_n \cos nx + b_n \sin nx) \| \to 0 \quad \text{as} \quad m \to \infty.$$

This is known as convergence in norm. The reason this works is that the trigonometric functions form a maximal orthogonal set in the space of square-integrable functions. In general, we get a Fourier-type series and convergence in norm whenever we have a maximal orthogonal set in any Hilbert space [9, Chapter 4]. The convergence in norm of the Fourier-Legendre series and the Fourier-Bessel series are instances of this general result.

Recall: Suppose V is a finite-dimensional inner product space and $\{v_1, \dots, v_k\}$ is an orthogonal basis. If $v = \sum_{i=1}^{k} a_i v_i$, then $\|v\|^2 = \sum_{i=1}^{k} a_i^2 \|v_i\|^2$. This is the *Pythagoras theorem.* There is an infinite-dimensional analogue which says that the square of the norm of f is the sum of the squares of the norms of its components wrt any maximal orthogonal set. Thus, we have

$$\|f\|^2 = a_0^2 + \frac{1}{2} \sum_{n \geq 1} (a_n^2 + b_n^2). \tag{5.5}$$

This is known as *Parseval's identity.*

Theorem 5.3. *Given any a_0, a_n and b_n such that the sum in the rhs of (5.5) is finite, there is a square-integrable function f with these Fourier coefficients.*

Further, f is unique up to a values on a set of measure zero (such as the rationals).

This is known as the *Riesz-Fischer theorem.*

5.2.2. Pointwise convergence. Pointwise convergence is more delicate. There are two issues here: Does the series on the right in (5.3) converge at x? If yes, then does it converge to $f(x)$? A convergence result is given below. To discuss what happens at the endpoints, it is better to deal with periodic functions.

Definition 5.4. A function $f : \mathbb{R} \to \mathbb{R}$ is *periodic* (of period 2π) if

$$f(x + 2\pi) = f(x)$$

for all x.

A function defined on $[-\pi, \pi]$ can be extended to a periodic function. There may be a problem if $f(\pi) \neq f(-\pi)$. In this case, we can choose say one of the two values for the extension. Note that changing the value of f at one point does not change its Fourier series (since the Fourier coefficients stay the same as before).

Theorem 5.5. *Let $f(x)$ be a periodic function of period 2π which is integrable on $[-\pi, \pi]$. Then at a point x, if the left and right derivative exist, then the Fourier series of f converges to*

$$\frac{1}{2}[f(x^+) + f(x^-)].$$

By definition, the right derivative at x exists if $f(x^+)$ exists and

$$\lim_{y \to x, y > x} \frac{f(y) - f(x^+)}{y - x}$$

exists. The left derivative at x is defined similarly.

Example 5.6. Consider the function

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < \pi, \\ -1 & \text{if } -\pi < x < 0. \end{cases}$$

The value at 0, π and $-\pi$ is left unspecified. Its periodic extension is the square-wave.

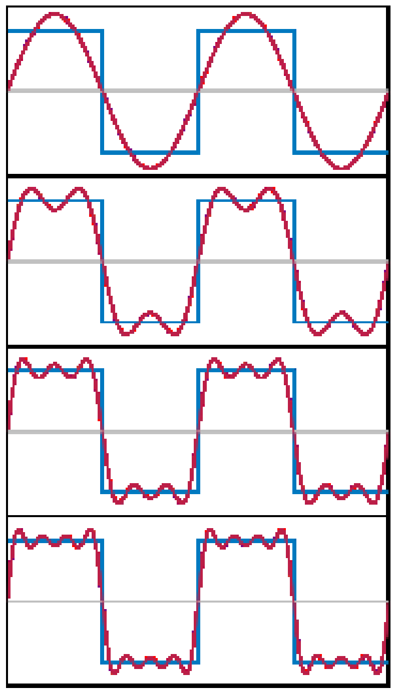
Since f is an odd function, a_0 and all the a_n are zero. The b_n for $n \geq 1$ can be calculated as follows.

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{2}{\pi} \int_0^{\pi} \sin nx \, dx = \frac{2}{n\pi}(1 - \cos n\pi) = \begin{cases} \frac{4}{n\pi} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Thus the Fourier series of $f(x)$ is

$$\frac{4}{\pi}\big(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots\big).$$

By Theorem 5.5, this series converges to $f(x)$ at all points except integer multiples of π where it converges to 0. The partial sums of the Fourier series wiggle around the square wave.



In particular, evaluating at $x = \pi/2$,

$$f(\frac{\pi}{2}) = 1 = \frac{4}{\pi}\big(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\big).$$

Rewriting,

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}.$$

(This can also be proved by evaluating the Taylor series of $\tan^{-1}(x)$ at $x = 1$.) What would we get if we applied Parseval's identity to f?

Example 5.7. Consider the function

$$f(x) = x^2, \quad -\pi \le x \le \pi.$$

Since f is an even function, the b_n are zero.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{\pi^2}{3}.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx = \frac{4}{n^2}\cos n\pi = \begin{cases} \frac{4}{n^2} & \text{if } n \text{ is even,} \\ -\frac{4}{n^2} & \text{if } n \text{ is odd.} \end{cases}$$

Thus the Fourier series of $f(x)$ is

$$\frac{\pi^2}{3} - 4\big(\cos x - \frac{\cos 2x}{4} + \frac{\cos 3x}{9} - \dots\big).$$

By Theorem 5.5, this series converges to $f(x)$ at all points. Evaluating at $x = \pi$,

$$\pi^2 = \frac{\pi^2}{3} + 4\big(1 + \frac{1}{4} + \frac{1}{9} + \dots\big).$$

This yields the identity

$$\sum_{n\geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

5.2.3. Fourier sine and cosine series. If f is an odd (even) function, then its Fourier series has only sine (cosine) terms. This allows us to do something interesting. Suppose f is defined on the interval $(0, \pi)$. Then we can extend it as an odd function on $(-\pi, \pi)$ and expand it in a Fourier sine series, or extend it as an even function on $(-\pi, \pi)$ and expand it in a Fourier cosine series. For instance, consider the function

$$f(x) = x, \quad 0 < x < \pi.$$

Then the Fourier sine series of $f(x)$ is

$$2\big(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots\big)$$

while the Fourier cosine series of $f(x)$ is

$$\frac{\pi}{2} - \frac{4}{\pi}\big(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \dots\big)$$

The two series are equal on $0 < x < \pi$ (but different on $-\pi < x < 0$). What happens if we put $x = \pi/2$ in either series?

The Fourier cosine series above is the same as the Fourier series of $g(x) = |x|$. Note that $g'(x)$ equals the function $f(x)$ in Example 5.6, and the Fourier series of the square wave is precisely the term-by-term derivative of the Fourier series of $g(x)$. This is a general fact which can be seen by applying derivative transfer on the Euler formulas.

One can deduce from the maximal orthogonality of the trigonometric functions that

$$\{\sin x, \sin 2x, \dots\} \quad \text{and} \quad \{1, \cos x, \cos 2x, \dots\}$$

are maximal orthogonal sets in $(0, \pi)$. This should not be surprising.

Every functions $f(x)$ can be expressed uniquely as a sum of an even function and an odd function:

$$f(x) = [\frac{f(x) + f(-x)}{2}] + [\frac{f(x) - f(-x)}{2}].$$

The Fourier series of $f(x)$ is the sum of the Fourier cosine series of its even part and the Fourier sine series of its odd part.

One can also expand a function on $0 < x < \pi$ in a Fourier sine quarter-wave series and a Fourier cosine quarter-wave series using the fact that

$$\{\sin x/2, \sin 3x/2, \dots\} \quad \text{and} \quad \{\cos x/2, \cos 3x/2, \dots\}$$

are maximal orthogonal sets in $(0, \pi)$. This can be done by extending sin as even function, and cos as odd function around $x = \pi$, and then extending them as a odd function and as an even function respectively around $x = 0$. So quarter-wave expansions are also special cases of usual Fourier expansions.

5.2.4. Fourier series for arbitrary periodic functions. One can also consider Fourier series for functions of any period not necessarily 2π. Suppose the period is 2ℓ. Then the Fourier series is of the form

$$a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\ell} + b_n \sin\frac{n\pi x}{\ell}.$$

The Fourier coefficients are given by

$$a_0 = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)dx, \quad a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\frac{n\pi x}{\ell}\,dx, \quad \text{and}$$

$$b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\frac{n\pi x}{\ell}\,dx, \quad n \geq 1.$$

It should be clear that by scaling the independent variable, one can transform the given periodic function to a 2π-periodic function, and then apply the standard theory.

CHAPTER 6

Heat equation

The heat equation is a PDE in the space variables and one time variable which governs how heat condution occurs in solids (under some idealized conditions). We derive the heat equation using Fourier's law. However the main emphasis here to introduce the method of separation of variables, and use it to solve the one-dimensional heat equation of a thin rod of finite length under a variety of boundary conditions. The solution is expressed as a Fourier expansion in the space variable, whose coefficients decay exponentially with time.

All the above ideas first occurred in work of Fourier around 1805. He later published them in 1822 in a book titled 'Heat conduction in solids'. However, Fourier paid no attention to the mathematical aspects of his theory. The first result along the lines of Theorem 5.5 was obtained by Dirichlet in 1830 (who also clarified the notion of a function).

We also consider briefly the two-dimensional heat equation and its solution on the circular disc using separation of variables which leads to a Fourier-Bessel series. (Fourier apparently had also looked at this.) The form of the heat equation on other domains and in higher dimensions is also discussed.

6.1. Method of separation of variables

Given a linear PDE in two variables say x and t, it is often very fruitful to first look for solutions of the type

$$u(x,t) = X(x)T(t) \tag{6.1}$$

and combine them to find the general solution. In many classical equations, the variables 'separate' and we are reduced to solving two linear ODEs, one each for $X(x)$ and $T(t)$.

Give some examples.

6.2. Fourier's law of heat condution

Let $u(x,t)$ denote the temperature at point x at time t of a solid body. Here x stands for three spatial directions.

Fourier's law of thermal conduction says that the heat flux density $\overrightarrow{q}$ is equal to the product of thermal conductivity K of the solid, and the negative of the temperature gradient $-\nabla u$:

$$\overrightarrow{q} = -K\nabla u. \tag{6.2}$$

For simplicity, we assume that K is a constant.

By Fourier's law, the amount of heat energy that flows out of an oriented surface $\overrightarrow{S}$ per unit time is

$$\int_S \overrightarrow{q} \cdot d\overrightarrow{S} = -K \int_S \nabla u \cdot d\overrightarrow{S}.$$

Now suppose S is the boundary of a solid region D. Then, using the divergence theorem of Gauss,

$$-K \int_S \nabla u \cdot d\overrightarrow{S} = -K \int_D \operatorname{div}(\nabla u) dx = -K \int_D \Delta u dx.$$

In the last step, we used that the divergence of the gradient is the Laplacian operator.

The specific heat σ of a material is the amount of heat energy required to raise the temperature of a unit mass by one temperature unit. Let δ denote the density of the material. Then the heat density is

$$\sigma \delta u_t(x, t)$$

and the amount of heat energy that flows out of $\overrightarrow{S}$ per unit time is

$$-\sigma\delta \int_D u_t(x, t) dx.$$

By equating the integrals, we obtain

$$K \int_D \Delta u dx = \sigma\delta \int_D u_t(x, t) dx. \tag{6.3}$$

This is the integral form of the heat equation. Since D is arbitrary, we obtain the differential form

$$u_t(x, t) = k\Delta u, \qquad k = K/\sigma\delta. \tag{6.4}$$

This is provided there are no external sources. If there is an external source, then the equation needs to be modified to

$$u_t(x, t) = k\Delta u + f(x, t), \qquad k = K/\sigma\delta.$$

The function $f(x, t)$ is positive or negative depending on whether it is a source or a sink.

6.3. One-dimensional heat equation

We now solve the one-dimensional heat equation

$$u_t = k u_{xx}, \quad 0 < x < \ell,\ t > 0. \tag{6.5}$$

This describes the temperature evolution of a thin rod of length ℓ. The temperature at $t = 0$ is specified. This is the initial condition. We write it as

$$u(x, 0) = u_0(x). \tag{6.6}$$

Adopting the method of separation of variables, write $u(x, t) = X(x)T(t)$ as in (6.1). Substitution in (6.5) does indeed separate the variables:

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)} = \lambda \text{ (say)}.$$

The equality is between a function of x and a function of t. So both must be constant. Denote this constant by λ.

In addition to the initial condition, there are conditions specified at the two endpoints of the rod. These are the boundary conditions. We consider four different kinds of boundary conditions one by one.

6.3.1. Dirichlet boundary conditions. The Dirichlet boundary conditions require

$$u(0,t) = u(\ell,t) = 0. \tag{6.7}$$

In other words, the endpoints of the rod are maintained at temperature 0 at all times t. (The rod is isolated from the surroundings except at the endpoints from where heat will be lost to the surroundings.)

We need to consider three cases:

(1) $\lambda > 0$: Write $\lambda = \mu^2$ with $\mu > 0$. Then

$$X(x) = Ae^{\mu x} + Be^{-\mu x} \quad \text{and} \quad T(t) = Ce^{\mu^2 kt}.$$

Hence

$$u(x,t) = e^{\mu^2 kt}(Ae^{\mu x} + Be^{-\mu x}),$$

where the constant C has been absorbed in A and B. The boundary conditions (6.7) imply that $A = 0 = B$. So there is no nontrivial solution of this form.

(2) $\lambda = 0$: In this case we have $X(x) = Ax + B$ and $T(t) = C$. Hence

$$u(x,t) = Ax + B.$$

The boundary conditions (6.7) give $A = 0 = B$. Thus this case also does not yield a nontrivial solution.

(3) $\lambda < 0$: Write $\lambda = -\mu^2$ with $\mu > 0$. It follows that

$$X(x) = A\cos\mu x + B\sin\mu x \quad \text{and} \quad T(t) = Ce^{-\mu^2 kt}.$$

Hence

$$u(x,t) = e^{-\mu^2 kt}[A\cos\mu x + B\sin\mu x].$$

The boundary conditions (6.7) now imply that $A = 0$. Also $B = 0$ unless $\mu = n\pi/\ell,\ n = 1, 2, 3, \ldots$. Thus

$$u_n(x,t) = e^{-n^2(\pi/\ell)^2 kt}\sin\frac{n\pi x}{\ell}, \quad n = 1, 2, 3, \ldots$$

are the nontrivial solutions.

The general solution is obtained by taking an infinite linear combination of these solutions:

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2(\pi/\ell)^2 kt}\sin\frac{n\pi x}{\ell}. \tag{6.8}$$

The coefficients b_n remain to be found. For this we finally make use of the initial condition (6.6) which can be written as

$$u(x,0) = u_0(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell}, \quad 0 < x < \ell.$$

It is natural to let the rhs be the Fourier sine series of $u_0(x)$ over the interval $(0,\ell)$. Hence the coefficients are

$$b_n = \frac{2}{\ell}\int_0^{\ell} u_0(x)\sin\frac{n\pi x}{\ell}\,dx. \tag{6.9}$$

Thus (6.8) with b_n defined by (6.9) is the unique solution to the heat equation with the Dirichlet boundary conditions. As t increases, the temperature of the rod rapidly approaches 0 everywhere.

6.3.2. Neumann boundary conditions. The Neumann boundary conditions are

$$u_x(0,t) = 0 = u_x(\ell,t). \tag{6.10}$$

In other words, there is no heat loss at the endpoints. Thus the rod is completely isolated from the surroundings.

As in the Dirichlet case, we need to consider three cases:

(1) $\lambda > 0$: Write $\lambda = \mu^2$ with $\mu > 0$. Then
$$X(x) = Ae^{\mu x} + Be^{-\mu x} \quad \text{and} \quad T(t) = Ce^{\mu^2 kt}.$$
The boundary conditions (6.10) imply that $A = 0 = B$. So there is no nontrivial solution of this form.

(2) $\lambda = 0$: In this case we have $X(x) = Ax + B$ and $T(t) = C$. Hence
$$u(x,t) = Ax + B.$$
The boundary conditions (6.10) give $A = 0$. Hence this case contributes the solution $u(x,t) = \text{constant}$.

(3) $\lambda < 0$: Write $\lambda = -\mu^2$ with $\mu > 0$. It follows that
$$u(x,t) = e^{-\mu^2 kt}[A\cos\mu x + B\sin\mu x].$$
The boundary conditions (6.10) now imply that $B = 0$. Also $A = 0$ unless $\mu = n\pi/\ell,\ n = 1, 2, 3, \ldots$. Thus
$$u_n(x,t) = e^{-n^2(\pi/\ell)^2 kt}\cos\frac{n\pi x}{\ell}, \quad n = 1, 2, 3, \ldots$$
are the nontrivial solutions.

The general solution will now be of the form

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-n^2(\pi/\ell)^2 kt}\cos\frac{n\pi x}{\ell}. \tag{6.11}$$

The coefficients a_n remain to be determined. For this we finally make use of the initial condition (6.6) which can be written as

$$u(x,0) = u_0(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\ell}, \quad 0 < x < \ell.$$

We decide that the rhs must be the Fourier cosine series of $u_0(x)$ over the interval $(0,\ell)$ and calculate the coefficients accordingly:

$$a_0 = \frac{1}{\ell}\int_0^\ell u_0(x)dx, \quad a_n = \frac{2}{\ell}\int_0^\ell u_0(x)\cos\frac{n\pi x}{\ell}dx. \tag{6.12}$$

Thus (6.11) with a_n as in (6.12) is the unique solution to the heat equation with the Neumann boundary conditions.

Remark 6.1. In the solution, all terms except for the first one tend rapidly to zero as $t \to \infty$. So one is left with a_0, which is the mean or average value of u_0. Physically, this means that an isolated rod will eventually assume a constant temperature, which is the mean of the initial temperature distribution.

6.3.3. Mixed boundary conditions. Consider the mixed boundary conditions

$$u(0,t) = 0 = u_x(\ell,t). \tag{6.13}$$

Thus, the left endpoint is maintained at temperature 0 (so there will be heat loss from that end), while there is no heat loss at the right endpoint.

We proceed as before and using the boundary conditions (6.13) conclude that there is no contributions from cases (1) and (2). Case (3) forces

$$X(x) = \sin\frac{(n+1/2)\pi x}{\ell}, \quad n \geq 0 \quad \text{and} \quad T(t) = e^{-(n+1/2)^2(\pi/\ell)^2 kt}.$$

The general solution then is a *sine quarter-wave*:

$$u(x,t) = \sum_{n\geq 0} b_n e^{-(n+1/2)^2(\pi/\ell)^2 kt} \sin\frac{(n+1/2)\pi x}{\ell}.$$

The coefficients are computed using the initial condition (6.6) and are given by

$$b_n = \frac{2}{\ell}\int_0^\ell u_0(x)\sin\frac{(n+1/2)\pi x}{\ell}dx.$$

If the mixed boundary conditions are instead

$$u_x(0,t) = 0 = u(\ell,t),$$

then we need the *cosine quarter-wave expansion* on $(0,\ell)$, namely

$$\sum_{n\geq 0} a_n \cos\frac{(n+1/2)\pi x}{\ell},$$

where

$$a_n = \frac{2}{\ell}\int_0^\ell u_0(x)\cos\frac{(n+1/2)\pi x}{\ell}dx.$$

6.3.4. Periodic boundary conditions. Consider the periodic boundary conditions

$$u(0,t) = u(\ell,t), \quad u_x(0,t) = u_x(\ell,t). \tag{6.14}$$

We proceed as before. Yet again there is no contribution from case (1). Case (2) yields constant solutions and case (3) yields (after a small calculation) a nontrivial solution for each $\mu = 2n\pi/\ell$:

$$u_n(x,t) = e^{-4n^2(\pi/\ell)^2 kt}\left[A\cos\frac{2n\pi x}{\ell} + B\sin\frac{2n\pi x}{\ell}\right], \quad n = 1,2,3,\ldots.$$

The general solution then is

$$u(x,t) = a_0 + \sum_{n\geq 1} e^{-4n^2(\pi/\ell)^2 kt}\left[a_n\cos\frac{2n\pi x}{\ell} + b_n\sin\frac{2n\pi x}{\ell}\right].$$

The coefficients are determined from the initial condition (6.6) using the *full* Fourier expansion on $(0,\ell)$, or equivalently on $(-\ell/2,\ell/2)$:

$$a_0 = \frac{1}{\ell}\int_0^\ell u_0(x)dx, \quad a_n = \frac{2}{\ell}\int_0^\ell u_0(x)\cos\frac{2n\pi x}{\ell}dx$$

and

$$b_n = \frac{2}{\ell}\int_0^\ell u_0(x)\sin\frac{2n\pi x}{\ell}dx.$$

6.4. Nonhomogeneous case

We now consider the nonhomogeneous one-dimensional heat equation

$$u_t - ku_{xx} = f(x,t), \quad 0 < x < \ell,\ t > 0, \tag{6.15}$$

with initial condition (6.6). We indicate how to solve this for each of the boundary conditions discussed earlier. For simplicity, we assume $\ell = 1$.

6.4.1. Dirichlet boundary conditions.

Due to (homogeneous) Dirichlet boundary conditions, we expand everything in a Fourier sine series over $(0,1)$. Thus let

$$u(x,t) = \sum_{n\geq 1} Y_n(t) \sin n\pi x,$$

$$f(x,t) = \sum_{n\geq 1} B_n(t) \sin n\pi x, \qquad u_0(x) = \sum_{n\geq 1} b_n \sin n\pi x,$$

where the $B_n(t)$ and the b_n are known while the $Y_n(t)$ are to be determined. Substituting, we obtain

$$\sum_{n\geq 1} [\dot{Y}_n(t) + kn^2\pi^2 Y_n(t)] \sin n\pi x = \sum_{n\geq 1} B_n(t) \sin n\pi x.$$

This implies that $Y_n(t)$ solves the following IVP

$$\dot{Y}_n(t) + kn^2\pi^2 Y_n(t) = B_n(t), \quad Y_n(0) = b_n.$$

To do a concrete example, suppose

$$f(x,t) = \sin \pi x \sin \pi t \quad \text{and} \quad u_0(x) = 0.$$

Thus, the initial temperature of the rod is 0 everywhere but there is a heat source. (Due to the $\sin \pi t$ term, it a sink as well depending on the value of t.) Then $b_n = 0$ for all $n \geq 1$, and

$$B_n(t) = \begin{cases} 0 & \text{for } n \neq 1, \\ \sin \pi t & \text{for } n = 1. \end{cases}$$

Therefore, for $n \neq 1$,

$$\dot{Y}_n(t) + kn^2\pi^2 Y_n(t) = 0, \quad Y_n(0) = 0$$

which implies $Y_n \equiv 0$. For $n = 1$,

$$\dot{Y}_1(t) + k\pi^2 Y_1(t) = \sin \pi t, \quad Y_1(0) = 0.$$

By the method of *undetermined coefficients*,

$$Y_1(t) = Ce^{-\pi^2 kt} + A \cos \pi t + B \sin \pi t.$$

The initial condition $Y_1(0) = 0$ implies $C + A = 0$. Substituting this back into the ODE yields

$$k\pi^2 B - \pi A = 1 \quad \text{and} \quad k\pi^2 A + \pi B = 0.$$

Solving for A and B, we get

$$u(x,t) = \frac{1}{\pi(k^2\pi^2 + 1)} \left[e^{-\pi^2 kt} - \cos \pi t + k\pi \sin \pi t \right] \sin \pi x.$$

6.4.2. Neumann boundary conditions. Due to (homogeneous) Neumann boundary conditions, we expand everything in a cosine series over $(0, 1)$. Thus let

$$u(x,t) = \sum_{n \geq 0} Y_n(t) \cos n\pi x,$$

$$f(x,t) = \sum_{n \geq 0} A_n(t) \cos n\pi x, \qquad u_0(x) = \sum_{n \geq 0} a_n \cos n\pi x,$$

where the $A_n(t)$ and the a_n are known while the $Y_n(t)$ are to be determined. Substituting, we obtain

$$\sum_{n \geq 0} [\dot{Y}_n + kn^2\pi^2 Y_n] \cos n\pi x = \sum_{n \geq 0} A_n(t) \cos n\pi x.$$

This implies that Y_n solves the following linear IVP

$$\dot{Y}_n(t) + kn^2\pi^2 Y_n(t) = A_n(t), \quad Y_n(0) = a_n.$$

The case $n = 0$ is included.

To do a concrete example, suppose

$$f(x,t) = x(x-1) \quad \text{and} \quad u_0(x) = 0.$$

Thus, the initial temperature of the rod is 0 everywhere but there is a heat sink. Due to Neumann conditions, the rod is isolated, so we expect the temperature to decrease with time. This is verified by the precise calculations below.

Note that $a_n = 0$, and since $f(x,t)$ is independent of t, the A_n are constants (with no dependence on t). Explicitly, $A_0 = -1/6$ and for $n \geq 1$,

$$A_n = 2\int_0^1 x(x-1) \cos n\pi x dx = \begin{cases} \frac{4}{n^2\pi^2} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

This check is left to the tired reader. We now need to solve

$$\dot{Y}_n(t) + kn^2\pi^2 Y_n(t) = A_n, \quad Y_n(0) = 0.$$

For $n = 0$, $\dot{Y}_0(t) = A_0$ and $Y_0(0) = 0$, so $\dot{Y}_0(t) = -\frac{t}{6}$. By the method of undetermined coefficients,

$$Y_n(t) = C_n + D_n e^{-n^2\pi^2 t}, \quad n \geq 1.$$

Solving we get $C_n + D_n = 0$ and $C_n = \frac{A_n}{kn^2\pi^2}$. Thus

$$u(x,t) = -\frac{t}{6} - \frac{1}{4k\pi^4} \sum_{n \geq 1} \frac{1 - e^{-4kn^2\pi^2 t}}{n^4} \cos 2n\pi x.$$

6.4.3. Mixed boundary conditions. In the case when the left endpoint is maintained at 0, while the right endpoint is insulated, we require *sine quarter-wave* expansion on $(0, 1)$. Thus we let

$$u(x,t) = \sum_{n \geq 0} Y_n(t) \sin(n + \frac{1}{2})\pi x,$$

$$f(x,t) = \sum_{n \geq 0} B_n(t) \sin(n + \frac{1}{2})\pi x, \qquad u_0(x) = \sum_{n \geq 0} b_n \sin(n + \frac{1}{2})\pi x.$$

As before, the $B_n(t)$ and the b_n are known quantities, while the $Y_n(t)$ are unique solutions of

$$\dot{Y}_n(t) + k(n + \frac{1}{2})^2\pi^2 Y_n(t) = B_n(t), \quad Y_n(0) = b_n.$$

If instead left endpoint is insulated, while the right endpoint is maintained at 0, then we need *cosine quarter-wave* expansion on $(0,1)$.

6.4.4. Periodic boundary conditions. In this case, we require the full Fourier expansion on $(0,1)$, or equivalently on $(-\frac{1}{2}, \frac{1}{2})$. So let

$$u(x,t) = Y_0(t) + \sum_{n\geq 1}[Y_n(t)\cos 2n\pi x + Z_n(t)\sin 2n\pi x]$$

$$f(x,t) = A_0(t) + \sum_{n\geq 1}[A_n(t)\cos 2n\pi x + B_n(t)\sin 2n\pi x]$$

$$u_0(x) = a_0 + \sum_{n\geq 1}[a_n\cos 2n\pi x + b_n\sin 2n\pi x].$$

These yield the linear first order ODEs

$$\dot{Y}_n(t) + 4kn^2\pi^2 Y_n(t) = A_n(t), \quad Y_n(0) = a_n, \text{ for all } n \geq 0.$$

$$\dot{Z}_n(t) + 4kn^2\pi^2 Z_n(t) = B_n(t), \quad Y_n(0) = b_n, \text{ for all } n \geq 1.$$

Thus $Y_n(t)$, $Z_n(t)$ and hence $u(x,t)$ is uniquely determined.

6.5. Temperature in a circular plate

We now consider the two-dimensional heat equation. The heat equation in a two-dimensional region is given by

$$u_t = k(u_{xx} + u_{yy}), \tag{6.16}$$

assuming that there are no sources or sinks inside the region.

6.5.1. Heat equation in polar coordinates. We solve this equation when the region is the disc of radius R centered at the origin (under suitable initial and boundary conditions). Due to the rotational symmetry, it is convenient to use polar coordinates (r,θ). Further we assume for simplicity that the initial temperature is independent of the angle θ. We write

$$u(r,\theta) = f(r).$$

That is, the temperature at a point at distance r from the origin is $f(r)$. The heat equation in polar coordinates (assuming no dependence on θ) is given by

$$u_t = k(u_{rr} + r^{-1}u_r). \tag{6.17}$$

This can be derived from (6.16) by repeated application of the chain rule.

There are many different boundary conditions one can consider. Let us assume the Dirichlet boundary condition

$$u(R,t) = 0.$$

Thus the temperature on the boundary circle of radius R is maintained at 0 at all times. As in the case of the thin rod, there is heat loss from the boundary, and we expect that the temperature of the disc will go to zero everywhere.

6.5.2. Boundary conditions leading to the Bessel function. We employ the method of separation of variables. Accordingly, let $u(r,t) = X(r)T(t)$. Substituting and separating variables, we obtain

$$\frac{X''(r)}{X(r)} + \frac{1}{r}\frac{X'(r)}{X(r)} = \frac{T'(t)}{kT(t)}.$$

Since lhs is a function of r and rhs that of t only, both must have a common constant value, say λ. Write $\lambda = -\mu^2,\ \mu > 0$. (The reason why this constant cannot be zero or positive is explained in detail in the context of the wave equation. See Section 7.4.) Then the equation in the r variable is

$$r^2X''(r) + rX'(r) + \mu^2r^2X(r) = 0$$

This is the scaled Bessel equation of order 0. The general solution is

$$X(r) = AJ_0(\mu r) + BY_0(\mu r),$$

where J_0 and Y_0 are the Bessel and Neumann functions of order 0. Due to boundedness at $r = 0$, $B = 0$. Moreover, the condition $X(R) = 0$ implies $\mu = \frac{z}{R}$ for $z \in Z^{(0)}$. For these values of μ, we have

$$T(t) = De^{-\mu^2kt}.$$

Therefore, the general solution of the heat equation under Dirichlet boundary condition is

$$u(r,t) = \sum_{z\in Z^{(0)}} c_z e^{-(z/R)^2kt} J_0\left(\frac{zr}{R}\right).$$

6.5.3. Initial condition and Fourier-Bessel series. We now invoke the initial condition to find that

$$f(r) = \sum_{z\in Z^{(0)}} c_z J_0\left(\frac{zr}{R}\right)$$

is the Fourier-Bessel expansion of $f(r)$ over the interval $[0,R]$ in terms of J_0. This allows us to compute the coefficients

$$c_z = \frac{2}{R^2J_1(z)^2}\int_0^R f(r)J_0\left(\frac{zr}{R}\right) rdr = \frac{2}{z^2J_1(z)^2}\int_0^z f\left(\frac{Rt}{z}\right) J_0(t).tdt.$$

Example 6.2. For $f(r) = 100(1 - r^2/R^2)$, if the temperature along the circumference of the disc is suddenly raised to 100 and maintained at that value, then find the temperature in the disc subsequently.

Let the temperature distribution be

$$U(r,t) = 100u(r,t) + 100.$$

Then u must solve the equations

$$\begin{aligned} u_t = k(u_{rr} + r^{-1}u_r), &\qquad \text{(Heat equation)},\\ u(R,t) = 0 &\qquad \text{(Homogeneous boundary condition)},\\ u(r,0) = -r^2/R^2 &\qquad \text{(Initial condition)}. \end{aligned}$$

From the above analysis, it only remains to find $c_z,\ z \in Z^{(0)}$. We know that

$$\begin{aligned} \int_0^z t^2J_0(t).tdt = z^2.zJ_1(z) - \int_0^z 2t.tJ_1(t)dt &= z^3J_1(z) - 2z^2J_2(z)\\ &= z^3J_1(z) - 2z^2\frac{2}{z}J_1(z) = (z^3 - 4z)J_1(z). \end{aligned}$$

On substitution,

$$c_z = \frac{2}{z^2 J_1(z)^2} \int_0^z (-t^2/z^2) J_0(t).tdt = \frac{8-2z^2}{z^3 J_1(z)}, \quad z \in Z^{(0)}.$$

Thus

$$U(r,t) = 100 + 100 \sum_{z \in Z^{(0)}} \frac{8-2z^2}{z^3 J_1(z)} e^{-z^2 k R^{-2} t} J_0\left(\frac{zr}{R}\right).$$

6.6. Heat equation in general. Laplacian

One can consider the two-dimensional heat equation on other regions in $\mathbb{R}^2$, or even on other surfaces in $\mathbb{R}^3$. In this case, the heat equation is of the form

$$u_t = k(\Delta(u)),$$

where Δ is the Laplacian operator. The Laplacian on some important domains in special coordinate systems are discussed below. In each setting, the method of separation of variables can be employed to decribe temperature evolution in that domain.

The abstract setting to define a Laplacian is a Riemannian manifold.

6.6.1. Laplacian on the plane. The Laplacian on the plane is

$$\Delta(u) = u_{xx} + u_{yy}. \tag{6.18}$$

In polar coordinates, it is given by

$$\Delta(u) = u_{rr} + r^{-1} u_r + r^{-2} u_{\theta\theta}. \tag{6.19}$$

This is convenient if there is a rotational symmetry in the problem. In the example of the circular plate, we had assumed θ independence, so the last term had dropped out simplifying the Laplacian further.

The expression (6.19) can be derived from (6.18) by repeated use of the chain rule. It has a radial part and an angular part. The angular part is entirely analogous to a term such as u_{ww} with $w = r\theta$. Since r is fixed while moving on the angular part, this equals $r^{-2}u_{\theta\theta}$. In the radial part, we always get the u_{rr} term. In addition, we get a u_r term whose coefficient is given by

$$\frac{d}{dr}\log(2\pi r) = \frac{2\pi}{2\pi r} = \frac{1}{r}.$$

(Note that $2\pi r$ is the length of the circle of radius r.)

6.6.2. Laplacian on the sphere. The Laplacian on the sphere of radius R in terms of spherical polar coordinates is

$$\Delta = \frac{1}{R^2}\left[\frac{\partial^2}{\partial \varphi^2} + \cot\varphi \frac{\partial}{\partial \varphi} + \frac{1}{\sin^2\varphi}\frac{\partial^2}{\partial \theta^2}\right], \tag{6.20}$$

where θ is the azimuthal angle and φ is the polar angle.

This formula can be understood in a similar manner to the polar coordinates formula above. Consider geodesic circles from the north pole. These are precisely the latitudes. Consider the latitude corresponding to φ. Its geodesic radius is along the longitude and given by $R\varphi$. Further the geodesic circle is a usual circle of radius $R\sin\varphi$.

The Laplacian can be broken into two parts. The azimuthal part given by $r^{-2}u_{\theta\theta}$ where $r = R\sin\varphi$. The radial or polar part we always get the $R^{-2}u_{\varphi\varphi}$, and the coefficient of $R^{-1}u_{\varphi}$ is

$$R^{-1}\frac{d}{d\varphi}\log(2\pi R\sin\varphi) = R^{-1}\cot\varphi.$$

6.6.3. Laplacian in three-dimensional space. The Laplacian in in three-dimensional space standard coordinates is given by

$$\Delta(u) = u_{xx} + u_{yy} + u_{zz}. \tag{6.21}$$

This is a convenient form to solve the heat equation in a cube. If the region is a solid sphere, then it is convenient to use the Laplacian in spherical polar coordinates. It is given by

$$\Delta_{\mathbb{R}^3} = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left[\frac{\partial^2}{\partial\varphi^2} + \cot\varphi\frac{\partial}{\partial\varphi} + \frac{1}{\sin^2\varphi}\frac{\partial^2}{\partial\theta^2}\right]. \tag{6.22}$$

This can be derived from (6.21) after a long calculation. Alternatively, one can apply the method of geodesic spheres (since we are in three dimensions). A geodesic sphere at distance r is the usual sphere of radius r, whose Laplacian was explained above. This is one of the term. In the radial part, we always have u_{rr}. In addition, we get a u_r term whose coefficient is given by

$$\frac{d}{dr}\log(4\pi r^2) = \frac{8\pi}{4\pi r^2} = \frac{2}{r}.$$

(Note that $4\pi r^2$ is the area of the sphere of radius r.)

The Laplacian in cylindrical coordinates is

$$\Delta(u) = (u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta}) + u_{zz}. \tag{6.23}$$

CHAPTER 7

Wave equation

The wave equation is a PDE in the space variables and one time variable which governs vibrations in solids (under some idealized conditions). The solid could be a string, or a membrane, or an air column, or a mechanical shaft (which vibrates torsionally). We do not derive the wave equation, most books give a derivation. The one-dimensional equation was first considered and solved by D'Alembert in 1747. His solution reveals many interesting phenomena that we associate with waves. We also independently proceed using the method of separation of variables, and solve the wave equation under a variety of boundary conditions (always assuming that the domain is of finite length).

7.1. D'Alembert's solution

The one-dimensional wave equation is defined by

$$u_{tt} = c^2 u_{xx}, \quad c \neq 0.$$

We think of x as the space variable and t as the time variable. To begin with, we do not impose any restriction on x, thus we are looking at the wave equation on the real line.

Imagine an infinite string stretched along the x-axis vibrating in the xy-plane, $u(x,t)$ represents the deflection from the mean position of the particle at position x at time t.

7.1.1. General solution. Let (ξ, η) be another coordinate system. Consider the linear change of variables

$$\xi = x - ct, \qquad \eta = x + ct.$$

(This is guesswork.) The determinant of this linear transformation is nonzero (since $c \neq 0$), so we can pass back and forth between the two coordinate systems. Let us express the wave equation in the (ξ, η)-coordinate system.

By the chain rule,

$$u_x = u_\xi + u_\eta, \qquad u_{xx} = u_{\xi\xi} + u_{\xi\eta} + u_{\eta\xi} + u_{\eta\eta} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}.$$

Similarly,

$$u_t = -cu_\xi + cu_\eta, \qquad u_{tt} = c^2 u_{\xi\xi} - c^2 u_{\xi\eta} - c^2 u_{\eta\xi} + c^2 u_{\eta\eta} = c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}).$$

Substituting in the wave equation,

$$u_{\xi\eta} = \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi}\right) = 0.$$

From here we see that $\frac{\partial u}{\partial \xi}$ is a function of ξ alone, and integrating once more, we see that

$$u(\xi, \eta) = \varphi(\xi) + \psi(\eta),$$

where φ and ψ are arbitrary twice differentiable functions. Plugging back the original variables, we see that

$$u(x,t) = \varphi(x - ct) + \psi(x + ct) \tag{7.1}$$

is a solution of the wave equation.

It is worthwhile to visualize the above solution. First take $\psi = 0$, so that $u(x,t) = \varphi(x - ct)$. Hence $u(x,0) = \varphi(x)$. Observe that the graph of $u(x,t)$ has the same shape as $\varphi(x)$ but it is pushed ct units to the right. Hence the term $\varphi(x - ct)$ describes a wave moving to the right along the x-axis with constant speed c. Similarly, $\psi(x + ct)$ describes a wave moving to the left along the x-axis with constant speed c. And the general solution is the superposition of two such travelling waves.

Imagine $\varphi(x)$ and $\psi(x)$ to be functions of compact support (that is, they are zero outside of a finite interval.) Further suppose that the support of $\varphi(x)$ lies entirely to the left of the support of $\psi(x)$. At $t = 0$, $u(x,0) = \varphi(x) + \psi(x)$, so we see $\varphi(x)$ on the left and $\psi(x)$ on the right. With the passage of time $\varphi(x)$ keeps moving to the right, while $\psi(x)$ keeps moving to the left. For a brief passage of time, the graphs overlap causing interference, and then later again the two break free of each other. Their original shape is restored and they move in opposite directions never to see each other again.

Since the functions only translate without altering their shape, there is no data loss with the passage of time. So the solution can also be run backwards in time without running into any singularity.

This is in sharp constrast to the heat equation. At $t = 0$, the initial temperature can in principle be any square-integrable function. But for any $t > 0$ however small, because of the exponential factor, the temperature distribution becomes smooth, and it rapidly becomes uniform. You start looking like your neighbor and society becomes homogeneous very quickly. In this sense, there is individuality or data loss.

7.1.2. Initial value problem. We now consider the wave equation as before (with x unrestricted) but subject to the initial conditions

$$\begin{aligned} u(x,0) &= f(x) \qquad \text{(Initial Position)} \\ u_t(x,0) &= g(x) \qquad \text{(Initial Velocity).} \end{aligned} \tag{7.2}$$

Imposing these conditions on the general solution (7.1), we obtain

$$f(x) = \varphi(x) + \psi(x) \quad \text{and} \quad g(x) = -c\varphi'(x) + c\psi'(x).$$

Let $G(x)$ be an antiderivative of $g(x)$. Integrating the second equation yields

$$\frac{1}{c}G(x) - K = -\varphi(x) + \psi(x),$$

where K is an arbitrary constant. Solving for φ and ψ,

$$\varphi(x) = \frac{1}{2}\big(f(x) - \frac{G(x)}{c} - K\big) \quad \text{and} \quad \psi(x) = \frac{1}{2}\big(f(x) + \frac{G(x)}{c} + K\big)$$

Substituting in the general solution, we obtain

$$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{G(x+ct) - G(x-ct)}{2c} \tag{7.3}$$

$$= \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds$$

This is known as D'Alembert's formula.

Consider the special case when $g \equiv 0$. (The string is held in some initial position and released.) The solution in this case is

$$u(x,t) = \frac{f(x+ct)+f(x-ct)}{2}. \tag{7.4}$$

Imagine f to be of compact support. Then f breaks into two equal pieces, one piece moves to the right and the other piece moves to the left, both with velocity c after a while the pieces break free of each other never to see each other again. For a concrete example, take

$$f(x) = \frac{1}{1+x^2}.$$

This is not of compact support, but it is concentrated near the origin. The evolution with time is given by

$$u(x,t) = \frac{1}{2}\left(\frac{1}{1+(x+ct)^2} + \frac{1}{1+(x-ct)^2}\right).$$

Visualize this as a function of x evolving in time.

7.1.3. Semi-infinite string with one end fixed. We now consider the wave equation with initial conditions as in (7.2), but x is now restricted to the interval $x > 0$. In particular, the functions f and g are only defined for $x > 0$. We impose the Dirichlet boundary condition $u(0,t) = 0$. This models the semi-infinite string with one end fixed.

The boundary condition implies

$$\varphi(-ct) + \psi(ct) = 0.$$

Hence $\varphi(y) = -\psi(-y)$. So the general solution can be rewritten as

$$u(x,t) = \psi(x+ct) - \psi(ct-x). \tag{7.5}$$

Though x is restricted, the function ψ must be defined for all real numbers since x and t couple with each other and t is unbounded.

The initial conditions yield

$$f(x) = \psi(x) - \psi(-x) \quad \text{and} \quad g(x) = -c\psi'(-x) + c\psi'(x). \tag{7.6}$$

Both the rhs are odd functions, the first is twice the odd part of ψ, while the second is c times the derivative of the twice the odd part of ψ. Note that the functions f and g are only defined for $x > 0$. So extend them as odd functions, so that the above equations are valid for all x.

The general solution to this initial and boundary value problem is given by (7.3) with f and g extended as odd functions. (Check directly that (7.3) does satisfy the boundary condition if f and g are odd.)

Now let us consider the consider the special case when $g \equiv 0$. Recall that the solution in this case is given by (7.4). Imagine f to be a small positive pulse centered around the point $x = A$. The pulse breaks into two equal pulses with half the amplitude. One moves to the right and the other to the left with the same speed. The one moving to the left reaches the endpoint $x = 0$, and gets reflected (in such a manner that the value at 0 is always 0) becoming a negative pulse. It then keeps moving to the right always lagging the other pulse by distance $2A$.

7.1.4. Finite string with both ends fixed. We now consider another boundary value problem: restrict x to the interval $0 < x < \ell$, and impose the Dirichlet boundary conditions

$$u(0,t) = u(\ell,t) = 0. \tag{7.7}$$

Now f and g are only defined on this finite interval.

We proceed as above. Substituting the extra boundary condition $u(\ell,t) = 0$ in (7.5) yields $\psi(ct+\ell) = \psi(ct-\ell)$ showing that ψ in addition must be periodic of period 2ℓ. In order for (7.6) to be valid for all x, we extend f and g as odd functions of period 2ℓ, and the general solution to this initial and boundary value problem is again given by (7.3).

Now let us consider the special case when $g \equiv 0$. Suppose f is a pulse. Then it breaks into two pieces one travelling to the right and the other to the left. Eventually both get reflected (keeping the endpoint values at 0) and this process continues. After two reflections and travelling a distance of 2ℓ we will be back at the starting configuration. Since the speed of travel is c, this happens at time $t = 2\ell/c$. Thus the solution is periodic in t with period $2\ell/c$. This can also be seen directly from the formula (7.4).

For certain f, the two traveling waves can create a very interesting interference in which certain points on the string called nodes never move. This is called a standing wave. For example, let $\ell = \pi$, $c = 1$, and $f = \sin nx$, where n is a fixed positive integer. Then the solution is

$$u(x,t) = \frac{1}{2}\big(\sin(n(x-t)) + \sin(n(x+t))\big) = \sin nx \cos nt.$$

There are $n-1$ points (apart from the two endpoints) which never move.

7.1.5. Semi-infinite or finite string with Neumann boundary conditions. Now consider the same initial value problem for the semi-infinite string but with the Neumann boundary condition $u_x(0,t) = 0$. The boundary condition implies $\varphi'(-ct) + \psi'(ct) = 0$. Hence $\varphi(y) = \psi(-y) + k$, where k is a constant. So the general solution can be rewritten as

$$u(x,t) = \psi(x+ct) + \psi(ct-x) + k.$$

The initial conditions yield

$$f(x) = \psi(x) + \psi(-x) + k \quad \text{and} \quad g(x) = c(\psi'(-x) + \psi'(x)).$$

Both the rhs are even functions. The general solution to this initial and boundary value problem is given by (7.3) with f and g extended as even functions.

In the special case when $g \equiv 0$, and f is a small positive pulse, everything works as before except the way in which the pulse gets reflected. The reflected wave now stays positive.

One can also do a similar analysis for the finite string.

There are nice animations on the internet of waves reflecting by odd extensions or by even extensions.

7.1.6. Inhomogeneous wave equation. If the equation has an external source term $s(x,t)$, we can still apply D'Alembert's method. Consider

$$u_{tt} - c^2 u_{xx} = s(x,t),$$

with *trivial* initial conditions:

$$u(x,0) = 0 = u_t(x,0).$$

The solution is given by

$$u(x,t) = \frac{1}{2c} \int \int_{\Delta} s(\zeta,\tau) d\zeta d\tau,$$

where Δ is the *region of influence* at (x,t) for the time interval $[0,t]$ which is defined as the triangle PQR with vertices $P(x,t)$, $Q(x-ct,0)$ and $R(x+ct,0)$. In other words,

$$u(x,t) = \frac{1}{2c} \int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} s(\zeta,\tau) d\zeta d\tau.$$

The above result can be verified directly or more naturally by integrating the wave equation itself over Δ and applying Green's theorem to the lhs. This gives $2cu(x,t)$ on the lhs and validates the *causality principle* at the same time.

Finally by combining the homogeneous case with arbitrary initial conditions and the inhomogeneous case with trivial initial conditions, we can write the general solution as

$$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c}\Big[\int_{x-ct}^{x+ct} g(s) ds + \int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} s(\zeta,\tau) d\zeta d\tau \Big].$$

7.2. Solution by separation of variables

We now solve the one-dimensional wave equation

$$u_{tt} - c^2 u_{xx} = 0, \quad 0 < x < \ell,\ t > 0. \tag{7.8}$$

The initial conditions are

$$u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x). \tag{7.9}$$

Adopting the method of separation of variables, let $u(x,t) = X(x)T(t)$ be as in (6.1). Substitution in (7.8) does indeed separate the variables:

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{c^2 T(t)} = \lambda \text{ (say)}.$$

The equality is between a function of x and a function of t. So both must be constant. We denote this constant by λ.

In addition to the initial condition, there are conditions specified at the two endpoints of the rod. These are the boundary conditions.

7.2.1. Dirichlet boundary conditions. The Dirichlet boundary conditions require

$$u(0,t) = u(\ell,t) = 0. \tag{7.10}$$

This is the problem of the vibrating string stretched between the points 0 and ℓ of the x-axis, $u_0(x)$ describes the initial position of the string and $u_1(x)$ describes the initial velocity. The constant c^2 is the ratio T/ρ, where T is the tension in the string and ρ is the linear density. In general, ρ will depend on x. We assume for simplicity that the mass is uniformly distributed and ρ is a constant. Note that c has the physical units of velocity.

Another physical interpretation is of a shaft vibrating torsionally, $u(x,t)$ is the angle through which the cross-section at x has rotated at time t from its equilibrium position. Also note that in this setting $u_t(x,t)$ represents angular velocity. The boundary conditions say that the two ends of the shaft are clamped and cannot rotate. In this situation, the constant c^2 is given in terms of physical constants such as modulus of elasticity, etc [14].

The boundary conditions can be rewritten as

$$X(0) = X(\ell) = 0.$$

As in the case of the heat equation we consider three cases.

(1) $\lambda > 0$: Write $\lambda = \mu^2$ with $\mu > 0$. Then

$$X(x) = Ae^{\mu x} + Be^{-\mu x}.$$

The boundary conditions imply that $A = 0 = B$. So there is no nontrivial solution of this form.

(2) $\lambda = 0$: In this case we have $X(x) = Ax + B$. Again the boundary conditions give $A = 0 = B$. Thus this case also does not yield a nontrivial solution.

(3) $\lambda < 0$: Write $\lambda = -\mu^2$ with $\mu > 0$. It follows that

$$X(x) = A\cos\mu x + B\sin\mu x.$$

The boundary conditions now imply that $A = 0$. Also $B = 0$ unless $\mu = n\pi/\ell,\ n = 1, 2, 3, \dots$. For any such μ,

$$T(t) = C\cos c\mu t + D\sin c\mu t.$$

Thus

$$u_n(x,t) = [C\cos\frac{cn\pi t}{\ell} + D\sin\frac{cn\pi t}{\ell}]\sin\frac{n\pi x}{\ell}, \quad n = 1, 2, 3, \dots.$$

are the nontrivial solutions.

The general solution (without considering the initial conditions) is

$$u(x,t) = \sum_{n\geq 1}[C_n\cos\frac{cn\pi t}{\ell} + D_n\sin\frac{cn\pi t}{\ell}]\sin\frac{n\pi x}{\ell}.$$

Now from the initial conditions,

$$C_n = \frac{2}{\ell}\int_0^\ell u_0(x)\sin\frac{n\pi x}{\ell}dx \quad \text{and} \quad D_n = \frac{2}{cn\pi}\int_0^\ell u_1(x)\sin\frac{n\pi x}{\ell}dx.$$

Thus the Fourier sine series of $u_0(x)$ and $u_1(x)$ enter into the solution.

Suppose $u_1(x) = 0$. In other words, we place the string in position $u_0(x)$ and then let go. In this case, the sine terms in the t variable will be absent. So the solution is

$$u(x,t) = \sum_{n\geq 1} C_n \cos \frac{cn\pi t}{\ell} \sin \frac{n\pi x}{\ell}, \tag{7.11}$$

where C_n are the Fourier sine coefficients of $u_0(x)$. This can be rewritten using the trigonometric addition formulas as

$$\begin{aligned} u(x,t) &= \frac{1}{2} \sum_{n\geq 1} C_n \sin \frac{n\pi}{\ell}(x+ct) + C_n \sin \frac{n\pi}{\ell}(x-ct) \\ &= \frac{u_0(x+ct) + u_0(x-ct)}{2}. \end{aligned}$$

This agrees with D'Alembert's formula (7.4).

Further, if $u_0(x) = \sin \frac{n\pi x}{\ell}$, then the solution will be

$$\cos \frac{cn\pi t}{\ell} \sin \frac{n\pi x}{\ell}.$$

This was the standing wave discussed in D'Alembert's solution (with $\ell = \pi$ and $c = 1$).

Remark 7.1. Bernoulli claimed (correctly) that the most general solution to the wave equation for a string released from rest is given by (7.11). However Euler and D'Alembert (wrongly) argued that this was impossible because a function of the form $x(\ell - x)$ could never be of the form (7.11). See [13, first page of Chapter IX].

7.2.2. Neumann boundary conditions. The Neumann boundary conditions require

$$u_x(0,t) = 0 = u_x(\ell,t). \tag{7.12}$$

These boundary conditions describe a shaft vibrating torsionally in which the ends are held in place by frictionless bearings so that rotation at the ends is permitted but all other motion is prevented. One can imagine a vibrating string free to slide vertically at both ends. Another possibility is that of longitudinal waves in an air column open at both ends.

The nontrivial elementary solutions in this case are

$$u_n(x,t) = [C \cos \frac{cn\pi t}{\ell} + D \sin \frac{cn\pi t}{\ell}] \cos \frac{n\pi x}{\ell}, \quad n = 0, 1, 2, \dots,$$

and the general solution is

$$u(x,t) = C_0 + D_0 t + \sum_{n\geq 1} [C_n \cos \frac{cn\pi t}{\ell} + D_n \sin \frac{cn\pi t}{\ell}] \cos \frac{n\pi x}{\ell}.$$

The initial conditions now imply

$$C_0 = \frac{1}{\ell} \int_0^\ell u_0(x) dx, \qquad C_n = \frac{2}{\ell} \int_0^\ell u_0(x) \cos \frac{n\pi x}{\ell} dx, \quad n \geq 1$$

and

$$D_0 = \frac{1}{\ell} \int_0^\ell u_1(x) dx, \qquad D_n = \frac{2}{cn\pi} \int_0^\ell u_1(x) \cos \frac{n\pi x}{\ell} dx, \quad n \geq 1.$$

Note the presence of the linear term $D_0 t$. It says that the whole wave or the vibrating interval drifts in the direction of u-axis at a uniform rate. Imposing the condition $\int_0^1 u_1(x)dx = 0$ can prevent this drift. At the other extreme, if $u_0(x) = 0$ and $u_1(x)$ is a constant, then $u(x,t) = D_0 t$, in the shaft example, the shaft will rotate with uniform angular velocity D_0, there will be no vibrational motion.

Work out an example.

7.2.3. Mixed boundary conditions. The mixed boundary conditions require

$$u(0,t) = 0 = u_x(\ell, t). \tag{7.13}$$

Again similar to as in the heat equation case, the general solution is of the form

$$u(x,t) = \sum_{n\geq 0} [C_n \cos c(n+1/2)\pi t + D_n \sin c(n+1/2)\pi t] \sin(n+1/2)\pi x.$$

Invoking the initial conditions now, we are lead to

$$C_n = \frac{2}{\ell}\int_0^\ell u_0(x)\sin(n+1/2)\frac{\pi}{\ell}x dx$$

and

$$D_n = \frac{2}{c(n+1/2)\pi}\int_0^\ell u_1(x)\sin(n+1/2)\frac{\pi}{\ell}x dx.$$

7.2.4. Periodic boundary conditions. The periodic boundary conditions require

$$u(0,t) = u(\ell,t) \quad \text{and} \quad u_x(0,t) = u_x(\ell,t). \tag{7.14}$$

The general solution is

$$u(x,t) = C_0 + D_0 t + \sum_{n\geq 1}[C_n \cos 2\frac{cn\pi t}{\ell} + D_n \sin 2\frac{cn\pi t}{\ell}][A_n \cos\frac{2n\pi x}{\ell} + B_n \sin\frac{2n\pi x}{\ell}].$$

The initial conditions yield

$$C_0 = \frac{1}{\ell}\int_0^\ell u_0(x)dx,$$

$$C_n A_n = \frac{2}{\ell}\int_0^\ell u_0(x)\cos\frac{2n\pi x}{\ell}dx, \quad \text{and} \quad C_n B_n = \frac{2}{\ell}\int_0^\ell u_0(x)\sin\frac{2n\pi x}{\ell}dx.$$

And also

$$D_0 = \frac{1}{\ell}\int_0^\ell u_1(x)dx,$$

and

$$D_n A_n = \frac{1}{cn\pi}\int_0^\ell u_1(x)\cos\frac{2n\pi x}{\ell}dx, \quad \text{and} \quad D_n B_n = \frac{1}{cn\pi}\int_0^\ell u_1(x)\sin\frac{2n\pi x}{\ell}dx.$$

Note that A_n, B_n, C_n and D_n are not uniquely defined, but the above products are. Note again the presence of the term $D_0 t$ in the solution.

7.3. Nonhomogeneous case

We now consider the nonhomogeneous one-dimensional wave equation

$$u_{tt} - u_{xx} = f(x,t), \quad 0 < x < 1,\ t > 0 \tag{7.15}$$

with initial conditions (7.9). For simplicity, we are taking $c = \ell = 1$. We indicate how to solve this for each of the boundary conditions discussed earlier.

7.3.1. Dirichlet boundary conditions. Due to the homogeneous Dirichlet boundary conditions, we expand in the sine series on $(0,1)$. So let

$$f(x,t) = \sum_{n\geq 1} B_n(t) \sin n\pi x \quad \text{and} \quad u(x,t) = \sum_{n\geq 1} Y_n(t) \sin n\pi x.$$

Then the functions $Y_n(t)$ must satisfy

$$\ddot{Y}_n(t) + n^2\pi^2 Y_n(t) = B_n(t), \quad n = 1, 2, 3, \ldots.$$

Also let

$$u_0(x) = \sum_{n\geq 1} b_n \sin n\pi x \quad \text{and} \quad u_1(x) = \sum_{n\geq 1} b_{1n} \sin n\pi x.$$

These lead to the initial conditions

$$Y_n(0) = b_n \quad \text{and} \quad \dot{Y}_n(0) = b_{1n}.$$

They determine the $Y_n(t)$ uniquely.

To do a concrete example, suppose

$$f(x,t) = \sin \pi x \sin \pi t \quad \text{and} \quad u_0(x) = u_1(x) = 0.$$

This problem has homogeneous Dirichlet boundary conditions and zero initial conditions. Thus $b_n = 0 = b_{1n}$ for all $n \geq 1$, and

$$B_n(t) = \begin{cases} 0 & \text{for } n \neq 1, \\ \sin \pi t & \text{for } n = 1. \end{cases}$$

Therefore we have $Y_n(t) = 0$ for $n \geq 2$ while $Y_1(t)$ is the solution to the IVP

$$\ddot{Y}_1(t) + \pi^2 Y_1(t) = \sin \pi t, \quad Y_1(0) = 0 = \dot{Y}_1(0).$$

Solving by the method of undetermined coefficients, we find $Y_1(t) = \frac{\sin \pi t - \pi t \cos \pi t}{2\pi^2}$ and hence

$$u(x,t) = \frac{(\sin \pi t - \pi t \cos \pi t)}{2\pi^2} \sin \pi x.$$

7.3.2. Neumann boundary conditions. For the homogeneous Neumann boundary conditions, we expand in the cosine series on $(0,1)$. So let

$$f(x,t) = A_0(t) + \sum_{n\geq 1} A_n(t) \cos n\pi x \quad \text{and} \quad u(x,t) = Y_0(t) + \sum_{n\geq 1} Y_n(t) \cos n\pi x$$

Then the functions Y_n must satisfy

$$\ddot{Y}_n(t) + n^2\pi^2 Y_n(t) = A_n(t), \quad n = 0, 1, 2, \ldots.$$

Also let

$$u_0(x) = a_0 + \sum_{n\geq 1} a_n \cos n\pi x \quad \text{and} \quad u_1(x) = a_{10} + \sum_{n\geq 1} a_{1n} \cos n\pi x.$$

These lead to the initial conditions

$$Y_n(0) = a_n \quad \text{and} \quad \dot{Y}_n(0) = a_{1n}$$

They determine the $Y_n(t)$ uniquely.

To do a concrete example, suppose

$$f(x,t) = x(x-1) \quad \text{and} \quad u_0(x) = u_1(x) = 0.$$

Note that $a_n = 0$, and since $f(x,t)$ is independent of t, the A_n are constants (with no dependence on t). Explicitly, $A_0 = -1/6$ and for $n \geq 1$,

$$A_n = 2\int_0^1 x(x-1)\cos n\pi x dx = \begin{cases} \frac{4}{n^2\pi^2} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

This same calculation was done for the heat equation. This easily gives

$$Y_0(t) = -t^2/12 \quad \text{and} \quad Y_n(t) = 0 \ \text{ for } n \text{ odd.}$$

Let $n = 2k$ with $k \geq 1$ be even. Then we need to solve

$$\ddot{Y}_{2k}(t) + 4k^2\pi^2 Y_{2k}(t) = \frac{1}{k^2\pi^2}$$

with vanishing initial conditions. Again by the method of undetermined coefficients, $A_{2k} = \frac{\sin^2 k\pi t}{2k^4\pi^4}$. Thus

$$u(x,t) = -\frac{t^2}{12} + \frac{1}{2\pi^4}\sum_{k\geq 1}\frac{\sin^2 k\pi t}{k^4}\cos 2k\pi x.$$

7.3.3. Mixed boundary conditions. For the homogeneous mixed boundary conditions $u(0,t) = 0 = u_x(1,t)$, we expand in the quarter-wave sine series on $(0,1)$. So let

$$f(x,t) = \sum_{n\geq 0} B_n(t)\sin(n+\frac{1}{2})\pi x \quad \text{and} \quad u(x,t) = \sum_{n\geq 0} Y_n(t)\sin(n+\frac{1}{2})\pi x.$$

Then the functions Y_n must satisfy

$$\ddot{Y}_n(t) + (n+\frac{1}{2})^2\pi^2 Y_n(t) = B_n(t), \quad n = 0,1,2,\ldots.$$

Also let

$$u_0(x) = \sum_{n\geq 0} b_n \sin(n+\frac{1}{2})\pi x \quad \text{and} \quad u_1(x) = \sum_{n\geq 0} b_{1n}\sin(n+\frac{1}{2})\pi x.$$

These lead to the initial conditions

$$Y_n(0) = b_n \quad \text{and} \quad \dot{Y}_n(0) = b_{1n}.$$

They determine the $Y_n(t)$ uniquely.

7.3.4. Periodic boundary conditions. For the periodic boundary conditions, we write the full Fourier series on $(0,1)$, or equivalently on $(-\frac{1}{2},\frac{1}{2})$. Thus $\ell = \frac{1}{2}$. So let

$$f(x,t) = A_0(t) + \sum_{n\geq 1}[A_n(t)\cos 2n\pi x + B_n(t)\sin 2n\pi x]$$

and

$$u(x,t) = Y_0(t) + \sum_{n\geq 1}[Y_n(t)\cos 2\pi x + Z_n(t)\sin 2n\pi x].$$

Also let

$$u_0(x) = a_0 + \sum_{n\geq 1}[a_n\cos 2n\pi x + b_n\sin 2n\pi x]$$

and

$$u_1(x) = a_{10} + \sum_{n\geq 1}[a_{1n}\cos 2n\pi x + b_{1n}\sin 2n\pi x].$$

Then the functions Y_n, Z_n must satisfy

$$\ddot{Y}_n(t) + 4n^2\pi^2 Y_n(t) = A_n(t), \quad Y_n(0) = a_n,\ \dot{Y}_n(0) = a_{1n},\ n \geq 0,$$

and

$$\ddot{Z}_n(t) + 4n^2\pi^2 Z_n(t) = B_n(t), \quad Z_n(0) = b_n,\ \dot{Z}_n(0) = b_{1n},\ n \geq 1.$$

They determine the $Y_n(t)$ and $Z_n(t)$ uniquely.

7.4. Vibrations of a circular membrane

In the one-dimensional wave equation, the method of separation of variables resulted in two identical ODEs, one in the space variable x and one in the time variable t. This is the ODE of the simple harmonic oscillator whose solution is a linear combination of sine and cosine.

Now we look at the two-dimensional wave equation. The term u_{xx} must now be replaced by $u_{xx} + u_{yy}$. This is the two-dimensional Laplacian. The method of separation of variables now proceeds in two steps: we have a total of 3 variables and we need to separate one variable at a time.

The best way to proceed in such situations is determined by the shape of the domain.

Consider a circular membrane of radius R. The wave equation written in polar coordinates (using (6.19)) is

$$u_{tt} = c^2(u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta}) \tag{7.16}$$

in the domain

$$(r, \theta, t) \in [0, R] \times [0, 2\pi] \times \mathbb{R}.$$

The initial conditions are

$$u(r, \theta, 0) = f(r, \theta) \quad \text{and} \quad u_t(r, \theta, 0) = g(r, \theta). \tag{7.17}$$

We assume Dirichlet boundary conditions

$$u(R, t) = 0. \tag{7.18}$$

Physically u represents the displacement of the point (x, y) at time t in the z-direction. These are transverse vibrations.

7.4.1. Radially symmetric solutions.

We first find solutions which are radially symmetric. This will happen whenever the initial conditions have radial symmetry, that is, f and g are functions of r alone. Write $u = u(r, t)$. We apply the method of separation of variables. Accordingly, let $u(r, t) = X(r)Z(t)$. Substituting in the simplified wave equation

$$u_{tt} = c^2(u_{rr} + r^{-1}u_r)$$

and separating variables, we obtain

$$\frac{Z''(t)}{c^2 Z(t)} = \frac{X''(r) + r^{-1}X'(r)}{X(r)} = \lambda.$$

Let the constant $\lambda = -\mu^2$. The explanation why this must be strictly negative is given below. Then the equation in the r variable is

$$r^2 X''(r) + rX'(r) + \mu^2 r^2 X(r) = 0$$

This is the scaled Bessel equation of order 0. This implies that $X(r)$ is a scalar multiple of the scaled Bessel function of the first kind $J_0(\mu r)$. Any solution involving $Y_0(\mu r)$ has to be discarded since it is unbounded at $r = 0$, but we require $u(0, t)$

to be finite. The Dirichlet boundary condition now implies $J_0(\mu R) = 0$. So μ must be R^{-1} times one of the countably many positive zeroes of J_0. These are the fundamental modes of vibration of the membrane. For any such μ (there are countably many of them), $Z(t)$ is given by

$$Z(t) = A\cos c\mu t + B\sin c\mu t.$$

Thus the elementary solutions are

$$u_n(r,t) = (A_n \cos c\mu_n t + B_n \sin c\mu_n t)J_0(\mu_n r),$$

where μ_n is the n-th positive zero of J_0. The general solution is given by

$$u(r,t) = \sum_{n\geq 1}(A_n \cos c\mu_n t + B_n \sin c\mu_n t)J_0(\mu_n r). \tag{7.19}$$

Setting $t = 0$ we obtain,

$$f(r) = u(r,0) = \sum_{n\geq 1} A_n J_0(\mu_n r).$$

This is the Fourier-Bessel series of $f(r)$. Explicitly, the coefficient A_n is given by

$$A_n = \frac{2}{R^2 J_1^2(\mu_n R)} \int_0^R r f(r) J_0(\mu_n r) dr. \tag{7.20}$$

Differentiating (7.19) wrt t termwise and setting $t = 0$, we obtain

$$g(r) = u_t(r,0) = \sum_{n\geq 1} c\mu_n B_n J_0(\mu_n r).$$

This is the Fourier-Bessel series of $g(r)$. The coefficient $c\mu_n B_n$ is determined by the above formula with g instead of f. Dividing by $c\mu_n$,

$$B_n = \frac{2}{c\mu_n R^2 J_1^2(\mu_n R)} \int_0^R r g(r) J_0(\mu_n r) dr. \tag{7.21}$$

[Remember: J_i^2 must be evaluated at a zero of J_0. So the argument $\mu_n R$ is correct.] Thus, the solution to (7.16) under radially symmetric initial conditions is given by (7.19) with the coefficients given by (7.20) and (7.21).

The reasons for not allowing λ to be positive or zero are given below.

- Suppose the constant $\lambda = \mu^2 > 0$. Then we get the Bessel equation scaled by the imaginary number $i\mu$. The general solution is $AJ_0(i\mu r) + BY_0(i\mu r)$. Since u must be bounded as $r \to 0$, B must vanish. Further, since

$$J_0(i\mu r) = \sum_{k\geq 0} \frac{(\mu r)^{2k}}{4^k (k!)^2}$$

  is a series of positive terms only, it cannot satisfy $X(R) = 0$. Hence $A = 0$ too. (J_0 is an entire function on the complex plane. On the real axis, it behaves like a damped cosine function and is bounded. On the imaginary axis, it behaves like the exponential function and is bounded. Observe that the above power series is similar to that of e^x. Also by Liouville's theorem, entire functions can never be bounded.)
- Suppose the constant $\lambda = 0$. Then we need to solve $r^2X'' + rX' = 0$. The general solution is $A\log r + B$ and again we get $A = B = 0$.

7.4.2. General solution. We solve for $u(r, \theta, t) = X(r)Y(\theta)Z(t)$. Substituting in the wave equation (7.16) and separating variables, we obtain

$$\frac{Y''(\theta)}{Y(\theta)} = -\frac{r^2 X''(r) + rX'(r)}{X(r)} + \frac{r^2 Z''(t)}{c^2 Z(t)}.$$

Since $Y(\theta)$ as well as its derivative must be 2π-periodic (that is, we have periodic boundary conditions in the θ-variable), the common constant has to be $-n^2$ for $n = 0, 1, 2, 3, \dots$ and correspondingly,

$$Y(\theta) = A\cos n\theta + B\sin n\theta.$$

Now the rhs gives

$$\frac{Z''(t)}{c^2 Z(t)} = \frac{X''(r) + r^{-1}X'(r)}{X(r)} - \frac{n^2}{r^2},$$

with variables again separated. Let the common constant be $\lambda = -\mu^2$. Then the equation in the r variable is

$$r^2 X''(r) + rX'(r) + (\mu^2 r^2 - n^2)X(r) = 0.$$

This is the scaled Bessel equation of order n. This implies that $X(r)$ is a scalar multiple of the scaled Bessel function of the first kind $J_n(\mu r)$ (the other solutions involving the Neumann function Y_n being discarded for the same reason as before). The Dirichlet boundary condition implies that $X(r) = 0$ which means that μR must be a zero of J_n. For any such μ (there are countably many of them), $Z(t)$ is given by

$$Z(t) = C\cos c\mu t + D\sin c\mu t,$$

and we have the elementary solution

$$(A\cos n\theta + B\sin n\theta)(C\cos c\mu t + D\sin c\mu t)J_n(\mu r).$$

The general solution is given by summing these. Note that there are two quantizations here: n and μ. The two initial conditions can be used to determine the products AC, BC, AD and BD.

In any of the $n = 0$ modes, the center of the membrane is vibrating with the maximum amplitude. (This is the radially symmetric case.) In any of the $n \neq 0$ modes, the center of the membrane is a node. Try to visualize the amplitude $\sin\theta J_1(\mu r)$, where μ is the first zero of J_1. There is a nodal line along a diameter, when the membrane on one side of the diameter is up, the membrane on the other side is down.

There is a must-see animation on wikipedia of these initial modes of vibration of a circular membrane.

The reasons for not allowing λ to be positive or zero are similar to the $n = 0$ case explained above.

- Suppose the constant $\lambda = \mu^2 > 0$. Then we get the Bessel equation scaled by the imaginary number $i\mu$. The general solution is $AJ_n(i\mu r) + BY_n(i\mu r)$. Since u must be bounded as $r \to 0$, B must vanish. Further, since $J_n(i\mu r)$ is a series of positive terms only, it cannot satisfy $X(R) = 0$. Hence $A = 0$ too.
- Suppose the constant $\lambda = 0$. Then we need to solve $r^2X'' + rX' - n^2X = 0$. The $n = 0$ case was done earlier. This is a Cauchy-Euler equation, roots of its indicial equation are n and $-n$, so for $n \neq 0$, r^n and r^{-n} are two

independent solutions. The first cannot be zero at $r = R$ while the second has a singularity at $r = 0$.

The frequencies of the vibrations of a drum arise from the zeroes of the Bessel functions J_n. These frequencies are not well-separated. As a result, the sound of a drum is a complicated noise in contrast to the sound of a string instrument. In the latter, there is only one quantization and the frequencies are well-separated.

7.5. Vibrations of a spherical membrane

We now look at another two-dimensional wave equation. We consider the vibrations of a spherical membrane of unit radius. The vibrations are in the radial direction. The term $u_{xx} + u_{yy}$ must now be replaced by the Laplacian of the sphere (6.20). For simplicity, let us take the radius to be 1 and also c to be 1. Thus we need to solve the wave equation

$$u_{tt} = \Delta_{\mathbb{S}^2} u,$$

where

$$\Delta_{\mathbb{S}^2} u = u_{\varphi\varphi} + \cot\varphi u_\varphi + \frac{1}{\sin^2\varphi} u_{\theta\theta}.$$

Terminology: θ is called the azimuthal angle, and φ is called the polar angle. Physically, u denotes the radial displacement from the mean position. There are issues with this interpretation.

A possible physical situation: the sphere is the surface of the earth, and the phenomenon is waves with high tides and low tides.

The sphere $\mathbb{S}^2$ has no boundary, so there are no boundary conditions to consider. Let us first find the pure harmonics and their associated frequencies. (So we do not worry about initial conditions for the moment.) These are solutions of the form

$$u(\theta, \varphi; t) = X(\theta)Y(\varphi)Z(t).$$

As usual on substituting in

$$u_{\varphi\varphi} + \cot\varphi u_\varphi + \frac{1}{\sin^2\varphi} u_{\theta\theta} = u_{tt},$$

we get

$$\frac{X''(\theta)}{X(\theta)} = \sin^2\varphi \frac{Z''(t)}{Z(t)} - \sin^2\varphi \left(\frac{Y''(\varphi)}{Y(\varphi)} + \cot\varphi \frac{Y'(\varphi)}{Y(\varphi)} \right).$$

Since the lhs is depending only on θ while rhs only on (φ, t), so the two sides must be equal to a common constant. Further due to the implicit periodic boundary conditions $X(0) = X(2\pi)$ and $X'(0) = X'(2\pi)$, the common constant must be $-m^2$, $m = 0, 1, 2, \ldots$ and

$$X(\theta) = A_m \cos m\theta + B_m \sin m\theta.$$

Next from the rhs we also have a further separation of variables

$$\frac{Z''(t)}{Z(t)} = \frac{Y''(\varphi)}{Y(\varphi)} + \cot\varphi \frac{Y'(\varphi)}{Y(\varphi)} - \frac{m^2}{1 - \cos^2\varphi} = \mu.$$

Again both sides equal a common constant μ, say.

7.5.1. Latitudinally symmetric solution. Let us first look at the case $m = 0$ in detail. The equation in $Y(\varphi)$ that we obtain is

$$Y''(\varphi) + \cot\varphi Y'(\varphi) - \mu Y(\varphi) = 0.$$

This is the polar form of the Legendre equation. One obtains the standard form by the substitution $x = \cos\varphi$, with $\varphi = \pi$ and $\varphi = 0$ corresponding to $x = -1$ and $x = 1$. (The substitution $x = \cos\varphi$ is natural, x represents the z-axis variable. In other words, the φ-variable travels along the longitude, the x-variable travels along the axis.) Since we want the solution to be bounded at the north and south poles, and the Legendre polynomials are the only bounded solutions in $(-1, 1)$, we see that $\mu = -n(n+1)$ for some integer n, in which case

$$Y(\varphi) = P_n(\cos\varphi),$$

the n-th Legendre polynomial. The corresponding $Z(t)$ for $n \geq 1$ are

$$Z(t) = C_n \cos(\sqrt{n(n+1)}t) + D_n \sin(\sqrt{n(n+1)}t)$$

yielding the pure harmonics

$$(C_n \cos(\sqrt{n(n+1)}t) + D_n \sin(\sqrt{n(n+1)}t))P_n(\cos\varphi).$$

They have no dependence on the azimuthal angle, that is, they have rotational symmetry around the z-axis. The maximum amplitudes of vibrations are at the north and south poles. There are n values of φ for which $P_n(\cos\varphi) = 0$, the corresponding latitudes are the nodal lines. The case $n = 0$ yields the solution

$$C_0 + D_0 t.$$

This is a linear (not oscillatory) motion; if $D_0 \neq 0$, then the membrane is expanding/shrinking at the rate D_0. I do not find this realistic.

You should visualize these vibrations. For $n = 1$, the equator is a nodal line, the maximum amplitudes are at the pole. When the north pole starts moving in towards the center along the z-axis, the south pole starts moving down the z-axis. The opposite happens in the other part of the cycle.

Suppose the initial position and initial velocity have no θ dependence and are given by $f(\cos\varphi)$ and $g(\cos\varphi)$. Then the C_n and D_n are determined by the Fourier-Legendre series of f and g. In particular,

$$C_0 = \int_{-1}^{1} f(x)dx \quad \text{and} \quad D_0 = \int_{-1}^{1} g(x)dx.$$

To get a physically realistic solution, we should have both these to be zero. Mathematically everything is fine. We do not need such assumptions. The action is happening in $\mathbb{S}^2 \times \mathbb{R}$ (which does not isometrically embed in $\mathbb{R}^3$).

7.5.2. General solution. Now we consider the case of nonzero integer m. This gives an associated Legendre equation for $Y(\varphi)$ in polar form, and nontrivial bounded solutions can exist iff $\mu = -n(n+1)$ with $n = m, m+1, m+2, \dots$. Recall that these solutions are

$$(1-x^2)^{m/2} D^m P_n(x)$$

in the standard form, which in polar form is

$$Y(\varphi) = \sin^m \varphi P_n^{(m)}(\cos\varphi).$$

For each n, $Z(t)$ is as before. So the pure harmonics are

$$(A\cos m\theta + B\sin m\theta)\sin^m \varphi P_n^{(m)}(\cos\varphi)(C_n\cos(\sqrt{n(n+1)}t) + D_n\sin(\sqrt{n(n+1)}t)),$$

where $0 \leq m \leq n$ are nonnegative integers and $n \neq 0$. The $n = 0$ case was already discussed above. Thus for each frequency $\omega_n := \sqrt{n(n+1)}$, there are $2n+1$ linearly independent amplitudes with a basis

$$\cos m\theta \sin^m \varphi P_n^{(m)}(\cos\varphi);\ m = 0, 1, 2, \ldots n$$

$$\sin m\theta \sin^m \varphi P_n^{(m)}(\cos\varphi);\ m = 1, 2, \ldots n.$$

The above amplitudes are eigenfunctions for the Laplacian operator on the sphere. The eigenvalues are $n(n+1)$.

Explicitly, for $n = 1$, we get three amplitudes

$$\cos\varphi,\ \sin\theta\sin\varphi \quad \text{and} \quad \cos\theta\sin\varphi.$$

These are nothing but z, y and x in spherical-polar coordinates. The nodal lines for these vibration modes are $z = 0$ (which is the equator), $y = 0$ and $x = 0$. These are the eigenfunctions for the Laplacian with eigenvalue 2.

Another *more useful* (but complex) basis for the eigenfunctions is

$$\{e^{im\theta}\sin^m \varphi P_n^{|(m)|}(\cos\varphi):\ m = 0, \pm 1, \pm 2, \cdots \pm n\}.$$

Try to visualize these modes of vibrations. For $m \geq 1$, the north and south poles are nodes, and we start getting longitudinal nodal lines from the $\cos m\theta$ or $\sin m\theta$ factors.

Remark 7.2. The integer n corresponding to the frequency ω_n is known as the *principal quantum number*. Confirm terminology. The numbers $m \in \{-n, -n+1, \ldots, 0, \ldots, n\}$ which describe the (complex) amplitudes are known as the *magnetic quantum numbers* underlying the corresponding principal quantum number n.

CHAPTER 8

Laplace equation

The Laplace equation $\Delta u = 0$ governs any kind of *steady state* of any system. An evolving conservative system is governed by $u_t = \Delta u$ which is known variously as heat equation, diffusion equation or evolution equation. In the steady state, the system does not change with time t, and we get the Laplace equation.

In one variable, the Laplacian is the second derivative. Recall that

$$f''(x) \approx \frac{1}{h^2}(f(x+h) + f(x-h) - 2f(x)).$$

Note that the Laplacian is zero iff $f(x) = ax + b$. In this case, the average value at two points is precisely the value at their midpoint.

The Laplacian in higher dimensions has a similar interpretation. For instance, in two variables, the Laplacian of a function f at a point x, namely $f_{xx} + f_{yy}$, is proportional to the average value of f on a small circle around x minus $f(x)$. One can show that the solutions to the Laplace equation are precisely those functions f where the average value of f on any circle around x equals $f(x)$, which is the value at the center. Such functions are called harmonic functions.

8.1. Solution by separation of variables

The Laplace equation in the plane is

$$u_{xx} + u_{yy} = 0. \tag{8.1}$$

We solve it on a rectangular domain $[0, a] \times [0, b]$ by employing the method of separation of variables.

8.1.1. General boundary conditions. The boundary of the rectangle consists of four edges. The Dirichlet boundary conditions specifies the value of u on the boundary edges. The Neumann boundary conditions specifies the value of partial derivative of u on the boundary edges in the direction normal to the boundary edge. More generally, one can take a linear combination of these two types of conditions:

$$\begin{aligned}
\alpha_1 u(x,0) + \beta_1 u_y(x,0) &= f_1(x),\\
\alpha_2 u(a,y) + \beta_2 u_x(a,y) &= f_2(y),\\
\alpha_3 u(x,b) + \beta_3 u_y(x,b) &= f_3(x),\\
\alpha_4 u(0,y) + \beta_4 u_x(0,y) &= f_4(y),
\end{aligned}$$

where $\alpha_i,\ \beta_i$ are constants with $(\alpha_i, \beta_i) \neq (0,0)$ for all i.

Let us solve this general problem. In particular, it will solve the problem with Dirichlet or Neumann boundary conditions. The idea is quite simple. We split the boundary conditions into two parts:

Part 1:

$$\begin{aligned}\alpha_1 u(x,0)+\beta_1 u_y(x,0)&=f_1(x),\\ \alpha_2 u(a,y)+\beta_2 u_x(a,y)&=0,\\ \alpha_3 u(x,b)+\beta_3 u_y(x,b)&=f_3(x),\\ \alpha_4 u(0,y)+\beta_4 u_x(0,y)&=0.\end{aligned}$$

Part 2:

$$\begin{aligned}\alpha_1 u(x,0)+\beta_1 u_y(x,0)&=0,\\ \alpha_2 u(a,y)+\beta_2 u_x(a,y)&=f_2(y),\\ \alpha_3 u(x,b)+\beta_3 u_y(x,b)&=0,\\ \alpha_4 u(0,y)+\beta_4 u_x(0,y)&=f_4(y).\end{aligned}$$

We seek solutions by separation of variables for Part 1, using the two homogeneous boundary conditions. As expected, we get elementary solutions of the type $X(x)Y(y)$, where $X(x)$ solves the Sturm-Liouville problem

$$X''(x)=\lambda X(x),\quad \alpha_4 X(0)+\beta_4 X'(0)=0=\alpha_2 X(a)+\beta_2 X'(a).$$

- The boundary conditions rule out the case $\lambda>0$.
- For $\lambda=0$, the general solution is $X(x)=A+Bx$. The boundary conditions yield

  $$\alpha_4 A+\beta_4 B=0\quad\text{and}\quad \alpha_2(A+Ba)+\beta_2 B=0.$$

  This has a nontrivial solution iff (α_4,β_4) and $(\alpha_2,\alpha_2 a+\beta_2)$ are multiples. The other function $Y(y)$ is also linear and given by $C+Dy$.
- For $\lambda<0$, say $\lambda=-\mu^2$, with $\mu>0$, the general solution is

  $$X(x)=A\cos\mu x+B\sin\mu x.$$

  The boundary conditions yield

  $$\alpha_4 A+\beta_4\mu B=0\quad\text{and}\quad \alpha_2(A\cos\mu a+B\sin\mu a)+\beta_2\mu(B\cos\mu a-A\sin\mu a)=0.$$

  This has a nontrivial solution, that is $(A,B)\neq(0,0)$ iff

  $$\begin{vmatrix}\alpha_4 & \mu\beta_4\\ \alpha_2\cos\mu a-\beta_2\mu\sin\mu a & \alpha_2\sin\mu a+\beta_2\mu\cos\mu a\end{vmatrix}=0. \tag{8.2}$$

  Whenever this happens, the pair (A,B) is determined up to a scalar multiple.

  The equation (8.2) is called the *characteristic equation* of the problem. By general Sturm-Liouville theory (which we have not discussed), this equation has a countable set of solutions

  $$\{\mu_n,\ n=1,2,3,\dots\}$$

  (and the resulting eigenfunctions are orthogonal). In the case that the boundary conditions are of Dirichlet or Neumann types only, the characterstic equation is easily solved by hand.

  The other function $Y(y)$ solves the equation

  $$Y''(y)=\mu^2 Y(y).$$

  So

  $$Y(y)=Ce^{\mu y}+De^{-\mu y}.$$

Hence the general solution is of the form

$$U_1(x,y) = (A_0+B_0x)(C_0+D_0x)+\sum_{n\geq 1}[A_n\cos\mu_n x+B_n\sin\mu_n x][C_n e^{\mu_n y}+D_n e^{-\mu_n y}].$$

The coefficients A_n and B_n have already been determined (up to a scalar). The remaining coefficients C_n and D_n are now determined by the remaining nonhomogeneous boundary conditions. (These work somewhat like initial conditions in the wave equation.)

In an exactly analogous manner we solve Part 2 and get a series solution $U_2(x,y)$, say. Finally due to linearity of the Laplace equation, we see that

$$u(x,y) = U_1x,y) + U_2(x,y)$$

is the required solution.

Akhil has worked out an example.

8.1.2. Dirichlet boundary conditions. The Dirichlet boundary conditions for the rectangle specify the value of $u(x,y)$ on the four boundary edges. For simplicity, let us consider the unit square $[0,1]\times[0,1]$, and for concreteness, consider the boundary conditions

$$u(x,0) = 0 = u(x,1) \quad \text{and} \quad u(0,y) = \sin\pi y = u(1,y).$$

This matches Part 2 rather than Part 1. Interchange the roles of x and y if you want to match Part 1.

Observe that the boundary conditions in the y variable are homogeneous. Hence it is obvious that any elementary solution will be

$$u_n(x,y) = X_n(x)\sin n\pi y, \quad n = 1,2,3,\ldots,$$

with

$$X_n''(x) = n^2\pi^2 X_n(x), \quad X_n(0) = a_n,\ X_n(1) = b_n.$$

The general solution then is

$$u(x,y) = \sum_{n\geq 1} X_n(x)\sin n\pi y.$$

From the boundary conditions in x variable,

$$\sin\pi y = \sum_{n\geq 1} X_n(0)\sin n\pi y = \sum_{n\geq 1} X_n(1)\sin n\pi y.$$

This implies that $a_n = 0 = b_n$ for $n\geq 2$. This in turn implies $X_n = 0$, $n\geq 2$. While $a_1 = 1 = b_1$ implies

$$X_1(x) = \frac{\sinh\pi x+\sinh\pi(1-x)}{\sinh\pi} \approx 0.08659(\sinh\pi x+\sinh\pi(1-x)).$$

Thus

$$\begin{aligned} u(x,y) &= \frac{(\sinh\pi x+\sinh\pi(1-x))\sin\pi y}{\sinh\pi} \\ &= \frac{\cosh[\pi(x-1/2)]\sin\pi y}{\cosh(\pi/2)} \\ &\approx 0.3985368\cosh[\pi(x-1/2)]\sin\pi y. \end{aligned}$$

8.1.3. Neumann boundary conditions. The Neumann boundary conditions specify the values of the normal derivatives along the four boundary edges. Consider the following special case.

$$u_x(0,y) = u_x(a,y) = 0, \quad u_y(x,0) = f(x),\ u_y(x,b) = g(x).$$

In other words, the boundary conditions in the first variable x are homogeneous. Therefore by the method of separation of variables, we have

$$u(x,y) = Y_0(y) + \sum_{n\geq 1} \cos\frac{n\pi x}{a} Y_n(y).$$

The $Y_n(y)$ satisfy the linear ODE

$$Y_n''(y) = \frac{n^2\pi^2}{a^2} Y_n(y), \quad Y_n'(0) = a_n,\ Y_n'(b) = b_n,\ n \geq 0,$$

where a_n, b_n are the coefficients of the Fourier cosine series of f, g respectively over the domain $(0,a)$. Note that for $n = 0$

$$Y_0(y) = a_0 y + C = b_0 y + D,$$

so no solution will exist if $a_0 \neq b_0$. Assuming $a_0 = b_0 = c$ say, $Y_0(y) = cy + D$. For $n \geq 1$,

$$Y_n(y) = C_n e^{n\pi y/a} + D_n e^{-n\pi y/a},$$

where C_n, D_n are uniquely determined from the system:

$$C_n - D_n = \frac{a a_n}{n\pi} \quad \text{and} \quad e^{n\pi} C_n - e^{-n\pi} D_n = \frac{a b_n}{n\pi}.$$

Therefore,

$$u(x,y) = D + (a_0 = b_0)y + \sum_{n\geq 1} \cos\frac{n\pi x}{a} [C_n e^{n\pi y/a} + D_n e^{-n\pi y/a}]$$

We see that solution is unique only upto an additive constant D. The condition $a_0 = b_0$ shows that the total energy or fluid or whatever flowing in through the lower edge equals that flowing out through the upper edge, since the PDE $\Delta u = 0$ is a steady state equation. (The side edges are sealed as per the homogeneous Neumann boundary conditions in x.)

As a concrete example, take $f(x) = x(x-a)$ and $g(x) = 0$. Then $g(x) = 0$ implies $b_n = 0$ for all n, while

$$a_0 = \frac{1}{a}\int_0^a x(x-a)dx = -\frac{a^2}{6} \neq b_0.$$

Hence no solution can exist.

8.1.4. Partly Dirichlet and partly Neumann boundary conditions. Now consider

$$u(0,y) = u(a,y) = 0, \quad u_y(x,0) = f(x),\ u_y(x,b) = g(x).$$

Thus we have Dirichlet boundary conditions on two edges, and Neumann boundary conditions on the other two edges. It is by now routine to deduce that

$$u(x,y) = \sum_{n\geq 1} \sin\frac{n\pi x}{a} Y_n(y),$$

where

$$Y_n''(y) = (n^2\pi^2/a^2) Y_n(y), \quad Y_n'(0) = c_n,\ Y_n'(a) = d_n.$$

Here c_n and d_n are defined by

$$f(x) = \sum_{n\geq 1} c_n \sin\frac{n\pi x}{a}, \quad \text{and} \quad g(x) = \sum_{n\geq 1} d_n \sin\frac{n\pi x}{a}.$$

The initial conditions in the ODE involve derivatives since

$$u_y(x,y) = \sum_{n\geq 1} \sin\frac{n\pi x}{a} Y_n'(y).$$

The unique solution to the ODE is

$$Y_n(y) = C_n e^{n\pi y/a} + D_n e^{-n\pi y/a},$$

where the constants are determined from

$$C_n - D_n = \frac{ac_n}{n\pi} \quad \text{and} \quad e^{n\pi}C_n - e^{-n\pi}D_n = \frac{ad_n}{n\pi}.$$

As a concrete example, take $f(x) = x(x-a)$ and $g(x) = 0$.

$$c_n = \frac{2}{a}\int_0^a x(x-a)\sin\frac{n\pi x}{a}dx = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \frac{-8a^2}{n^3\pi^3} & \text{if } n \text{ is odd,} \end{cases}$$

while $d_n = 0$ for all n. This implies that $Y_n = 0$ for even n. For odd n,

$$C_n - D_n = \frac{-8a^3}{n^4\pi^4} \quad \text{and} \quad e^{n\pi}C_n - e^{-n\pi}D_n = 0,$$

which yields

$$C_n = \frac{4a^3e^{-n\pi}}{n^4\pi^4\sinh n\pi} \quad \text{and} \quad D_n = \frac{4a^3e^{n\pi}}{n^4\pi^4\sinh n\pi}.$$

8.2. Laplacian in three dimensions

The three-dimensional Laplacian in polar coordinates is given by

$$\Delta_{\mathbb{R}^3} = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left[\frac{\partial^2}{\partial\varphi^2} + \cot\varphi\frac{\partial}{\partial\varphi} + \frac{1}{\sin^2\varphi}\frac{\partial^2}{\partial\theta^2}\right].$$

Consider the steady state equation

$$\Delta u(r,\theta,\varphi) = 0.$$

As usual we apply the method of separation of variables. So let $u = X(r)Y(\theta)Z(\varphi)$. The domain of the equation is

$$[0,\infty)\times[0,2\pi]\times[0,\pi]$$

and the implicit homogeneous boundary conditions are

(i) $u(r,\theta,\varphi)$ remains bounded as $\varphi \to 0, \pi$, and

(ii) $u(r,0,\varphi) = u(r,2\pi,\varphi)$, $u_\theta(r,0,\varphi) = u_\theta(r,2\pi,\varphi)$.

On substituting $u = XYZ$ in the steady state equation and simplifying, we get

$$\frac{r^2X''(r)}{X(r)} + \frac{2rX'(r)}{X(r)} = -\left[\frac{Z''(\varphi)}{Z'(\varphi)} + \cot\varphi\frac{Z'(\varphi)}{Z(\varphi)} + \frac{1}{\sin^2\varphi}\frac{Y''(\theta)}{Y(\theta)}\right].$$

The lhs is a function of r alone while rhs is that of (θ,φ), hence both sides must equal to a common constant. This constant must be of the form $n(n+1)$, where n is a nonnegative integer. (This is forced by the rhs equation.) The lhs leads to the equation

$$r^2X''(r) + 2rX'(r) - n(n+1)X(r) = 0.$$

This is a Cauchy-Euler equation, whose independent solutions are r^n and $\frac{1}{r^{n+1}}$.

The rhs leads to another separation of variables,

$$\frac{Y''(\theta)}{Y(\theta)} = -\sin^2\varphi\left[\frac{Z''(\varphi)}{Z(\varphi)} + \cot\varphi\frac{Z'(\varphi)}{Z(\varphi)} + n(n+1)\right].$$

Solving for $Y(\theta)$ and $Z(\varphi)$ under the boundary conditions stated above is solving for the amplitudes of the fundamental harmonics of the unit sphere $\mathbb{S}^2$. This would be clearer if we separate out the time variable first in the harmonics problem, the equation in φ and θ would then be the same as what we have above.

The periodic boundary conditions in θ imply that the common constant is $-m^2$, for $m = 0, 1, 2, \ldots$. The equation for $Z(\varphi)$ is the associated Legendre equation in polar form. This has bounded solutions at $\varphi = 0, \pi$ since we chose the previous constant to be $n(n+1)$.

Let us concentrate on the elementary solutions for $m = 0$, that is, no θ dependence. They are given by

$$(A_n r^n + B_n \frac{1}{r^{n+1}})P_n(\cos\varphi), \quad n = 0, 1, 2, \ldots.$$

Suppose on the sphere of radius R, we are given

$$u(R, \varphi) = f(r).$$

Then (since the solution must be bounded near the origin) the steady state in the interior of the sphere is given by

$$\sum_{n\geq 0} A_n r^n P_n(\cos\varphi),$$

where the A_n are the coefficients of the Fourier-Legendre series of f. Explicitly, they are given by

$$A_n = \frac{2n+1}{2R^n}\int_0^\pi f(\varphi)P_n(\cos\varphi)\sin\varphi d\varphi.$$

Similarly, the steady state in the exterior of the sphere is given by

$$\sum_{n\geq 0} B_n \frac{1}{r^{n+1}} P_n(\cos\varphi),$$

with coefficients

$$B_n = \frac{2n+1}{2}R^{n+1}\int_0^\pi f(\varphi)P_n(\cos\varphi)\sin\varphi d\varphi.$$

Remark 8.1. The gravitational potential due to a point mass at distance a on the z-axis can be determined using the above method. The potential on the sphere of radius a is the reciprocal of $2a\sin\varphi/2$ (undefined at $\varphi = 0$. Expand this into a Legendre series, the coefficients will be related to integer powers of a. Now relate this to the generating function of the Legendre polynomials.

APPENDIX A

Gamma function

We provide here some background on the gamma function which extrapolates the factorial function on nonnegative integers to all real numbers except the non-positive integers.

Define for all $p > 0$,

$$\Gamma(p) := \int_0^\infty t^{p-1} e^{-t} dt. \tag{A.1}$$

(Note that there is a problem at $p = 0$ since $1/t$ is not integrable in an interval containing 0. The same problem persists for $p < 0$. For large values of p, there is no problem because e^{-t} is rapidly decreasing.) Note that

$$\Gamma(1) = \int_0^\infty e^{-t} dt = 1.$$

For any integer $n \geq 1$,

$$\begin{aligned}\Gamma(n+1) = \lim_{x\to\infty} \int_0^x t^n e^{-t} dt &= \lim_{x\to\infty} \left(-t^n e^{-t}\big|_0^x + n \int_0^x t^{n-1} e^{-t} dt \right) \\ &= n \left(\lim_{x\to\infty} \int_0^x t^{n-1} e^{-t} dt \right) = n\Gamma(n).\end{aligned}$$

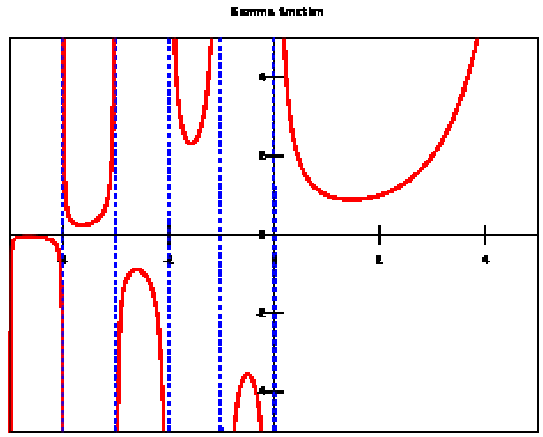
(The boundary term vanishes. So we have transferred the derivative from e^{-t} to t^n as in (2.7). This process can be iterated.) It follows from this identity that

$$\Gamma(n) = (n-1)!.$$

Thus the gamma function extends the factorial function to all positive real numbers. The above calculation is valid for any real $p > 0$, so

$$\Gamma(p+1) = p\,\Gamma(p). \tag{A.2}$$

We use this identity to extend the gamma function to all real numbers except 0 and the negative integers: First extend it to the interval $(-1, 0)$, then to $(-2, -1)$, and so on. The graph is shown below.



Though the gamma function is now defined for all real numbers (except the nonpositive integers), remember that formula (A.1) is valid only for $p > 0$. The identity (A.2) can be rewritten in the form

$$\frac{1}{\Gamma(p)} = \frac{p}{\Gamma(p+1)}. \tag{A.3}$$

If we impose the natural condition that the reciprocal of Γ evaluated at a nonpositive integer is 0, then (A.3) holds for all p.

A well-known value of the gamma function at a non-integer point is

$$\Gamma(1/2) = \int_0^\infty t^{-1/2} e^{-t} dt = 2\int_0^\infty e^{-s^2} ds = \sqrt{\pi}.$$

(We used the substitution $t = s^2$.) Using the identity (A.2), we deduce that

$$\begin{array}{lll} \Gamma(-3/2) & = \frac{4}{3}\sqrt{\pi} & \approx 2.363 \\ \Gamma(-1/2) & = -2\sqrt{\pi} & \approx -3.545 \\ \Gamma(3/2) & = \frac{1}{2}\sqrt{\pi} & \approx 0.886 \\ \Gamma(5/2) & = \frac{3}{4}\sqrt{\pi} & \approx 1.329 \\ \Gamma(7/2) & = \frac{15}{8}\sqrt{\pi} & \approx 3.323. \end{array}$$

More information on the gamma function can be found in[13, Chapter XII], also see wikipedia.

APPENDIX B

Euler constant

We show that

$$\gamma_0 := -\int_0^\infty (\log t)e^{-t}dt = \gamma, \tag{B.1}$$

the Euler constant (4.14).

We divide the proof into four steps.

(1) Since $\frac{1}{n} = \int_0^\infty e^{-nt}dt$, by summing a geometric progression,

$$H_n := \sum_{j=1}^{n} \frac{1}{j} = \int_0^\infty \frac{e^{-t}(1-e^{-nt})}{1-e^{-t}}dt = \int_0^\infty \frac{1-e^{-nt}}{e^t-1}dt.$$

(2) Next we claim that

$$\log n = \int_0^\infty \frac{e^{-t}-e^{-nt}}{t}dt.$$

This is a Frullani integral. It can be expressed as a double integral

$$\int_0^\infty \frac{1}{t}\left[\int_t^{nt} e^{-s}ds\right]dt.$$

On changing the order of integration we obtain

$$\int_0^\infty \left[\int_{s/n}^{n} \frac{dt}{t}\right] e^{-s}ds$$

which is clearly $\log n$. This proves the claim.

Aliter: Let $f(n) = \int_0^\infty \frac{e^{-t}-e^{-nt}}{t}dt$. Then $f'(n) = 1/n$ and $f(1) = 0$. So $f(n) = \log n$. The technical point is that the truncated derived integral $\int_0^T e^{-nt}dt = \frac{1}{n}\left(1-e^{-nT}\right)$ converges to $\frac{1}{n}$ *uniformly* for $n \in [a,b] \subseteq (0,\infty)$ as $T \to \infty$.

(3) Now we can write

$$H_n - \log n = \int_0^\infty \left[\frac{1}{e^t-1} - \frac{e^{-t}}{t}\right] - \int_0^\infty \left[\frac{e^{-nt}}{t} - \frac{e^{-nt}}{e^t-1}\right]dt$$

As $n \to \infty$, the second integral converges to 0 by the dominated convergence Theorem. [It can be easily seen that $f_n(t) := \frac{e^{-nt}}{t} - \frac{e^{-nt}}{e^t-1}$ is pointwise decreasing to 0 on $(0,\infty)$ in other words, monotonically decreasing and bounded below by 0.] Therefore we conclude that

$$\gamma = \int_0^\infty \left[\frac{1}{e^t-1} - \frac{e^{-t}}{t}\right].$$

(4) For the last step we first compute indefinite integrals.

$$\int \frac{1}{e^t - 1} dt = \int \frac{e^{-t/2}}{e^{t/2} - e^{-t/2}} dt = -2 \int \frac{x dx}{1 - x^2} = \log(1 - e^{-t}).$$

$$\int \frac{e^{-t} dt}{t} = e^{-t} \log t + \int e^{-t} \log t dt.$$

Putting together,

$$\begin{aligned} \gamma &= \Big[\log(1 - e^{-t}) - e^{-t} \log t \Big]_0^\infty - \int_0^\infty e^{-t} \log t dt \\ &= 0 - \Big[\log(1 - e^{-t}) - e^{-t} \log t \Big]_{t=0} + \gamma_0 \\ &= -\Big[\log \left(\frac{1 - e^{-t}}{t} \right) + (1 - e^{-t}) \log t \Big]_{t=0} + \gamma_0 \\ &= [0 + 0] + \gamma_0 = \gamma_0 \text{ as required.} \end{aligned}$$

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