B.Stat. (Hons.) & B.Math. (Hons.) Admission Test: 2014

Multiple-Choice Test
Time: 2 hours

1. The system of inequalities

$$a - b^2 \geq \frac{1}{4},\ b - c^2 \geq \frac{1}{4},\ c - d^2 \geq \frac{1}{4},\ d - a^2 \geq \frac{1}{4} \qquad \text{has}$$

(A) no solutions
(B) exactly one solution
(C) exactly two solutions
(D) infinitely many solutions.

2. Let $\log_{12} 18 = a$. Then $\log_{24} 16$ is equal to

(A) $\dfrac{8-4a}{5-a}$ (B) $\dfrac{1}{3+a}$ (C) $\dfrac{4a-1}{2+3a}$ (D) $\dfrac{8-4a}{5+a}$.

3. The number of solutions of the equation $\tan x + \sec x = 2\cos x$, where $0 \leq x \leq \pi$, is

(A) 0 (B) 1 (C) 2 (D) 3.

4. Using only the digits 2, 3 and 9, how many six digit numbers can be formed which are divisible by 6?

(A) 41 (B) 80 (C) 81 (D) 161

5. What is the value of the following integral?

$$\int_{\frac{1}{2014}}^{2014} \frac{\tan^{-1} x}{x}\, dx$$

(A) $\dfrac{\pi}{4}\log 2014$ (B) $\dfrac{\pi}{2}\log 2014$ (C) $\pi \log 2014$ (D) $\dfrac{1}{2}\log 2014$

6. A light ray travelling along the line $y = 1$, is reflected by a mirror placed along the line $x = 2y$. The reflected ray travels along the line

(A) $4x - 3y = 5$ (B) $3x - 4y = 2$ (C) $x - y = 1$ (D) $2x - 3y = 1$.

7. For a real number x, let $[x]$ denote the greatest integer less than or equal to x. Then the number of real solutions of $\big|\, 2x - [x] \,\big| = 4$ is

(A) 1 (B) 2 (C) 3 (D) 4.

8. What is the ratio of the areas of the regular pentagons inscribed inside and circumscribed around a given circle?

(A) $\cos 36^\circ$ (B) $\cos^2 36^\circ$ (C) $\cos^2 54^\circ$ (D) $\cos^2 72^\circ$

9. Let z_1, z_2 be nonzero complex numbers satisfying $|z_1 + z_2| = |z_1 - z_2|$. The circumcentre of the triangle with the points z_1, z_2, and the origin as its vertices is given by

(A) $\frac{1}{2}(z_1 - z_2)$ (B) $\frac{1}{3}(z_1 + z_2)$ (C) $\frac{1}{2}(z_1 + z_2)$ (D) $\frac{1}{3}(z_1 - z_2)$.

10. In how many ways can 20 identical chocolates be distributed among 8 students so that each student gets at least one chocolate and exactly two students get at least two chocolates each?

(A) 308 (B) 364 (C) 616 (D) $\binom{8}{2}\binom{17}{7}$

11. Two vertices of a square lie on a circle of radius r, and the other two vertices lie on a tangent to this circle. Then, each side of the square is

(A) $\frac{3r}{2}$ (B) $\frac{4r}{3}$ (C) $\frac{6r}{5}$ (D) $\frac{8r}{5}$.

12. Let P be the set of all numbers obtained by multiplying five distinct integers between 1 and 100. What is the largest integer n such that 2^n divides at least one element of P?

(A) 8 (B) 20 (C) 24 (D) 25

13. Consider the function $f(x) = ax^3 + bx^2 + cx + d$, where a, b, c and d are real numbers with $a > 0$. If f is strictly increasing, then the function $g(x) = f'(x) - f''(x) + f'''(x)$ is

(A) zero for some $x \in \mathbb{R}$ (B) positive for all $x \in \mathbb{R}$
(C) negative for all $x \in \mathbb{R}$ (D) strictly increasing.

14. Let A be the set of all points (h, k) such that the area of the triangle formed by (h, k), $(5, 6)$ and $(3, 2)$ is 12 square units. What is the least possible length of a line segment joining $(0, 0)$ to a point in A?

(A) $\frac{4}{\sqrt{5}}$ (B) $\frac{8}{\sqrt{5}}$ (C) $\frac{12}{\sqrt{5}}$ (D) $\frac{16}{\sqrt{5}}$

15. Let $P = \{abc : a, b, c \text{ positive integers, } a^2 + b^2 = c^2, \text{ and 3 divides } c\}$. What is the largest integer n such that 3^n divides every element of P?

(A) 1 (B) 2 (C) 3 (D) 4

16. Let $A_0 = \emptyset$ (the empty set). For each $i = 1, 2, 3, \ldots$, define the set $A_i = A_{i-1} \cup \{A_{i-1}\}$. The set A_3 is

(A) $\emptyset$ (B) $\{\emptyset\}$ (C) $\{\emptyset, \{\emptyset\}\}$ (D) $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$

17. Let $f(x) = \dfrac{1}{x-2}\cdot$ The graphs of the functions f and f^{-1} intersect at

(A) $(1+\sqrt{2}, 1+\sqrt{2})$ and $(1-\sqrt{2}, 1-\sqrt{2})$

(B) $(1+\sqrt{2}, 1+\sqrt{2})$ and $(\sqrt{2}, -1-\frac{1}{\sqrt{2}})$

(C) $(1-\sqrt{2}, 1-\sqrt{2})$ and $(-\sqrt{2}, -1+\frac{1}{\sqrt{2}})$

(D) $(\sqrt{2}, -1-\frac{1}{\sqrt{2}})$ and $(-\sqrt{2}, -1+\frac{1}{\sqrt{2}})$

18. Let N be a number such that whenever you take N consecutive positive integers, at least one of them is coprime to 374. What is the smallest possible value of N?

(A) 4 (B) 5 (C) 6 (D) 7

19. Let $A_1, A_2, \ldots, A_{18}$ be the vertices of a regular polygon with 18 sides. How many of the triangles $\triangle A_iA_jA_k$, $1 \le i < j < k \le 18$, are isosceles but not equilateral?

(A) 63 (B) 70 (C) 126 (D) 144

20. The limit $\displaystyle\lim_{x\to 0}\frac{\sin^\alpha x}{x}$ exists *only when*

(A) $\alpha \ge 1$ (B) $\alpha = 1$
(C) $|\alpha| \le 1$ (D) α is a positive integer.

21. Consider the region $R = \{(x, y) : x^2 + y^2 \le 100, \sin(x + y) > 0\}$. What is the area of R?

(A) 25π (B) 50π (C) 50 (D) $100\pi - 50$

22. Consider a cyclic trapezium whose circumcentre is on one of the sides. If the ratio of the two parallel sides is $1 : 4$, what is the ratio of the sum of the two oblique sides to the longer parallel side?

(A) $\sqrt{3} : \sqrt{2}$ (B) $3 : 2$ (C) $\sqrt{2} : 1$ (D) $\sqrt{5} : \sqrt{3}$

23. Consider the function $f(x) = \left\{\log_e\left(\dfrac{4+\sqrt{2x}}{x}\right)\right\}^2$ for $x > 0$. Then,

(A) f decreases upto some point and increases after that
(B) f increases upto some point and decreases after that
(C) f increases initially, then decreases and then again increases
(D) f decreases initially, then increases and then again decreases.

24. What is the number of ordered triplets (a, b, c), where a, b, c are positive integers (not necessarily distinct), such that $abc = 1000$?

(A) 64 (B) 100 (C) 200 (D) 560

25. Let $f : (0, \infty) \to (0, \infty)$ be a function differentiable at 3, and satisfying $f(3) = 3f'(3) > 0$. Then the limit

$$\lim_{x\to\infty} \left(\frac{f\left(3 + \dfrac{3}{x}\right)}{f(3)} \right)^x$$

(A) exists and is equal to 3
(B) exists and is equal to e
(C) exists and is always equal to $f(3)$
(D) need not always exist.

26. Let z be a non-zero complex number such that $\left| z - \frac{1}{z} \right| = 2$. What is the maximum value of $|z|$?

(A) 1 (B) $\sqrt{2}$ (C) 2 (D) $1 + \sqrt{2}$.

27. The minimum value of

$$\left| \sin x + \cos x + \tan x + \operatorname{cosec} x + \sec x + \cot x \right| \quad \text{is}$$

(A) 0 (B) $2\sqrt{2} - 1$ (C) $2\sqrt{2} + 1$ (D) 6

28. For any function $f : X \to Y$ and any subset A of Y, define

$$f^{-1}(A) = \{x \in X : f(x) \in A\}.$$

Let A^c denote the complement of A in Y. For subsets A_1, A_2 of Y, consider the following statements:

(i) $f^{-1}(A_1^c \cap A_2^c) = (f^{-1}(A_1))^c \cup (f^{-1}(A_2))^c$
(ii) If $f^{-1}(A_1) = f^{-1}(A_2)$ then $A_1 = A_2$.

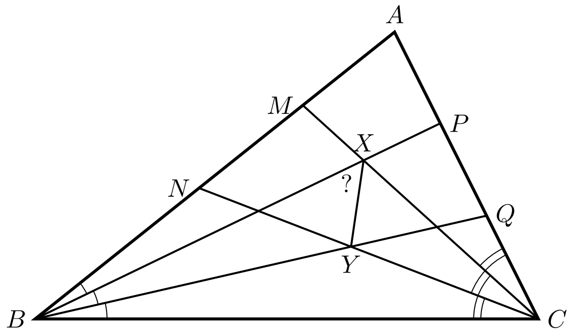
Then,

(A) both (i) and (ii) are always true
(B) (i) is always true, but (ii) may not always be true
(C) (ii) is always true, but (i) may not always be true
(D) neither (i) nor (ii) is always true.

29. Let f be a function such that $f''(x)$ exists, and $f''(x) > 0$ for all $x \in [a, b]$. For any point $c \in [a, b]$, let $A(c)$ denote the area of the region bounded by $y = f(x)$, the tangent to the graph of f at $x = c$ and the lines $x = a$ and $x = b$. Then

(A) $A(c)$ attains its minimum at $c = \frac{1}{2}(a + b)$ for any such f
(B) $A(c)$ attains its maximum at $c = \frac{1}{2}(a + b)$ for any such f
(C) $A(c)$ attains its minimum at both $c = a$ and $c = b$ for any such f
(D) the points c where $A(c)$ attains its minimum depend on f.

30. In $\triangle ABC$, the lines BP, BQ trisect $\angle ABC$ and the lines CM, CN trisect $\angle ACB$. Let BP and CM intersect at X and BQ and CN intersect at Y. If $\angle ABC = 45°$ and $\angle ACB = 75°$, then $\angle BXY$ is



(A) $45°$ (B) $47\frac{1}{2}°$ (C) $50°$ (D) $55°$

B.Stat. (Hons.) & B.Math. (Hons.) Admission Test: 2014

Short-Answer Type Test
Time: 2 hours

1. A class has 100 students. Let a_i, $1 \le i \le 100$, denote the number of friends the i-th student has in the class. For each $0 \le j \le 99$, let c_j denote the number of students having *at least* j friends. Show that

$$\sum_{i=1}^{100} a_i = \sum_{j=0}^{99} c_j .$$

2. It is given that the graph of $y = x^4 + ax^3 + bx^2 + cx + d$ (where a, b, c, d are real) has at least 3 points of intersection with the x-axis. Prove that either there are *exactly* 4 distinct points of intersection, or one of those 3 points of intersection is a local minimum or maximum.

3. Consider a triangle PQR in $\mathbb{R}^2$. Let A be a point lying on $\triangle PQR$ or in the region enclosed by it. Prove that, for any function $f(x, y) = ax + by + c$ on $\mathbb{R}^2$,

$$f(A) \le \max\{f(P), f(Q), f(R)\} .$$

4. Let f and g be two non-decreasing twice differentiable functions defined on an interval (a, b) such that for each $x \in (a, b)$, $f''(x) = g(x)$ and $g''(x) = f(x)$. Suppose also that $f(x)g(x)$ is linear in x on (a, b). Show that we must have $f(x) = g(x) = 0$ for all $x \in (a, b)$.

5. Show that the sum of 12 consecutive integers can never be a perfect square. Give an example of 11 consecutive integers whose sum is a perfect square.

6. Let A be the region in the xy-plane given by

$$A = \{(x, y) : x = u + v, y = v, u^2 + v^2 \le 1\} .$$

Derive the length of the longest line segment that can be enclosed inside the region A.

7. Let $f : [0, \infty) \to \mathbb{R}$ be a non-decreasing continuous function. Show then that the inequality

$$(z - x) \int_y^z f(u)du \ge (z - y) \int_x^z f(u)du$$

holds for any $0 \le x < y < z$.

8. Consider $n\,(>1)$ lotus leaves placed around a circle. A frog jumps from one leaf to another in the following manner. It starts from some selected leaf. From there, it skips exactly one leaf in the clockwise direction and jumps to the next one. Then it skips exactly two leaves in the clockwise direction and jumps to the next one. Then it skips three leaves again in the clockwise direction and jumps to the next one, and so on. Notice that the frog may visit the same leaf more than once. Suppose it turns out that if the frog continues this way, then all the leaves are visited by the frog sometime or the other. Show that n *cannot* be odd.