

**LECTURE NOTES**

**SUBJECT NAME: NUMERICAL METHODS**

**SUBJECT CODE:MA1251**

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# UNIT I

## SOLUTION OF EQUATIONS AND EIGENVALUE PROBLEMS

*Solution of equation*

*Fixed point iteration:  $x=g(x)$  method*

*Newton's method*

*Solution of linear system by Gaussian elimination*

*Gauss – Jordan method*

*Gauss – Jacobi method*

*Gauss – Seidel method*

*Inverse of a matrix by Gauss Jordan method*

*Eigenvalue of a matrix by power method*

*Jacobi method for symmetric matrix.*

## Solution of Algebraic and Transcendental Equations

The equation of the form

$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n = 0$  -----(A) is called *rational integral equation*

Here,  $a_0 \neq 0$ ,  $n$  is a positive integer,  $a_0, a_1, a_2, \dots, a_n$  are constants.

The rational integral equation is classified into two parts

1. Algebraic Equation
2. Transcendental Equation

### Algebraic Equation

In this equation,  $f(x)$  is a polynomial purely in  $x$  as in (A)

**Example:**  $x^3 - 3x + 1 = 0, x^4 + 2x^3 - 3x^2 + 2x + 1 = 0$

### Transcendental Equation

In this equation,  $f(x)$  contains some other functions such as trigonometric, logarithmic or exponential etc.,

**Example:**  $3x - \cos x - 1 = 0, x \log_{10} x - 1.2 = 0.$

### Properties

1. If  $f(a)$  and  $f(b)$  have opposite signs then one root of  $f(x)=0$  lies between  $a$  and  $b$
2. To find an equation whose roots are with opposites signs to those of the given equ. change  $x$  to  $(-x)$
3. To find an equation whose roots are reciprocals of the roots of of the given equ. change  $x$  to  $\left(\frac{1}{x}\right)$
4. Every equation of an odd degree has atleast one real root whose sign is opposite to that of its last term.
5. Every equation of an even degree with last term negative have atleast a pair of real roots one positive and other negative.

### Methods for solving Algebraic and Transcendental Equations

- Fixed point iteration:  $x = g(x)$  method (or) Method of successive approximation.
- Newton's method (or) Newton's Raphson method

### Fixed point iteration: $x = g(x)$ method (or) Method of successive approximation

Let  $f(x) = 0$  be the given equation whose roots are to be determined.

### Steps for this method

1. Use the first property find 'a' and 'b' where the roots lies between.

2. Write the given equation in the form  $x = \phi(x)$  with the condition  $|\phi'(x)| < 1$
3. Let the initial approximation be  $x_0$  which lies in the interval (a, b)
4. Continue the process using  $x_n = \phi(x_{n-1})$
5. If the difference between the two consecutive values of  $x_n$  is very small then we stop the process and that value is the root of the equation.

### Convergence of iteration method

The iteration process converges quickly if  $|\phi'(x)| < 1$  where  $x = \phi(x)$  is the given equation. If  $|\phi'(x)| > 1$ ,  $|x_n - \alpha|$  will become infinitely large and hence this process will not converge. The convergence is linear.

**Example:** Consider the equation  $f(x) = x^3 + x - 1 = 0$

we can write  $x = \phi(x)$  in three types

1.  $x = 1 - x^3$
2.  $x = \frac{1}{1+x^2}$
3.  $x = -x^3$

but we take the type which has the convergence property  $|\phi'(x)| < 1$

$$f(x) = x^3 + x - 1$$

$$f(0) = -ve \text{ and } f(1) = +ve$$

hence the root lies between 0 and 1

Now consider **The equation (1.)  $x = 1 - x^3$**

$$\text{Here } \phi(x) = 1 - x^3, \phi'(x) = -3x^2$$

$$\text{at } x = 0.9 \quad \phi'(x) = -3(0.9)^2$$

$$\phi'(x) = -3(0.81) = -2.81$$

$$\Rightarrow |\phi'(x)| > 1$$

$\Rightarrow$  this equation  $x = 1 - x^3$  will not converge

so the iteration will not work if we consider this equation

Now consider

$$\text{the equation } x = \frac{1}{1+x^2}$$

$$\text{Here } \phi'(x) = \frac{-2x}{(1+x^2)^2}$$

$$\text{at } x = 0.9$$

$$\phi'(x) = \frac{-2 \times 0.9}{(1+0.9^2)^2} = \frac{-1.8}{3.2761} = 0.5494$$

$$\Rightarrow |\phi'(x)| < 1$$

$\Rightarrow$  the equation is converge

$\therefore$  we use this equation.

no need to consider the third type.

### Problems based on Fixed point iteration

1. Find a real root of the equation  $x^3 + x^2 - 1 = 0$  by iteration method.

**Solution:**

Let  $f(x) = x^3 + x^2 - 1$

$f(0) = -ve$  and  $f(1) = +ve$

Hence a real root lies between 0 and 1.

Now can be written as  $x = \frac{1}{\sqrt{1+x}} = \varphi(x)$

In this type only  $|\varphi'(x)| < 1$  in  $(0, 1)$

Let the initial approximation be  $x_0 = 0.5$

$$x_1 = \varphi(x_0) = \frac{1}{\sqrt{0.5+1}} = 0.81649$$

$$x_2 = \varphi(x_1) = \frac{1}{\sqrt{0.81649+1}} = 0.74196$$

$$x_3 = \varphi(x_2) = \frac{1}{\sqrt{0.74196+1}} = 0.75767$$

$$x_4 = \varphi(x_3) = \frac{1}{\sqrt{0.75767+1}} = 0.75427$$

$$x_5 = \varphi(x_4) = \frac{1}{\sqrt{0.75427+1}} = 0.75500$$

$$x_6 = \varphi(x_5) = \frac{1}{\sqrt{0.75500+1}} = 0.75485$$

$$x_7 = \varphi(x_6) = \frac{1}{\sqrt{0.75485+1}} = 0.75488$$

Here the difference between  $x_6$  and  $x_7$  is very small.

therefore the root of the equation is **0.75488**

2. Find the real root of the equation  $\cos x = 3x - 1$ , using iteration method.

**Solution:**

Let  $f(x) = \cos x - 3x - 1$

$f(0) = +ve$  and  $f(1) = -ve$

$\therefore$  A root lies between 0 and  $\pi/2$

The given equation can be written as

$$x = \frac{1}{3}(1 + \cos x) = \varphi(x)$$

$$\varphi'(x) = \frac{-\sin x}{3}$$

Clearly,  $|\varphi'(x)| < 1$  in  $(0, \pi/2)$

Let the initial approximation be  $x_0 = 0$

$$x_1 = \varphi(x_0) = \frac{1}{3}(1 + \cos 0) = 0.66667$$

$$x_2 = \varphi(x_1) = \frac{1}{3}(1 + \cos 0.66667) = 0.59529$$

$$x_3 = \varphi(x_2) = \frac{1}{3}(1 + \cos 0.59529) = 0.60933$$

$$x_4 = \varphi(x_3) = \frac{1}{3}(1 + \cos 0.60933) = 0.60668$$

$$x_5 = \varphi(x_4) = \frac{1}{3}(1 + \cos 0.60668) = 0.60718$$

$$x_6 = \varphi(x_5) = \frac{1}{3}(1 + \cos 0.60718) = 0.60709$$

$$x_7 = \varphi(x_6) = \frac{1}{3}(1 + \cos 0.60709) = 0.60710$$

$$x_8 = \varphi(x_7) = \frac{1}{3}(1 + \cos 0.60710) = 0.60710$$

Since the values of  $x_7$  and  $x_8$  are equal, the required root is **0.60710**

-----3.  
Find the negative root of the equation  $x^3 - 2x + 5 = 0$

**Solution:**

The given equation is  $x^3 - 2x + 5 = 0$  -----(1)

we know that if  $\alpha, \beta, \gamma$  are the roots of the equation (1), then the equation whose roots are  $-\alpha, -\beta, -\gamma$  is  $x^3 + (-1)^0 x^2 + (-1)^2(-2x) + (-1)^3 5 = 0$  -----(2)

The negative root of the equation (1) is same as the positive root of the equation (2)

Let  $f(x) = x^3 - 2x + 5$

Now  $f(2) = -ve$  and  $f(3) = +ve$

Hence the root lies between 2 and 3. Equation (2) can be written as

$$x = (2x + 5)^{\frac{1}{3}} = \varphi(x)$$

where  $|\varphi'(x)| < 1$  in (2, 3)

Let the initial approximation be  $x_0 = 2$

Since the values of  $x_6$  and  $x_7$  are equal, the root is **2.09455**

Therefore the negative root of the given equation is **-2.09455**

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**Newton's method (or) Newton's Raphson method (Method of tangents)**

Let  $f(x) = 0$  be the given equ. whose roots are to be determined.

**FORMULA:**  $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

### Steps for this method

- i. Use the first property find 'a' and 'b' where the roots lies between.
- ii. The initial approximation  $x_0$  is 'a' if  $|f(a)| < |f(b)|$ ; the initial approximation  $x_0$  is 'b' if  $|f(b)| < |f(a)|$  in the interval (a, b).
- iii. Use the formula and continue the process
- iv. If the difference between the two consecutive values of  $x_{n+1}$  is very small then we stop the process and that value is the root of the equ.

### Note

- ✍ The process will evidently fail if  $f'(x) = 0$  in the neighbourhood of the root. In such cases Regula-Falsi method should be used.
- ✍ If we choose initial approximation  $x_0$  close to the root then we get the root of the equ. very quickly.
- ✍ The order of convergence is two

### Condition for convergence of Newton's Raphson method

$$\boxed{|f(x).f''(x)| < |f'(x)|^2}$$

### Problems based on Newton's Method

- 1 . Compute the real root of  $x \log_{10} x = 1.2$  correct to three decimal places using Newton's Raphson Method

Solution:

$$\text{Let } f(x) = x \log_{10} x - 1.2$$

$$\text{Now } f(2) = -ve \text{ and } f(3) = +ve$$

Hence the root lies between 2 and 3.

Let the initial approximation be  $x_0 = 3$

$$f(x) = x \log_{10} x - 1.2$$

$$f'(x) = \log_{10} x + x \cdot \frac{1}{x} \log_{10} e$$

$$= \log_{10} x + 0.4343$$

$$\left[ \because \frac{d}{dx} (\log_a x) = \frac{1}{x} \log_a e \right]$$

$$\left[ \log_{10} e = 0.4343 \right]$$

iteration	value of $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
initial iteration $x_0$	3
1	$x_1 = 3 - \frac{x_0 \log_{10} x_0 - 1.2}{\log_{10} x_0 + 0.4343} = 2.746$
2	$x_2 = 2.746 - \frac{x_1 \log_{10} x_1 - 1.2}{\log_{10} x_1 + 0.4343} = 2.741$
3	$x_3 = 2.741 - \frac{x_2 \log_{10} x_2 - 1.2}{\log_{10} x_2 + 0.4343} = 2.741$

Hence the real root of  $f(x)=0$ , correct to three decimal places is 2.741  
-----2.

Evaluate  $\sqrt{12}$  to four decimal places by Newton's Raphson Method

**Solution:**

Let  $x = \sqrt{12} \Rightarrow x^2 = 12 \Rightarrow x^2 - 12 = 0$

Let  $f(x) = x^2 - 12$  and  $f'(x) = 2x$

Now  $f(3) = -ve$  and  $f(4) = +ve$ . Hence the root lies between 3 and 4. Here  $|f(3)| < |f(4)|$  the root is nearer to 3. Therefore the initial approximation is  $x_0 = 3$

iteration	value of $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
initial iteration $x_0$	3
1	$x_1 = 3 - \frac{f(3)}{f'(3)} = 3.5$
2	$x_2 = 3.5 - \frac{f(3.5)}{f'(3.5)} = 3.4642$
3	$x_3 = 3.4642 - \frac{f(3.4642)}{f'(3.4642)} = 3.4641$
4	$x_4 = 3.4641 - \frac{f(3.4641)}{f'(3.4641)} = 3.4641$

Hence the value of  $\sqrt{12}$  is 3.4641  
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## Solving Simultaneous Equations with two variables using Newton's Method

Let the simultaneous equations with two variables be  $f(x, y) = 0$  and  $g(x, y) = 0$   
 $x_1 = x_0 + h$  and  $y_1 = y_0 + k$

$$h = \frac{-D_1}{D} \quad \text{and} \quad k = \frac{D_2}{D}$$

$$\text{where } D = \begin{vmatrix} f_x & f_y \\ g_x & g_y \end{vmatrix}$$

$$D_1 = \begin{vmatrix} f_0 & f_y \\ g_0 & g_y \end{vmatrix}$$

$$D_2 = \begin{vmatrix} f_x & f_0 \\ g_x & g_0 \end{vmatrix}$$

### Problems

- Find the solution of the equation  $4x^2 + 2xy + y^2 = 30$  and  $2x^2 + 3xy + y^2 = 3$  correct to 3 places of decimals, using Newton's Raphson method, given that  $x_0 = -3$  and  $y_0 = 2$ .

Solution:

$$\text{Let } f(x, y) = 4x^2 + 2xy + y^2 - 30 \quad \text{and} \quad g(x, y) = 2x^2 + 3xy + y^2 - 3$$

$$f_x = 8x + 2y, \quad f_y = 2x + 2y, \quad g_x = 4x + 3y, \quad g_y = 3x + 2y$$

$x_0$	$y_0$	$f_x$	$f_y$	$g_x$	$g_y$	$f_0$	$g_0$
-3	2	-20	-2	-6	-5	-2	1

$$D = \begin{vmatrix} f_x & f_y \\ g_x & g_y \end{vmatrix} = \begin{vmatrix} -20 & -2 \\ -6 & -5 \end{vmatrix} = 88$$

$$D_1 = \begin{vmatrix} f_0 & f_y \\ g_0 & g_y \end{vmatrix} = \begin{vmatrix} -2 & -2 \\ 1 & -5 \end{vmatrix} = 12$$

$$D_2 = \begin{vmatrix} f_x & f_0 \\ g_x & g_0 \end{vmatrix} = \begin{vmatrix} -20 & -2 \\ -6 & 1 \end{vmatrix} = -32$$

$$h = \frac{-D_1}{D} = \frac{-12}{88} = -0.1364$$

$$k = \frac{-D_2}{D} = \frac{32}{88} = 0.3636$$

$$x_1 = x_0 + h \quad \text{and} \quad y_1 = y_0 + k$$

$$\Rightarrow x_1 = -3.1364 \text{ and } y_1 = 2.364$$

$x_1$	$y_1$	$\begin{matrix} \curvearrowright \\ x_1 \end{matrix}$	$\begin{matrix} \curvearrowright \\ y_1 \end{matrix}$	$\begin{matrix} \curvearrowright \\ g_{x_1} \end{matrix}$	$\begin{matrix} \curvearrowright \\ g_{y_1} \end{matrix}$	$f_1$	$g_1$
-3.1364	2.364	-20.360	-1.544	-5.452	-4.680	0.0995	0.0169

$$D = \begin{vmatrix} \begin{matrix} \curvearrowright \\ x_1 \end{matrix} & \begin{matrix} \curvearrowright \\ y_1 \end{matrix} \\ \begin{matrix} \curvearrowright \\ g_{x_1} \end{matrix} & \begin{matrix} \curvearrowright \\ g_{y_1} \end{matrix} \end{vmatrix} = 86.8669$$

$$D_1 = \begin{vmatrix} f_1 & \begin{matrix} \curvearrowright \\ y_1 \end{matrix} \\ g_1 & \begin{matrix} \curvearrowright \\ g_{y_1} \end{matrix} \end{vmatrix} = -0.4395$$

$$D_2 = \begin{vmatrix} \begin{matrix} \curvearrowright \\ x_1 \end{matrix} & f_1 \\ \begin{matrix} \curvearrowright \\ g_{x_1} \end{matrix} & g_1 \end{vmatrix} = 0.1984$$

$$h = \frac{-D_1}{D} = 0.0051$$

$$k = \frac{-D_2}{D} = -0.0023$$

$$\Rightarrow x_2 = -3.131 \text{ and } y_2 = 2.362$$

$x_2$	$y_2$	$\begin{matrix} \curvearrowright \\ x_2 \end{matrix}$	$\begin{matrix} \curvearrowright \\ y_2 \end{matrix}$	$\begin{matrix} \curvearrowright \\ g_{x_2} \end{matrix}$	$\begin{matrix} \curvearrowright \\ g_{y_2} \end{matrix}$	$f_2$	$g_2$
-3.131	2.362	-20.324	-1.538	-5.438	-4.669	0.0008	-0.0009

$$D = \begin{vmatrix} \begin{matrix} \curvearrowright \\ x_2 \end{matrix} & \begin{matrix} \curvearrowright \\ y_2 \end{matrix} \\ \begin{matrix} \curvearrowright \\ g_{x_2} \end{matrix} & \begin{matrix} \curvearrowright \\ g_{y_2} \end{matrix} \end{vmatrix} = 86.5291$$

$$D_1 = \begin{vmatrix} f_2 & \begin{matrix} \curvearrowright \\ y_2 \end{matrix} \\ g_2 & \begin{matrix} \curvearrowright \\ g_{y_2} \end{matrix} \end{vmatrix} = -0.0051$$

$$D_2 = \begin{vmatrix} \begin{matrix} \curvearrowright \\ x_2 \end{matrix} & f_2 \\ \begin{matrix} \curvearrowright \\ g_{x_2} \end{matrix} & g_2 \end{vmatrix} = 0.0226$$

$$h = \frac{-D_1}{D} = 0.0001$$

$$k = \frac{-D_2}{D} = -0.0003$$

$\Rightarrow x_3 = -3.1309$  and  $y_3 = 2.3617$ . Since the two consecutive values of  $x_2, x_3$  and  $y_2, y_3$  are approximately equal, the correct solution can be taken as  $x = -3.1309$  and  $y = 2.3617$ .

## Method of False Position (or) Regula Falsi Method (or) Method of Chords

$$\text{FORMULA: } x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

- This is the first approximation to the actual root.
- Now if  $f(x_1)$  and  $f(a)$  are of opposite signs, then the actual root lies between  $x_1$  and  $a$ .
- Replacing  $b$  by  $x_1$  and keeping  $a$  as it is we get the next approx.  $x_2$  to the actual root.
- Continuing this manner we get the real root.

### Note:

- ✍ The convergence of the root in this method is slower than Newton's Raphson Method

### Problems based on Regula Falsi Method

1. Solve the equation  $x \tan x = -1$  by Regula-Falsi method starting with  $x_0 = 2.5$  and  $x_1 = 3.0$  correct to 3 decimal places.

Solution:

$$\text{Let } f(x) = x \tan x + 1$$

$$f(2.5) = -ve \text{ and } f(3) = +ve$$

Let us take  $a = 3$  and  $b = 2.5$

iteration	a	b	$x = \frac{af(b) - bf(a)}{f(b) - f(a)}$	
1	3	2.5	$x_1 = \frac{3f(2.5) - 2.5f(3)}{f(2.5) - f(3)} = 2.8012$	$f(2.8012) = +ve$
2	2.8012	2.5	$x_2 = \frac{2.8012 f(2.5) - 2.5 f(2.8012)}{f(2.5) - f(2.8012)} = 2.7984$	$f(2.7984) = +ve$
3	2.7984	2.5	$x_3 = \frac{2.7984 f(2.5) - 2.5 f(2.7984)}{f(2.5) - f(2.7984)} = 2.7984$	

Since the two consecutive values of  $x_2$  and  $x_3$  are approximately equal, the required root of the equation  $f(x) = 0$  is 2.7984

2. Find the root of  $xe^x = 3$  by Regula-Falsi method correct to 3 decimal places.

Solution:

$$\text{Let } f(x) = xe^x - 3$$

$$f(1) = +ve \text{ and } f(1.5) = -ve$$

∴ The root lies between 1 and 1.5

Take  $a = 1$  and  $b = 1.5$

iteration	a	b	$x = \frac{af(b) - bf(a)}{f(b) - f(a)}$	
1	1	1.5	$x_1 = \frac{1f(1.5) - 1.5f(1)}{f(1.5) - f(1)} = 1.035$	f(1.035)=-ve
2	1.035	1.5	$x_2 = \frac{1.035 f(1.5) - 1.5 f(1.035)}{f(1.5) - f(1.035)} = 1.045$	f(1.045)=-ve
3	1.045	1.5	$x_3 = \frac{1.045 f(1.5) - 1.5 f(1.045)}{f(1.5) - f(1.045)} = 1.048$	f(1.045)=-ve
4	1.048	1.5	$x_4 = \frac{1.048 f(1.5) - 1.5 f(1.048)}{f(1.5) - f(1.048)} = 1.048$	

Since the two consecutive values of  $x_3$  and  $x_4$  are equal, the required root of the equation  $f(x)=0$  is 1.048

### Solutions of linear algebraic equations

A system of  $m$  linear equations (or a set of  $m$  simultaneous linear equations) in ‘ $n$ ’ unknowns  $x_1, x_2, \dots, x_n$  is a set of equations of the form,

$$\left. \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \dots\dots\dots \dots\dots\dots \dots\dots\dots \dots\dots\dots \dots\dots\dots \dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{array} \right\} \quad (1)$$

Where the coefficients of  $x_1, x_2, \dots, x_n$  and  $b_1, b_2, \dots, b_m$  are constants.

The left hand side members of (1) may be specified by the square array of the coefficients, known as the coefficient matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Whereas the complete set may be specified by the rectangular array

$$M = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix}$$

is known as the **augmented matrix**.

There are two methods to solve such a system by numerical methods.

- **Direct methods**
- **Iterative or indirect methods.**

**Gaussian elimination method, Gauss-Jordan method, belongs to Direct methods,**

**Gauss-Seidel iterative method and relaxation method belongs to iterative methods.**

**Back Substitution**

Let A be a given square matrix of order ‘n’, b a given n-vector. We wish to solve the linear system.

$$Ax=b$$

For the unknown n-vector x. The solution vector x can be obtained without difficulty in case A is upper-triangular with all diagonal entries are non-zero. In that case the system has the form

$$\left. \begin{aligned}
 a_{11}x_1 + a_{12}x_2 + \dots + a_{1,n-1}x_{n-1} + a_{1n}x_n &= b_1 \\
 a_{22}x_2 + \dots + a_{2,n-1}x_{n-1} + a_{2n}x_n &= b_2 \\
 \dots\dots\dots \dots\dots\dots \dots\dots\dots \dots\dots\dots & \\
 a_{n-2,n-2}x_{n-2} + a_{n-2,n-1}x_{n-1} &= b_{n-2} \\
 a_{n-1,n-1}x_{n-1} + a_{n-1,n}x_n + a_{n-2}x_n &= b_{n-1} \\
 a_{nn}x_n &= b_n
 \end{aligned} \right\} \tag{1}$$

In particular, the last equation involves only  $x_n$ ; hence, since  $a_{nn} \neq 0$ , we must have

$$x_n = \frac{b_n}{a_{nn}}$$

Since we now know, the second last equation

$$a_{n-1,n-1}x_{n-1} + a_{n-1,n}x_n = b_{n-1}$$

Involves only one unknown, namely,  $x_{n-1}$ .

As  $a_{n-1,n-1} \neq 0$ , it follows that

$$x_{n-1} = \frac{b_{n-1} - a_{n-1,n}x_n}{a_{n-1,n-1}}$$

With  $x_n$  and  $x_{n-1}$  now determined, the third from last equation

$$a_{n-2,n-2}x_{n-2} + a_{n-2,n-1}x_{n-1} + a_{n-2,n}x_n = b_{n-2}$$

Contains only one true unknown, namely,  $x_{n-2}$ . Once again, wince  $a_{n-2,n-2} \neq 0$ , we can solve for  $x_{n-2}$ .

$$x_{n-2} = \frac{b_{n-2} - a_{n-2,n-1}x_{n-1} - b_{n-1} - a_{n-2,n}x_n}{a_{n-2,n-2}} \text{ and so on.}$$

This process of determining the solution of (1) is called **Back substitution**.

## Gauss elimination method

Basically the most effective direct solution techniques, currently being used are applications of Gauss elimination, method which Gauss proposed over a century ago. In this method, the given system is transformed into an equivalent system with upper-triangular coefficient matrix i.e., a matrix in which all elements below the diagonal elements are zero which can be solved by back substitution.

### Note

- ✍ This method fails if the element in the top of the first column is zero. Therefore in this case we can interchange the rows so as to get the pivot element in the top of the first column.
- ✍ If we are not interested in the elimination of  $x, y, z$  in a particular order, then we can choose at each stage the numerically largest coefficient of the entire coefficient matrix. This requires an interchange of equations and also an interchange of the position of the variables.

### Problems based on Gauss Elimination Method

1. Solve  $2x+y+4z=12$ ;  $8x-3y+2z=20$ ;  $4x+11y-z=33$  by gauss elimination method

#### Solution:

The given equations are,

$$2x+y+4z=12$$

$$8x-3y+2z=20$$

$$4x+11y-z=33$$

above equations can be written as

$$\begin{bmatrix} 2 & 1 & 4 \\ 8 & -3 & 2 \\ 4 & 11 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 20 \\ 33 \end{bmatrix}$$

A      X      B

where, A-co efficient matrix; B-constants; X- unknown variables

The augmented matrix can be written as

$$\begin{aligned}
 [A, B] &= \begin{bmatrix} 2 & 1 & 4 & 12 \\ 8 & -3 & 2 & 20 \\ 4 & 11 & -1 & 33 \end{bmatrix} \\
 \sim & \begin{bmatrix} 2 & 1 & 4 & 12 \\ 0 & 7 & 14 & 28 \\ 4 & 11 & -1 & 33 \end{bmatrix} & R_2 \rightarrow 4R_1 - R_2 \\
 \sim & \begin{bmatrix} 2 & 1 & 4 & 12 \\ 0 & 7 & 14 & 28 \\ 0 & -9 & 9 & -9 \end{bmatrix} & R_3 \rightarrow 2R_1 - R_3 \\
 \sim & \begin{bmatrix} 2 & 1 & 4 & 12 \\ 0 & 7 & 14 & 28 \\ 0 & 0 & 189 & 189 \end{bmatrix} & R_3 \rightarrow 9R_1 + 7R_3
 \end{aligned}$$

solutions are obtained from above matrix by back substitution method as

$$\begin{aligned}
 2x+y+4z &= 12 & \longrightarrow (1) \\
 7y+14z &= 28 & \longrightarrow (2) \\
 189z &= 189 & \longrightarrow (3)
 \end{aligned}$$

from the above equations we get  $z=1, y=2, x=3$

thus the solution of the equations are  $x=3 ; y= 2 ; z=1$

2. Solve  $3x+4y+5z=18; 2x-y+8z=13; 5x-2y+7z=20$  by gauss elimination method

Solution:

The given equations are,

$$3x+4y+5z=18$$

$$2x-y+8z=13$$

$$5x-2y+7z=20$$

above equations can be written as

$$\begin{bmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 18 \\ 13 \\ 20 \end{bmatrix}$$

A    X    B

where, A-co efficient matrix; B-constants; X- unknown variables

The augmented matrix can be written as

$$\begin{aligned} [A, B] &= \begin{bmatrix} 3 & 4 & 5 & 18 \\ 2 & -1 & 8 & 13 \\ 5 & -2 & 7 & 20 \end{bmatrix} \\ &\sim \begin{bmatrix} 3 & 4 & 5 & 18 \\ 0 & 11 & -14 & 3 \\ 0 & 26 & 4 & 30 \end{bmatrix} & R_2 \rightarrow 2R_1 - 3R_2, R_3 \rightarrow 5R_1 - 3R_3 \\ &\sim \begin{bmatrix} 3 & 4 & 115 & 18 \\ 0 & 11 & -14 & -3 \\ 0 & 0 & -408 & -408 \end{bmatrix} & R_3 \rightarrow 26R_2 - 11R_3 \end{aligned}$$

solutions are obtained from above matrix by back substitution method as

$$\begin{aligned} 3x+4y+5z &= 18 & \longrightarrow & (1) \\ 11y-14z &= -3 & \longrightarrow & (2) \\ -408z &= -408 & \longrightarrow & (3) \end{aligned}$$

from the above equations we get  $z=1, y=1, x=3$   
thus the solution of the equations are  $x=3 ; y=1 ; z=1$

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### Gauss – Jordan Method

This method is a modified from Gaussian elimination method. In this method, the coefficient matrix is reduced to a diagonal matrix (or even a unit matrix) rather than a triangular matrix as in the Gaussian method. Here the elimination of the unknowns is done not only in the equations below, but also in the equations above the leading diagonal. Here we get the solution without using the back substitution method since after completion of the Gauss – Jordan method the equations become

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ \dots \\ x_n \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \dots \\ \dots \\ a_n \end{bmatrix}$$

#### Note:

- ✍ This method involves more computation than in the Gaussian method.
- ✍ In this method we can find the values of  $x_1, x_2, \dots, x_n$  immediately without using back substitution.
- ✍ Iteration method is self-correcting method, since the error made in any computation is corrected in the subsequent iterations.



## Problems based on Gauss Jordan Method

1. Solve  $3x+4y+5z=18$ ;  $2x-y+8z=13$ ;  $5x-2y+7z=20$  by gauss elimination method

Solution:

The given equations are,

$$3x+4y+5z=18$$

$$2x-y+8z=13$$

$$5x-2y+7z=20$$

above equations can be written as

$$\begin{bmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 18 \\ 13 \\ 20 \end{bmatrix}$$

A    X    B

where, A-co efficient matrix; B-constants; X- unknown variables

The augmented matrix can be written as

$$\begin{aligned} [A, B] &= \begin{bmatrix} 3 & 4 & 5 & 18 \\ 2 & -1 & 8 & 13 \\ 5 & -2 & 7 & 20 \end{bmatrix} \\ &\sim \begin{bmatrix} 3 & 4 & 5 & 18 \\ 0 & 11 & -14 & 3 \\ 0 & 26 & 4 & 30 \end{bmatrix} & R_2 \rightarrow 2R_1 - 3R_2, R_3 \rightarrow 5R_1 - 3R_3 \\ &\sim \begin{bmatrix} 33 & 0 & 111 & 210 \\ 0 & 11 & -14 & -3 \\ 0 & 0 & -408 & -408 \end{bmatrix} & R_1 \rightarrow 11R_1 - 4R_2, R_3 \rightarrow 26R_2 - 11R_3 \\ &\sim \begin{bmatrix} 13464 & 0 & 0 & 40392 \\ 0 & -4488 & 0 & -4488 \\ 0 & 0 & -408 & -408 \end{bmatrix} & R_1 \rightarrow 408R_1 + 111R_3, R_2 \rightarrow -408R_1 + 14R_3 \\ &\sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} & R_1 \rightarrow R_1 / 13464, R_2 \rightarrow R_2 / -4488, R_3 \rightarrow R_3 / -408 \end{aligned}$$

without back substitution method we get  $z=1, y=1, x=3$

Thus the solution of the equations are  $x=3$  ;  $y=1$  ;  $z=1$

2. Solve  $10x+y+z=12$ ;  $2x+10y+z=13$ ;  $2x+2y+10z=14$  by gauss Jordan method

Solution:

The given equations are,

$$\begin{aligned} 10x+y+z &= 12 \\ 2x+10y+z &= 13 \\ 2x+2y+10z &= 14 \end{aligned}$$

above equations can be written as

$$\begin{bmatrix} 10 & 1 & 1 \\ 2 & 10 & 1 \\ 2 & 2 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 13 \\ 14 \end{bmatrix}$$

A            X            B

A-co efficient matrix

B-constants

X- unknown variables

The augmented matrix can be written as

$$\begin{aligned} \underline{A}, \underline{B} &= \begin{bmatrix} 10 & 1 & 1 & 12 \\ 2 & 10 & 1 & 13 \\ 2 & 2 & 10 & 14 \end{bmatrix} \\ \sim & \begin{bmatrix} 10 & 1 & 1 & 12 \\ 0 & -49 & -4 & -53 \\ 0 & -9 & -49 & -58 \end{bmatrix} \quad R_2 \rightarrow R_1 - 5R_2, R_3 \rightarrow R_1 - 5R_3 \\ \sim & \begin{bmatrix} 490 & 0 & 45 & 535 \\ 0 & -49 & -4 & -53 \\ 0 & 0 & 2365 & 2365 \end{bmatrix} \quad R_1 \rightarrow 49R_1 + R_2, R_3 \rightarrow 9R_2 - 49R_3 \\ \sim & \begin{bmatrix} 1158850 & 0 & 0 & 1158850 \\ 0 & -115885 & 0 & -115885 \\ 0 & 0 & 2365 & 2365 \end{bmatrix} \quad R_1 \rightarrow 2365R_1 - 45R_3, R_2 \rightarrow 2365R_2 + 4R_3 \quad \text{witho} \\ \sim & \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad R_1 \rightarrow R_1 / 1158850, R_2 \rightarrow R_2 / -115885, R_3 \rightarrow R_3 / 2365 \end{aligned}$$

at the back substitution  $x=1$  ;  $y= 1$  ;  $z=1$

### Jacobi's (or Gauss – Jacobi's) iteration method

Let the system of simultaneous equations be

$$\left. \begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned} \right\} \quad (1)$$

This system of equations can also be written as

$$\left. \begin{aligned} x &= \frac{1}{a_1} (d_1 - b_1 y - c_1 z) \\ y &= \frac{1}{b_2} (d_2 - a_2 x - c_2 z) \\ z &= \frac{1}{c_3} (d_3 - a_3 x - b_3 y) \end{aligned} \right\} \quad (2)$$

Let the first approximation be  $x_0, y_0$  and  $z_0$ . Substituting  $x_0, y_0$  and  $z_0$  in (2) we get,

$$x_1 = \frac{1}{a_1} (d_1 - b_1 y_0 - c_1 z_0)$$

$$y_1 = \frac{1}{b_2} (d_2 - a_2 x_0 - c_2 z_0)$$

$$z_1 = \frac{1}{c_3} (d_3 - a_3 x_0 - b_3 y_0)$$

Substitution the values of  $x_1, y_1$  and  $z_1$  in (2) we get the second approximations  $x_2, y_2$  and  $z_2$  as given below

$$x_2 = \frac{1}{a_1} (d_1 - b_1 y_1 - c_1 z_1)$$

$$y_1 = \frac{1}{b_2} (d_2 - a_2 x_1 - c_2 z_1)$$

$$z_1 = \frac{1}{c_3} (d_3 - a_3 x_1 - b_3 y_1)$$

Substituting the values of  $x_2, y_2$  and  $z_2$  in (2) we get the third approximations  $x_3, y_3$  and  $z_3$ .

This process may be repeated till the difference between two consecutive approximations is negligible.

### **Problems based on Gauss Jacobi Method**

1. Solve the following equations by Gauss Jacobi's iteration method,  $20x+y-2z=17$ ,  $3x+20y-z=-18$ ,  $2x-3y+20z=25$ .

Solution:

The given equations are

$$20x+y-2z=17,$$

$$3x+20y-z=-18,$$

$$2x-3y+20z=25.$$

The equations can be written as,

$$x = \frac{1}{20} [7 - y + 2z]$$

$$y = \frac{1}{20} [18 - 3x + z]$$

$$z = \frac{1}{20} [5 - 2x + 3y]$$

Iteration	$x = \frac{1}{20} [7 - y + 2z]$	$y = \frac{1}{20} [18 - 3x + z]$	$z = \frac{1}{20} [5 - 2x + 3y]$
Initial	$x_0=0$	$y_0=0$	$z_0=0$
1	$x_1 = \frac{1}{20} [7] = 0.85$	$y_1 = \frac{1}{20} [18] = -0.9$	$z_1 = \frac{1}{20} [5] = 1.25$
2	$x_2 = \frac{1}{20} [7 + 0.9 + 2.5] = 1.02$	$y_2 = \frac{1}{20} [18 - 3(0.85) + 1.25] = -0.965$	$z_2 = \frac{1}{20} [5 - 1.7 - 2.7] = 1.03$
3	$x_3 = \frac{1}{20} [7 + 0.965 + 2(1.3)] = 1.00125$	$y_3 = \frac{1}{20} [18 - 3(1.02) + 1.03] = -1.0015$	$z_3 = \frac{1}{20} [5 - 2(1.02) + 3(0.965)] = 1.00325$
4	$x_4 = \frac{1}{20} [7 + 1.0015 + 2(1.00325)] = 1.0004$	$y_4 = \frac{1}{20} [18 - 3(1.00125) + 1.00325] = -1.000025$	$z_4 = \frac{1}{20} [5 - 2(1.00125) + 3(-1.0015)] = 0.99965$

$x_3 \sim x_4$ ;  $y_3 \sim y_4$ ;  $z_3 \sim z_4$

Therefore the solution is  $x=1$ ;  $y= -1$ ;  $z=1$

2. Solve the following equations by Gauss Jacobi's method  $9x+2y+4z=20$ ;  
 $x+10y+4z=6$ ;  $2x-4y+10z=-15$

Solution:

The given system of equations is

$$9x+2y+4z=20;$$

$$x+10y+4z=6;$$

$$2x-4y+10z=-15$$

The equations can be written as

$$x = \frac{1}{9} [20 - 2y - 4z]$$

$$y = \frac{1}{10} [6 - x - 4z]$$

$$z = \frac{1}{10} [-15 - 2x + 4y]$$

Let the initial values be  $x_0=y_0=z_0=0$

Iteration	$x = \frac{1}{9} [0 - 2y - 4z]$	$y = \frac{1}{10} [-x - 4z]$	$z = \frac{1}{10} [15 - 2x + 4y]$
Initial	$x_0=0$	$y_0=0$	$z_0=0$
1	$x_1 = \frac{20}{9} = 2.222$	$y_1 = \frac{6}{10} = 0.6$	$z_1 = \frac{-15}{10} = -1.5$
2	$x_2=2.7556$	$y_2=0.9778$	$z_2=-1.7044$
3	$x_3=2.762$	$y_3=1.0062$	$z_3=-1.66$

$x_2 \sim x_3$ ;  $y_2 \sim y_3$ ;  $z_2 \sim z_3$

Therefore the solution is  $x=2.8$ ;  $y= 1$  ;  $z=-1.7$

### Gauss – Seidal Iterative Method

Let the given system of equations be

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = C_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = C_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = C_3$$

.....

.....

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots + a_{nn}x_n = C_n$$

Such system is often amenable to an iterative process in which the system is first rewritten in the form

$$x_1 = \frac{1}{a_{11}} [C_1 - a_{12}x_2 - a_{13}x_3 - \dots - a_{1n}x_n] \quad (1)$$

$$x_2 = \frac{1}{a_{22}} [C_2 - a_{21}x_1 - a_{23}x_3 - \dots - a_{2n}x_n] \quad (2)$$

$$x_3 = \frac{1}{a_{33}} [C_3 - a_{31}x_1 - a_{32}x_2 - \dots - a_{3n}x_n] \quad (3)$$

.....

$$x_n = \frac{1}{a_{nn}} [C_n - a_{n1}x_1 - a_{n2}x_2 - a_{n3}x_3 - \dots - a_{nn}x_n] \quad (4)$$

First let us assume that  $x_2 = x_3 = \dots = x_n = 0$  in (1) and find  $x_1$ . Let it be  $x_1^*$ . Putting  $x_1^*$  for  $x_1$  and  $x_3 = x_4 = \dots = x_n = 0$  in (2) we get the value for  $x_2$  and let it be  $x_2^*$ . Putting  $x_1^*$  for  $x_1$  and  $x_2^*$  for  $x_2$  and  $x_3 = x_4 = \dots = x_n = 0$  in (3) we get the value for  $x_3$  and let it be  $x_3^*$ . In this way we can find the first approximate values for  $x_1, x_2, \dots, x_n$ . Similarly we can find the better approximate value of  $x_1, x_2, \dots, x_n$  by using the relation

$$x_1^* = \frac{1}{a_{11}} (C_1 - a_{12}x_2 - a_{13}x_3 - \dots - a_{1n}x_n)$$

$$x_2^* = \frac{1}{a_{22}} (C_2 - a_{21}x_1^* - a_{23}x_3 - \dots - a_{2n}x_n)$$

$$x_3^* = \frac{1}{a_{33}} (C_3 - a_{31}x_1^* - a_{32}x_2^* - \dots - a_{3n}x_n)$$

$$x_n^* = \frac{1}{a_{nn}} (C_n - a_{n1}x_1^* - a_{n2}x_2^* - \dots - a_{n,n-1}x_{n-1}^*)$$

### Note:

- ✍ This method is very useful with less work for the given systems of equation whose augmented matrix have a large number of zero elements.
- ✍ We say a matrix is **diagonally dominant** if the numerical value of the leading diagonal element in each row is greater than or equal to the sum of the numerical values of the other elements in that row.
- ✍ For the Gauss – Seidal method to coverage quickly, the coefficient matrix must be diagonally dominant. If it is not so, we have to rearrange the equations in such a way that the coefficient matrix is diagonally dominant and then only we can apply Gauss – Seidal method.

### Problems based on Gauss Seidal Method

1. Solve  $x+y+54z=110$ ,  $27x+6y-z=85$ ,  $6x+15y+2z=72$ , by using Gauss Seidal method.

#### Solution:

The system of equations is

$$x+y+54z=110,$$

$$27x+6y-z=85,$$

$$6x+15y+2z=72,$$

The co-efficient matrix is

$$\begin{bmatrix} 1 & 1 & 54 \\ 27 & 6 & -1 \\ 6 & 15 & 2 \end{bmatrix} \begin{array}{l} 1 \not> 1+54 \\ 6 \not> 27+1 \\ 2 \not> 6+15 \end{array}$$

$$\begin{bmatrix} 27 & 6 & -1 \\ 1 & 1 & 54 \\ 6 & 15 & 2 \end{bmatrix} \begin{array}{l} 27 > 6+1 \\ 1 \not> 1+54 \\ 2 \not> 6+15 \end{array} \text{ Here } R_1 \Leftrightarrow R_2$$

$$\begin{bmatrix} 27 & 6 & -1 \\ 6 & 15 & 2 \\ 1 & 1 & 54 \end{bmatrix} \begin{array}{l} 27 > 6+1 \\ 15 > 6+2 \text{ Here } R_2 \leftrightarrow R_3 \\ 54 > 1+1 \end{array}$$

Here the matrix is diagonally dominant

The diagonally dominant matrix is

$$\begin{bmatrix} 27 & 6 & -1 \\ 6 & 15 & 2 \\ 1 & 1 & 54 \end{bmatrix}$$

Thus the matrix is diagonally dominant, now the system of equations is

$$27x+6y-z=85,$$

$$6x+15y+2z=72,$$

$$x+y+54z=110.$$

The equations can be written as

$$x = \frac{1}{27} (85-6y+z)$$

$$y = \frac{1}{15} (72-6x-2z)$$

$$z = \frac{1}{54} (110-x-y)$$

The initial values be  $x_0=y_0=z_0=0$ .

Iteration	$x = \frac{1}{27} (85-6y+z)$	$y = \frac{1}{15} (72-6x-2z)$	$z = \frac{1}{54} (110-x-y)$
Initial	$x_0=0$	$y_0=0$	$z_0=0$
1	$x_1=85/27=3.148$	$y_1 = \frac{1}{15} (72-18.888)$ $=3.5408$	$z_1 = \frac{1}{54} (110-3.148-$ $3.541)$ $=1.9132$
2	$x_2 = \frac{1}{27} (85-1.2448+1.9132)$ $=2.432$	$y_2 = \frac{1}{15} (72-14.592-3.8264)$ $=3.572$	$z_2 = \frac{1}{54} (110-2.432-$ $3.572)$ $=1.9259$
3	$x_3 = \frac{1}{27} (85-21.432+1.9258)$ $=2.4257$	$y_3 = \frac{1}{15} (72-14.5542-3.8516)$ $=3.5729$	$z_3 = \frac{1}{54} (110-2.4257-$ $3.5729)$ $=1.9259$
4	$x_4 = \frac{1}{27} (85-$ $21.4374+1.9259)$ $=2.4255$	$y_4 = \frac{1}{15} (72-14.553-3.8518)$ $=3.5730$	$z_4 = \frac{1}{54} (110-2.4255-$ $3.5730)$ $=1.9259$
5	$x_5 = \frac{1}{27} (85-21.438+1.9259)$ $=2.4255$	$y_5 = \frac{1}{15} (72-14.553-3.8518)$ $=3.5730$	$z_5 = \frac{1}{54} (110-2.4255-$ $3.5730)$ $=1.9259$

$x_4 \sim x_5$  ;  $y_4 \sim y_5$  ;  $z_4 \sim z_5$ , thus the solution is  $x=2.4255$ ,  $y=3.5730$ ,  $z=1.9259$ .

---

2. Solve  $8x-3y+2z=20$ ;  $6x+3y+12z=35$ ;  $4x+11y-z=33$  by Gauss Seidal method

Solution:

The system of equations is

$$8x-3y+2z=20;$$

$$6x+3y+12z=35;$$

$$4x+11y-z=33$$

The co-efficient matrix is  $\begin{bmatrix} 8 & -3 & 2 \\ 6 & 3 & 12 \\ 4 & 11 & -1 \end{bmatrix}$   $8 > 3+2$   
 $3 > 6+12$   
 $1 > 4+11$

Thus the matrix is not diagonally dominant

$$\begin{bmatrix} 8 & -3 & 2 \\ 4 & 11 & -1 \\ 6 & 3 & 12 \end{bmatrix} \begin{matrix} 8 > 3+2 \\ 11 > 4+1 \\ 12 > 6+3 \end{matrix}$$

Now the matrix is diagonally dominant

The system of equations is

$$8x-3y+2z=20$$

$$4x+11y-z=33$$

$$6x+3y+12z=35$$

The equations can be written as

$$x = \frac{1}{8} (20+3y-2z)$$

$$y = \frac{1}{11} (33-4x+z)$$

$$z = \frac{1}{12} (35-6x-3y)$$

Let the initial values be  $x_0=y_0=z_0=0$



Iteration	$x = \frac{1}{8} (20+3y-2z)$	$y = \frac{1}{11} (33-4x+z)$	$z = \frac{1}{12} (35-6x-3y)$
Initial	$x_0=0$	$y_0=0$	$z_0=0$
1	$x_1=20/8=2.5$	$y_1 = \frac{1}{11} (33-10)=2.0909$	$z_1 = \frac{1}{12} (35-15-6)=11.439$
2	$x_2 = \frac{1}{8} (20+6-2.334) = 2.9583$	$y_2 = \frac{1}{11} (33-11.833+1.607) = 2.0758$	$z_2 = \frac{1}{12} (35-17.7498-6.2274) = 0.91875$
3	$x_3 = \frac{1}{8} (20+6.2274-1.8371) = 3.0260$	$y_3 = \frac{1}{11} (33-12.104+0.91857) = 1.9825$	$z_3 = \frac{1}{12} (35-18.156-5.9475) = 0.9077$
4	$x_4 = \frac{1}{8} (20-5.9475-1.8154) = 3.0165$	$y_4 = \frac{1}{11} (33-12.066+0.9077) = 1.9856$	$z_4 = \frac{1}{12} (35-18.0996-5.9568) = 0.9120$
5	$x_5 = \frac{1}{8} (20+5.9568-1.824) = 3.0166$	$y_5 = \frac{1}{11} (33-12.0664+0.9120) = 1.9859$	$z_5 = \frac{1}{12} (35-18.0996-5.9577) = 0.9120$

$x_4 \sim x_5$  ;  $y_4 \sim y_5$  ;  $z_4 \sim z_5$ ,

Therefore the solution  **$x=3.0165$ ;  $y=1.9856$ ;  $z=0.9120$**

## **Inverse of a Matrix**

### **Gauss Jordan Method**

Let  $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$  be the given matrix

**Step 1:**

Write the augmented matrix  $[A/I] = \left[ \begin{array}{ccc|ccc} a_1 & b_1 & c_1 & 1 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 1 & 0 \\ a_3 & b_3 & c_3 & 0 & 0 & 1 \end{array} \right]$

**Step 2:**

Use either row or column operations make the augmented matrix  $[A/I]$  as  $[I/A^{-1}]$ . Here  $A^{-1}$  is the required inverse of the given matrix.

**Problems based on Inverse of a Matrix**

1. Point the inverse of a matrix  $\begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$  by using Gauss-Jordan Method.

Solution:

$$\begin{aligned} [A/I] &= \left[ \begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 3 & 0 & 1 & 0 \\ 1 & 4 & 9 & 0 & 0 & 1 \end{array} \right] \\ &\sim \left[ \begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -3 & 3 & -2 & 0 \\ 0 & -7 & -17 & 1 & 0 & -2 \end{array} \right] R_2 \rightarrow 3R_1 - 2R_2, R_3 \rightarrow R_1 - 2R_3 \\ &\sim \left[ \begin{array}{ccc|ccc} 2 & 0 & -2 & 4 & -2 & 0 \\ 0 & -1 & -3 & 3 & -2 & 0 \\ 0 & 0 & -4 & 20 & -14 & 2 \end{array} \right] R_1 \rightarrow R_1 + R_2, R_3 \rightarrow 7R_2 - R_3 \\ &\sim \left[ \begin{array}{ccc|ccc} 4 & 0 & 0 & -12 & 10 & -2 \\ 0 & -4 & 0 & -48 & 34 & -6 \\ 0 & 0 & -4 & 20 & -14 & 2 \end{array} \right] R_1 \rightarrow 2R_1 - R_3, R_2 \rightarrow R_2 - 3R_3 \\ &\sim \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 2.5 & -0.5 \\ 0 & 1 & 0 & 12 & -8.5 & 1.5 \\ 0 & 0 & 1 & -5 & 3.5 & -0.5 \end{array} \right] R_1 \rightarrow \frac{R_1}{4}, R_2 \rightarrow \frac{R_2}{-4}, R_3 \rightarrow \frac{R_3}{-4} \\ \therefore A^{-1} &= \begin{bmatrix} -3 & 2.5 & -0.5 \\ 12 & -8.5 & 1.5 \\ -5 & 3.5 & -0.5 \end{bmatrix} \end{aligned}$$

- 
2. Point the inverse of a matrix  $\begin{bmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{bmatrix}$  by using Gauss-Jordan Method.

Solution:

$$\begin{aligned}
 [A/I] &= \left[ \begin{array}{ccc|ccc} 8 & -4 & 0 & 1 & 0 & 0 \\ -4 & 8 & -4 & 0 & 1 & 0 \\ 0 & -4 & 8 & 0 & 0 & 1 \end{array} \right] \\
 &\sim \left[ \begin{array}{ccc|ccc} 8 & -4 & 0 & 1 & 0 & 0 \\ 0 & 12 & -8 & 1 & 2 & 0 \\ 0 & -4 & 8 & 0 & 0 & 1 \end{array} \right] R_2 \rightarrow R_1 + 2R_2 \\
 &\sim \left[ \begin{array}{ccc|ccc} 24 & 0 & -8 & 4 & 2 & 0 \\ 0 & 12 & -8 & 1 & 2 & 0 \\ 0 & 0 & 16 & 1 & 2 & 3 \end{array} \right] R_1 \rightarrow 3R_1 + R_2, R_3 \rightarrow R_2 + 3R_3 \\
 &\sim \left[ \begin{array}{ccc|ccc} 48 & 0 & 0 & 9 & 6 & 3 \\ 0 & 24 & 0 & 3 & 6 & 3 \\ 0 & 0 & 16 & 1 & 2 & 3 \end{array} \right] R_1 \rightarrow 2R_1 + R_3, R_2 \rightarrow 2R_2 + R_3 \\
 &\sim \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{16} & \frac{1}{8} & \frac{1}{16} \\ 0 & 1 & 0 & \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \\ 0 & 0 & 1 & \frac{1}{16} & \frac{1}{8} & \frac{3}{16} \end{array} \right] R_1 \rightarrow \frac{R_1}{48}, R_2 \rightarrow \frac{R_2}{24}, R_3 \rightarrow \frac{R_3}{16} \\
 \therefore A^{-1} &= \begin{bmatrix} \frac{3}{16} & \frac{1}{8} & \frac{1}{16} \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \\ \frac{1}{16} & \frac{1}{8} & \frac{3}{16} \end{bmatrix}
 \end{aligned}$$

### Eigen values and Eigenvectors

Let A be any square matrix of order n. then for any scalar  $\lambda$ , we can form a matrix  $[A - \lambda I]$  where I is the  $n^{\text{th}}$  order unit matrix. The determinant of this matrix equated to zero is called the characteristic equation of A. i.e., the characteristic equation of the matrix A is  $|A - \lambda I| = 0$ . Clearly this a polynomial of degree n in  $\lambda$  having n roots for  $\lambda$ , say  $\lambda_1, \lambda_2, \dots, \lambda_n$ . These values are called eigenvalues of the given matrix A.

For each of these eigenvalues, the system of equations  $(A - \lambda I)X = 0$  has a non-

trivial solution for the vector  $X = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$ . This solution  $X = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$  is called a latent

vector or eigen vector corresponding to the eigenvalue  $\lambda$ .

If A is of order n, then its characteristic equation is of  $n^{\text{th}}$  degree. If n is large, it is very difficult to find the exact roots of the characteristic equation and hence the eigenvalues are difficult to find. But there are numerical methods available for such cases. We list below two such methods called

- Power method
- Jacobi's method

The second method can be applied only for symmetric matrices.

### **Power method**

This method can be applied to find numerically the greatest eigenvalue of a square matrix (also called the dominant eigenvalue). The method is explained below.

Let  $\lambda_1, \lambda_2, \dots, \lambda_n$  be the eigenvalues of A and let  $\lambda_1$  be the dominant eigenvalue.

i.e.,  $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \dots \geq |\lambda_n|$

if the corresponding eigenvectors are  $x_0, x_1, x_2, \dots, x_n$ , then any arbitrary vector y can be written as  $y = a_0x_0 + a_1x_1 + \dots + a_nx_n$ , since the eigenvectors are linearly independent. Now

$$\begin{aligned} A^k y &= A^k (a_0x_0 + \dots + a_nx_n) \\ &= a_0\lambda_1^k x_0 + a_1\lambda_2^k x_1 + \dots + a_n\lambda_n^k x_n \quad [A^k X = \lambda^k X] \\ &= \lambda_1^k \left[ a_0x_0 + a_1 \left( \frac{\lambda_2}{\lambda_1} \right)^k x_1 + \dots \right] \end{aligned}$$

But  $\left| \frac{\lambda_i}{\lambda_1} \right| < 1, (i = 2, \dots, n)$ . Hence  $A^k y = \lambda_1^k a_0 x_0$  and  $A^{k+1} y = \lambda_1^{k+1} a_0 x_0$ .

Hence, if k is large,  $\lambda_1 = \frac{A^{k+1} y}{A^k y}$  where the division is carried out in the corresponding components.

Here y is quite arbitrary. But generally we choose it as the vector having all its components ones.

**Note:**

✎ If  $\lambda_1, \lambda_2, \dots, \lambda_n$  are the eigenvalues of A, then the eigenvalue  $\lambda_1$  is dominant if

$$|\lambda_1| > |\lambda_i| \text{ for } i = 2, 3, \dots, n.$$

✎ The eigenvector  $X = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$  corresponding to the eigenvalue  $\lambda_1$  is called the

dominant eigenvector.

✎ If the eigenvalues of A are -3, 1, 2, then -3 is dominant.

✎ If the eigen values of A are -4, 1, 4 then A has no dominant eigenvalue since

$$|-4| = |4|.$$

✎ The power method will work satisfactorily only if A has a dominant eigenvalue.

✎ Eigen vector may be  $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$  (or)  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$  (or)  $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$  for 3 x 3 matrix.

**Problems based on Eigen value of a matrix**

1. Using power method to find a dominant eigen value of a given matrix

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

Solution:

$$\text{Let } A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \text{ and } x_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$Ax_0 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1/2 \\ 0 \end{bmatrix}$$

$$\text{Here } x_1 = \begin{bmatrix} 1 \\ -1/2 \\ 0 \end{bmatrix}$$

$$Ax_1 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.5 \\ -2 \\ 0.5 \end{bmatrix} = 2.5 \begin{bmatrix} 1 \\ -0.8 \\ 0.2 \end{bmatrix}$$

$$\text{Here } x_2 = \begin{bmatrix} 1 \\ -0.8 \\ 0.2 \end{bmatrix}$$

$$Ax_2 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.8 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 2.8 \\ -2.8 \\ 1.2 \end{bmatrix} = 2.8 \begin{bmatrix} 1 \\ -1 \\ 0.428 \end{bmatrix}$$

$$\text{Here } x_3 = \begin{bmatrix} 1 \\ -1 \\ 0.428 \end{bmatrix}$$

$$Ax_3 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0.428 \end{bmatrix} = \begin{bmatrix} 3 \\ -3.428 \\ 1.856 \end{bmatrix} = -3.428 \begin{bmatrix} -0.875 \\ 1 \\ -0.541 \end{bmatrix}$$

$$\text{Here } x_4 = \begin{bmatrix} -0.875 \\ 1 \\ -0.541 \end{bmatrix}$$

$$Ax_4 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} -0.875 \\ 1 \\ -0.541 \end{bmatrix} = \begin{bmatrix} -2.75 \\ 3.416 \\ -2.082 \end{bmatrix} = 3.416 \begin{bmatrix} -0.805 \\ 1 \\ -0.609 \end{bmatrix}$$

$$\text{Here } x_5 = \begin{bmatrix} -0.805 \\ 1 \\ -0.609 \end{bmatrix}$$

$$Ax_5 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} -0.805 \\ 1 \\ -0.609 \end{bmatrix} = \begin{bmatrix} -2.61 \\ 3.414 \\ -2.3 \end{bmatrix} = 3.414 \begin{bmatrix} -0.764 \\ 1 \\ -0.65 \end{bmatrix}$$

$$\text{Here } x_6 = \begin{bmatrix} -0.764 \\ 1 \\ -0.65 \end{bmatrix}$$

$$Ax_6 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} -0.764 \\ 1 \\ -0.65 \end{bmatrix} = \begin{bmatrix} -2.528 \\ 3.414 \\ -2.3 \end{bmatrix} = 3.414 \begin{bmatrix} -0.74 \\ 1 \\ -0.674 \end{bmatrix}$$

$$\text{Here } x_7 = \begin{bmatrix} -0.74 \\ 1 \\ -0.674 \end{bmatrix}$$

$$Ax_7 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} -0.74 \\ 1 \\ -0.674 \end{bmatrix} = \begin{bmatrix} -2.48 \\ 3.414 \\ -2.348 \end{bmatrix} = 3.414 \begin{bmatrix} -0.726 \\ 1 \\ -0.68 \end{bmatrix}$$

Here  $x_7 = x_8$  approximately.

$$\therefore \lambda = 3.414$$

Eigen value = 3.414

$$\text{Eigen vector} = \begin{bmatrix} -0.72 \\ 1 \\ -0.68 \end{bmatrix}$$

## Steps for finding the smallest eigen value

- ✍ First obtained the largest eigen value  $\lambda_1$  of the given matrix.
- ✍ Let  $B=A-\lambda_1 I$ . Let  $\lambda$  be the largest eigen value of the matrix B then **the numerically smallest eigen value of A is  $\lambda+\lambda_1$ .**

2. Find the largest eigen value and the corresponding eigen vector of the

matrix  $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix}$  and hence find the remaining eigen values.

Solution:

$$\text{Let } \mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix}$$

### Step 1

To find the largest Eigen value of A

$$\text{Let } \mathbf{x}_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\mathbf{A} \mathbf{x}_0 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 7 \end{pmatrix} = 7 \begin{pmatrix} 0.43 \\ 0.29 \\ 1 \end{pmatrix}$$

$$\mathbf{x}_1 = \begin{pmatrix} 0.43 \\ 0.29 \\ 1 \end{pmatrix}$$

$$\mathbf{A} \mathbf{x}_1 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix} \begin{pmatrix} 0.43 \\ 0.29 \\ 1 \end{pmatrix} = \begin{pmatrix} 4.01 \\ 0.84 \\ 7 \end{pmatrix} = 7 \begin{pmatrix} 0.57 \\ 0.12 \\ 1 \end{pmatrix}$$

$$\mathbf{x}_2 = \begin{pmatrix} 0.57 \\ 0.12 \\ 1 \end{pmatrix}$$

$$\mathbf{A} \mathbf{x}_2 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix} \begin{pmatrix} 0.57 \\ 0.12 \\ 1 \end{pmatrix} = 7 \begin{pmatrix} 0.54 \\ 0.22 \\ 1 \end{pmatrix}$$

$$\mathbf{x}_3 = \begin{pmatrix} 0.54 \\ 0.22 \\ 1 \end{pmatrix}$$



$$A x_3 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix} \begin{pmatrix} 0.54 \\ 0.22 \\ 1 \end{pmatrix} = 7 \begin{pmatrix} 0.56 \\ 0.16 \\ 1 \end{pmatrix}$$

$$x_4 = \begin{pmatrix} 0.56 \\ 0.16 \\ 1 \end{pmatrix}$$

$$A x_4 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix} \begin{pmatrix} 0.56 \\ 0.16 \\ 1 \end{pmatrix} = 7 \begin{pmatrix} 0.556 \\ 0.19 \\ 1 \end{pmatrix}$$

$$x_5 = \begin{pmatrix} 0.556 \\ 0.19 \\ 1 \end{pmatrix}$$

$$A x_5 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix} \begin{pmatrix} 0.556 \\ 0.19 \\ 1 \end{pmatrix} = 7 \begin{pmatrix} 0.56 \\ 0.18 \\ 1 \end{pmatrix}$$

$$x_6 = \begin{pmatrix} 0.56 \\ 0.18 \\ 1 \end{pmatrix}$$

$x_5 = x_6$  approximately

$\therefore$  Eigen value  $= \lambda = 7$

$$\text{Eigen vector} = \begin{pmatrix} 0.56 \\ 0.18 \\ 1 \end{pmatrix}$$

## Step 2

To find the largest Eigen value of B

$$B = A - \lambda I = \begin{bmatrix} -6 & 2 & 3 \\ 0 & -11 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{Let } y_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$B y_0 = \begin{bmatrix} -6 & 2 & 3 \\ 0 & -11 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Here the we get eigen vector is zero . But the eigen vector should be non- zero. So

we consider th value of  $y_0$  be  $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$\mathbf{B} y_0 = \begin{bmatrix} -6 & 2 & 3 \\ 0 & -11 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -6 \\ 0 \\ 0 \end{pmatrix} = -6 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$y_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\text{Here } y_1 = y_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

The numerically largest eigen value of  $\mathbf{B} = -6 = \lambda$

Numerically smallest eigen value of  $\mathbf{A} = \lambda + \lambda_1 = -6 + 7 = 1 = \lambda_2$

$\lambda_1 + \lambda_2 + \lambda_3 =$  sum of the main diagonals of  $\mathbf{A} = 1 - 4 + 7 = 4$

ie)  $7 + 1 + \lambda_3 = 4$

$\Rightarrow \lambda_3 = -4$

The eigen values are 1, -4, 7.

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UNIT- I

PART-A.

1. Under the conditions that  $f(a)$  and  $f(b)$  have opposite signs and  $a < b$ , find the approximation to the root of  $f(x) = 0$  by the method of false position.

Ans:  $x = \frac{af(b) - bf(a)}{f(b) - f(a)}$

2. Find the approx. value of the root of  $f(x) = x^3 - 3x - 5 = 0$  by the method of false position, the root lying between 2 and 3.

Ans:  $a = 2$     $f(2) = -3$     $b = 3$     $f(3) = 13$

$$x = \frac{2(13) - 3(-3)}{13 + 3} = 2.1875$$

3. State the criterion for convergence in Newton-Raphson method?

Ans: Newton Raphson method converges if  $|f(x) f''(x)| < [f'(x)]^2$  in the interval considered

4. Show that Newton Raphson Method has quadratic convergence. (or) Define that Newton Raphson method's order of convergence. What is the order of convergence of Newton's method?

Ans:  
Let  $\varphi(x_n) = x_{n+1}$  be an iteration method for solving the equation  $x = \varphi(x)$ . If  $\alpha$  is a root of the equation, then  $x_n = \alpha + \epsilon_n$



$$\therefore x_{n+1} = \varphi(x_n) = \varphi(\alpha) + \epsilon_n \frac{\varphi'(\alpha)}{1!} + \epsilon_n^2 \frac{\varphi''(\alpha)}{2!} + \dots$$

The power of  $\epsilon_n$  in the first non vanishing term after  $\varphi(\alpha)$  is called the order of convergence.

Order of convergence is two in Newton's method.

5. If an approx. value of the root of equation  $x^x = 1000$  is 4.5, find a better approx. of root by Newton's method.

Ans: Taking log on both sides of  $x^x = 1000$ .

$$f(x) = x \log_e x - \log_e 1000 \Rightarrow f'(x) = 1 + \log_e x$$

$$\text{Approx. is } x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 4 + \frac{0.0605}{1.6532} = 4.5366$$

6. Write the Newton's formula to find the cube root of N.

Ans:  $x = \sqrt[3]{N} \Rightarrow x^3 - N = 0 \Rightarrow f(x) = x^3 - N, f'(x) = 3x^2$

By Newton's Method  $x_{n+1} = x_n - \frac{x_n^3 - N}{3x_n^2} = \frac{2x_n^3 + N}{3x_n^2}$

7. Establish an iteration formula to find the reciprocal of a positive number N by ~~Newton's~~ Newton's Method.

Ans:  $N = \frac{1}{x} \Rightarrow f(x) = -\frac{1}{x} + N$  and  $f'(x) = \frac{1}{x^2}$

$$x_{n+1} = x_n - \frac{[N - \frac{1}{x_n}]}{\frac{1}{x_n^2}} \Rightarrow x_{n+1} = x_n(2 - Nx_n)$$

8. What is the order of convergence of fixed point iteration  $x = g(x)$  method?

Ans: The order of convergence is one

9. What is the sufficient condition for the convergence of  $x = g(x)$  method.

Ans: The sufficient condition for the convergence is  $|\varphi'(x)| < 1$  for all  $x$  in the interval  $I$  containing the root  $x = \alpha$  of the equation  $f(x) = 0$ , which can be written as  $x = \varphi(x)$ .

10. How do you express the equation  $x^3 + x^2 - 1 = 0$  for the positive root by iteration method.

Ans:  $x^2(x+1) = 1 \Rightarrow x = \frac{1}{\sqrt{x+1}} = \varphi(x)$ .

11. Can we find a real root of the equation  $x^3 + x^2 - 1 = 0$  in the interval  $[0, 1]$  by iteration method.

Ans: we can express  $x^3 + x^2 - 1 = 0$  in the form  $x = \frac{1}{\sqrt{x+1}} = \varphi(x) \Rightarrow \varphi'(x) = -\frac{1}{2(x+1)^{3/2}}$  and  $|\varphi'(x)| < 1$

for all  $x \in [0, 1]$ .

$\therefore$  A real root with suitable initial approximation  $x_0$  can be found by iteration method.

12. Distinguish Gauss Elimination method and Gauss Jordan method.

Ans:

Gauss Elimination	Gauss Jordan
1. Coefficient matrix $A$ of the system reduces to upper triangular matrix	Coeff. matrix $A$ of the system reduces into diagonal of unit matrix
2. Back Substitution process gives solution	Solution obtained directly

13. State the principle involved in Gauss Elimination method of solving a system of equations.

Ans: Augmented matrix  $(A, B)$  reduces into  $(U, k)$  and solution is obtained from the equivalent upper triangular system of equation by back substitution.

14. Explain Gauss Jordan method to solve the system  $Ax = B$ .

Ans: Principle: Reduce augmented matrix  $(A, B)$  into  $[I, k]$  and obtain the solution directly, without back substitution.

15. By Gauss elimination, solve  $x + y = 2$ ,  $2x + 3y = 5$ .

Ans: 
$$\left[ \begin{array}{cc|c} 1 & 1 & 2 \\ 2 & 3 & 5 \end{array} \right] \rightsquigarrow \left[ \begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1 \end{array} \right] \Rightarrow \begin{array}{l} x + y = 2 \\ y = 1 \Rightarrow x = 1 \end{array}$$

$\therefore$  solution  $x = 1, y = 1$

16. Solve  $3x + 2y = 4$ ,  $2x - 3y = 7$  by Gauss Jordan method.

Ans: 
$$\left[ \begin{array}{cc|c} 3 & 2 & 4 \\ 2 & -3 & 7 \end{array} \right] \sim \left[ \begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array} \right] \Rightarrow x = 2, y = -1$$

17. When will the solution of  $Ax = B$  by Gauss Seidal converge quickly

Ans: The coefficient matrix of  $A$  is diagonally dominant

18. State the condition for convergence of Gauss Seidal

Ans: If the matrix  $A$  is diagonally dominant

$$\sum_{j=1}^n |a_{ij}| < |a_{ii}| \text{ for all } i, \text{ the Gauss-Seidal method}$$

converge, where  $A = (a_{ij})$  the coeff. matrix of the given matrix  $Ax = B$ .

19. Check whether Gauss-Seidal method can be used to solve  $2x - 3y + 20z = 35$ ,  $20x + y - 2z = 17$ ,  $3x + 20y - z = -18$  is less number of iteration? If possible solve them.

Ans:

$$\text{Coeff. matrix } \begin{pmatrix} 2 & -3 & 20 \\ 20 & 1 & -2 \\ 3 & 20 & -1 \end{pmatrix} \quad \begin{array}{l} 2 \neq 3+20 \\ 1 \neq 20+2 \\ 1 \neq 3+20 \end{array}$$

$\therefore$  The given system of equations is not diagonally dominant.

$$\begin{pmatrix} 20 & 1 & -2 \\ 3 & 20 & -1 \\ 2 & -3 & 20 \end{pmatrix} \quad \begin{array}{l} 20 > 1+2 \\ 20 > 3+1 \\ 20 > 2+3 \end{array}$$

$\therefore$  Now the given system is diagonally dominant.

So the system becomes  $20x + y - 2z = 17$ ,  $3x + 20y - z = -18$ ,  
 $2x - 3y + 20z = 35$ .

20. Distinguish between direct and indirect method of solving a system of equation  $Ax = B$ .

Ans: Direct Method: Involve a certain amount of fixed computation

Indirect Method: The solution is obtained by successive approx. and the amount of computation depends on the degree of required accuracy.

21. State the basic principle involved for finding  $A^{-1}$  by Gauss-Jordan.

Ans: Reduce the augmented matrix  $(A|I)$  into  $(I|x)$

Then  $X = A^{-1}$



22. How will you find the smallest eigen value of Square matrix A.

Ans: By power method, the largest eigen value of  $A^{-1}$  can be found. Then smallest eigen values of A is the reciprocal of the largest eigen values of  $A^{-1}$ .

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23. Find the Inverse of  $\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$  by Gauss-Jordan Method.

Ans:  $(A|I) = \left[ \begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array} \right]$   
 $\sim \left[ \begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{array} \right] \sim \left[ \begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & -2 \end{array} \right]$   
 $A^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$

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PART-B

1. Find an approx. root of  $x \log_{10} x - 12 = 0$  by false position method
2.  $A = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}$  find the largest eigen value of in magnitude and its corresponding eigen vector
3. Find the root  $4x - e^x = 0$  that lies between 2 & 3 by Newton's Method.
4. Apply Gauss-Seidal method to solve the following System of equation  $20x + y - 2z = 17$ ,  $3x + 20y - z = -18$ ,  $2x - 3y + 20z = 25$
5. Find the smallest positive root of the equation  $x e^{-2x} = \frac{1}{2} \sin x$  correct to 3 decimal places using Newton's Raphson method.

6. Find all eigen value of the matrix  $\begin{pmatrix} 2 & 1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$  by

Jacobi method.

7. Gauss-Seidal Method,  $2x + y + 6z = 9$ ,  $8x + 3y + 8z = 13$ ,  
 $x + 5y + z = 7$

8. Find a root of ~~the~~ the equation  $\cos x = 3x - 1$   
by iteration method.

9. Gauss Jordan method find the inverse of

$$A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{pmatrix}$$

10. Solve  $10x + y + z = 12$ ,  $2x + 10y + z = 13$ ,  $x + y + 5z = 7$  by  
Gauss Jordan method

11. Positive root of  $x e^x = 1$  correct to 4 decimal places  
using Regula-falsi method.

12. Find the numerically largest eigen value of

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \text{ by power method.}$$

— x —

## UNIT II INTERPOLATION AND APPROXIMATION

*Lagrangian Polynomials*

*Divided differences*

*Interpolating with a cubic spline*

*Newton's forward difference formula*

*Newton's backward difference formula*

**LAGRANGIAN POLYNOMIALS**

**Formula**

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)(x_0-x_5)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1-x_5)}y_1 +$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)(x-x_5)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)(x_2-x_5)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)(x-x_5)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)(x_3-x_5)}y_3 +$$

$$\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_5)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)(x_4-x_5)}y_4 + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_5-x_0)(x_5-x_1)(x_5-x_2)(x_5-x_3)(x_5-x_4)}y_5$$

**Problems based on Lagrange's Method**

- Using Lagrange's formula to calculate f(3) from the following table (A.U. N/D. 2007)

<b>x</b>	<b>0</b>	<b>1</b>	<b>2</b>	<b>4</b>	<b>5</b>	<b>6</b>
<b>f(x)</b>	<b>1</b>	<b>14</b>	<b>15</b>	<b>5</b>	<b>6</b>	<b>19</b>

Solution:

<b>x</b>	<b>x<sub>0</sub>=0</b>	<b>x<sub>1</sub>=1</b>	<b>x<sub>2</sub>=2</b>	<b>x<sub>3</sub>=4</b>	<b>x<sub>4</sub>=5</b>	<b>x<sub>5</sub>=6</b>
<b>f(x)</b>	<b>y<sub>0</sub>=1</b>	<b>y<sub>1</sub>=14</b>	<b>y<sub>2</sub>=15</b>	<b>y<sub>3</sub>=5</b>	<b>y<sub>4</sub>=6</b>	<b>y<sub>5</sub>=19</b>

We know that Lagrange's formula is

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)(x_0-x_5)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1-x_5)}y_1 +$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)(x-x_5)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)(x_2-x_5)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)(x-x_5)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)(x_3-x_5)}y_3 +$$

$$\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_5)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)(x_4-x_5)}y_4 + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_5-x_0)(x_5-x_1)(x_5-x_2)(x_5-x_3)(x_5-x_4)}y_5$$

$$y(x) = \frac{(x-1)(x-2)(x-4)(x-5)(x-6)}{(0-1)(0-2)(0-4)(0-5)(0-6)}(1) + \frac{(x-0)(x-2)(x-4)(x-5)(x-6)}{(1-0)(1-2)(1-4)(1-5)(1-6)}(14) +$$

$$\frac{(x-0)(x-1)(x-4)(x-5)(x-6)}{(2-0)(2-1)(2-4)(2-5)(2-6)}(15) + \frac{(x-0)(x-1)(x-2)(x-5)(x-6)}{(4-0)(4-1)(4-2)(4-5)(4-6)}(5) +$$

$$\frac{(x-0)(x-1)(x-2)(x-4)(x-6)}{(5-0)(5-1)(5-2)(5-4)(5-6)}(6) + \frac{(x-0)(x-1)(x-2)(x-4)(x-5)}{(6-0)(6-1)(6-2)(6-4)(6-5)}(19)$$

$$y(x) = \frac{\overbrace{(x-1)}^{\curvearrowright} \overbrace{(x-2)}^{\curvearrowright} \overbrace{(x-4)}^{\curvearrowright} \overbrace{(x-5)}^{\curvearrowright} \overbrace{(x-6)}^{\curvearrowright}}{\overbrace{(x-1)}^{\curvearrowright} \overbrace{(x-2)}^{\curvearrowright} \overbrace{(x-4)}^{\curvearrowright} \overbrace{(x-5)}^{\curvearrowright} \overbrace{(x-6)}^{\curvearrowright}} + \frac{\overbrace{(x-2)}^{\curvearrowright} \overbrace{(x-4)}^{\curvearrowright} \overbrace{(x-5)}^{\curvearrowright} \overbrace{(x-6)}^{\curvearrowright}}{\overbrace{(x-1)}^{\curvearrowright} \overbrace{(x-3)}^{\curvearrowright} \overbrace{(x-4)}^{\curvearrowright} \overbrace{(x-5)}^{\curvearrowright}} (14) +$$

$$\frac{\overbrace{(x-1)}^{\curvearrowright} \overbrace{(x-4)}^{\curvearrowright} \overbrace{(x-5)}^{\curvearrowright} \overbrace{(x-6)}^{\curvearrowright}}{\overbrace{(x-2)}^{\curvearrowright} \overbrace{(x-3)}^{\curvearrowright} \overbrace{(x-4)}^{\curvearrowright}} (15) + \frac{\overbrace{(x-1)}^{\curvearrowright} \overbrace{(x-2)}^{\curvearrowright} \overbrace{(x-5)}^{\curvearrowright} \overbrace{(x-6)}^{\curvearrowright}}{\overbrace{(x-4)}^{\curvearrowright} \overbrace{(x-6)}^{\curvearrowright} \overbrace{(x-1)}^{\curvearrowright} \overbrace{(x-2)}^{\curvearrowright}} (5) +$$

$$\frac{\overbrace{(x-1)}^{\curvearrowright} \overbrace{(x-2)}^{\curvearrowright} \overbrace{(x-4)}^{\curvearrowright} \overbrace{(x-6)}^{\curvearrowright}}{\overbrace{(x-6)}^{\curvearrowright} \overbrace{(x-1)}^{\curvearrowright}} (6) + \frac{\overbrace{(x-1)}^{\curvearrowright} \overbrace{(x-2)}^{\curvearrowright} \overbrace{(x-4)}^{\curvearrowright} \overbrace{(x-5)}^{\curvearrowright}}{\overbrace{(x-6)}^{\curvearrowright} \overbrace{(x-6)}^{\curvearrowright} \overbrace{(x-4)}^{\curvearrowright} \overbrace{(x-5)}^{\curvearrowright}} (19)$$

put  $x=3$  we get

$$y(3) = -\frac{(3-1)(3-2)(3-4)(3-5)(3-6)}{240} + \frac{3(3-2)(3-4)(3-5)(3-6)}{60} (14) - \frac{3(3-1)(3-4)(3-5)(3-6)}{48} (15)$$

$$+ \frac{3(3-1)(3-2)(3-5)(3-6)}{48} (5) - \frac{3(3-1)(3-2)(3-4)(3-6)}{60} (6) + \frac{3(3-1)(3-2)(3-4)(3-5)(3-6)}{240} (19)$$

$$= -\frac{2(1)(-1)(-2)(-3)}{240} + \frac{3(1)(-1)(-2)(-3)}{60} (14) - \frac{2(1)(-1)(-2)(-3)}{48} (15)$$

$$+ \frac{3(2)(1)(-2)(-3)}{48} (5) - \frac{3(2)(1)(-1)(-3)}{60} (6) + \frac{3(2)(1)(-1)(-2)}{240} (19)$$

$$= \frac{12}{240} - \frac{252}{60} + \frac{540}{48} + \frac{180}{48} - \frac{108}{60} + \frac{228}{240}$$

$$= \frac{12 - 1008 + 2700 + 900 - 432 + 228}{240}$$

$$= \frac{2400}{240} = 10$$

Answer:  $y(3) = 10$ .

2. Using Lagrange's formula fit a polynomial to the data (A.U. N/D. 2006)

$x$	0	1	3	4
$f(x)$	-12	0	6	12

Solution:

$x$	$x_0=0$	$x_1=1$	$x_2=3$	$x_3=4$
$f(x)$	$y_0=-12$	$y_1=0$	$y_2=6$	$y_3=12$

We know that Lagrange's formula is

$$y(x) = \frac{\overbrace{(x-x_1)}^{\curvearrowright} \overbrace{(x-x_2)}^{\curvearrowright} \overbrace{(x-x_3)}^{\curvearrowright} \overbrace{(x-x_4)}^{\curvearrowright} \overbrace{(x-x_5)}^{\curvearrowright}}{\overbrace{(x_0-x_1)}^{\curvearrowright} \overbrace{(x_0-x_2)}^{\curvearrowright} \overbrace{(x_0-x_3)}^{\curvearrowright} \overbrace{(x_0-x_4)}^{\curvearrowright} \overbrace{(x_0-x_5)}^{\curvearrowright}} y_0 + \frac{\overbrace{(x-x_0)}^{\curvearrowright} \overbrace{(x-x_2)}^{\curvearrowright} \overbrace{(x-x_3)}^{\curvearrowright} \overbrace{(x-x_4)}^{\curvearrowright} \overbrace{(x-x_5)}^{\curvearrowright}}{\overbrace{(x_1-x_0)}^{\curvearrowright} \overbrace{(x_1-x_2)}^{\curvearrowright} \overbrace{(x_1-x_3)}^{\curvearrowright} \overbrace{(x_1-x_4)}^{\curvearrowright} \overbrace{(x_1-x_5)}^{\curvearrowright}} y_1 +$$

$$\frac{\overbrace{(x-x_0)}^{\curvearrowright} \overbrace{(x-x_1)}^{\curvearrowright} \overbrace{(x-x_3)}^{\curvearrowright} \overbrace{(x-x_4)}^{\curvearrowright} \overbrace{(x-x_5)}^{\curvearrowright}}{\overbrace{(x_2-x_0)}^{\curvearrowright} \overbrace{(x_2-x_1)}^{\curvearrowright} \overbrace{(x_2-x_3)}^{\curvearrowright} \overbrace{(x_2-x_4)}^{\curvearrowright} \overbrace{(x_2-x_5)}^{\curvearrowright}} y_2 + \frac{\overbrace{(x-x_0)}^{\curvearrowright} \overbrace{(x-x_1)}^{\curvearrowright} \overbrace{(x-x_2)}^{\curvearrowright} \overbrace{(x-x_4)}^{\curvearrowright} \overbrace{(x-x_5)}^{\curvearrowright}}{\overbrace{(x_3-x_0)}^{\curvearrowright} \overbrace{(x_3-x_1)}^{\curvearrowright} \overbrace{(x_3-x_2)}^{\curvearrowright} \overbrace{(x_3-x_4)}^{\curvearrowright} \overbrace{(x_3-x_5)}^{\curvearrowright}} y_3$$

$$y(x) = \frac{(x-1)(x-3)(x-4)}{(0-1)(0-3)(0-4)}(-12) + \frac{(x-0)(x-1)(x-3)}{(1-0)(1-3)(1-4)}(0)$$

$$+ \frac{(x-0)(x-1)(x-4)}{(3-0)(3-1)(3-4)}(6) + \frac{(x-0)(x-1)(x-3)}{(4-0)(4-1)(4-3)}(12)$$

$$y(x) = \frac{(x-1)(x-3)(x-4)}{(-1)(-3)(-4)}(-12) + \frac{(x)(x-1)(x-4)}{(3)(2)(-1)}(6) + \frac{(x)(x-1)(x-3)}{(4)(3)(1)}(12)$$

$$y(x) = (x-1)(x-3)(x-4) - x(x-1)(x-4) + (x)(x-1)(x-3)$$

$$= (x-1)[x^2 - 3x - 4x + 12 - x^2 + 4x + x^2 - 3x]$$

$$= (x-1)[x^2 - 6x + 12]$$

$$= x^3 - 6x^2 + 12x - x^2 + 6x - 12$$

$$y(x) = x^3 - 7x^2 + 18x - 12 \text{----- (1)}$$

Substituting x=2 in (1), we get

$$y(2) = 2^3 - 7(2^2) + 18(2) - 12 = 8 - 28 + 36 - 12 = 44 - 40 = 4$$

Answer: y(2) = 4.

3. Using Lagrange's interpolation find the polynomial through (0, 0), (1, 1) and (2, 2) (A.U. M/J. 2007)

<b>x</b>	<b>0</b>	<b>1</b>	<b>2</b>
<b>f(x)</b>	<b>0</b>	<b>1</b>	<b>2</b>

**Solution:**

x	x <sub>0</sub> =0	x <sub>1</sub> =1	x <sub>2</sub> =2
f(x)	y <sub>0</sub> =0	y <sub>1</sub> =1	y <sub>2</sub> =2

We know that Lagrange's formula is

$$y(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}y_2$$

$$y(x) = \frac{(x-1)(x-2)}{(0-1)(0-2)} \cdot 0 + \frac{(x-0)(x-2)}{(1-0)(1-2)} \cdot 1 + \frac{(x-0)(x-1)}{(2-0)(2-1)} \cdot 2$$

$$y(x) = \frac{(x-2)}{(1)} \cdot 1 + \frac{(x-1)}{(2)} \cdot 2$$

$$= -x(x-2) + x(x-1)$$

$$= -x^2 + 2x + x^2 - x$$

$$y(x) = x$$

∴ The required polynomial is y = x

4. The following table gives certain corresponding values of  $x$  and  $\log_{10} x$ . Compute the value of  $\log_{10} 323.5$ , by using Lagrange's formula

$x$	321.0	322.8	324.2	325.0
$f(x)$	2.50651	2.50893	2.51081	2.51188

Solution:

$x$	$x_0=321.0$	$x_1=322.8$	$x_2=324.2$	$x_3=325.0$
$f(x)$	$y_0= 2.50651$	$y_1= 2.50893$	$y_2= 2.51081$	$y_3= 2.51188$

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)(x_0-x_5)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1-x_5)}y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)(x-x_5)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)(x_2-x_5)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)(x-x_5)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)(x_3-x_5)}y_3$$

Substituting these values in Lagrange's interpolation formula, we get,

$$\begin{aligned} f(323.5) &= \frac{(323.5-322.8)(323.5-324.2)(323.5-325)}{(321-322.8)(321-324.2)(321-325)}(2.5061) + \\ &\quad \frac{(323.5-321)(323.5-324.2)(323.5-325)}{(322.8-321)(322.8-324.2)(322.8-325)}(2.50893) \\ &\quad + \frac{(323.5-321)(323.5-322.8)(323.5-325)}{(324.2-321)(324.2-322.8)(324.2-325)}(2.51081) + \\ &\quad \frac{(323.5-321)(323.5-322.8)(323.5-324.2)}{(325-321)(325-322.8)(325-324.2)}(2.51188) \\ &= -0.07996 + 1.18794 + 1.83897 - 0.43708 \\ f(323.5) &= 2.50987 \end{aligned}$$

## Inverse Interpolation

The process of finding a value of  $x$  for the corresponding value of  $y$  is called inverse interpolation.

$$x = \frac{(y - y_1)(y - y_2) \dots (y - y_n)}{(y_0 - y_1)(y_0 - y_2) \dots (y_0 - y_n)} x_0 + \frac{(y - y_0)(y - y_2) \dots (y - y_n)}{(y_1 - y_0)(y_1 - y_2) \dots (y_1 - y_n)} x_1 + \dots + \frac{(y - y_0)(y - y_1) \dots (y - y_{n-1})}{(y_n - y_0)(y_n - y_1) \dots (y_n - y_{n-1})} x_n$$

### Problems based on Inverse Interpolation

1. Find the value of x when y=85, using Lagrange's formula from the following table.

x	2	5	8	14
y	94.8	87.9	81.3	68.7

Solution:

x	$x_0=2$	$x_1=5$	$x_2=8$	$x_3=14$
y	$y_0=94.8$	$y_1=87.9$	$y_2=81.3$	$y_3=68.7$

$$x = \frac{(y - y_1)(y - y_2)(y - y_3)}{(y_0 - y_1)(y_0 - y_2)(y_0 - y_3)} x_0 + \frac{(y - y_0)(y - y_2)(y - y_3)}{(y_1 - y_0)(y_1 - y_2)(y_1 - y_3)} x_1 + \frac{(y - y_0)(y - y_1)(y - y_3)}{(y_2 - y_0)(y_2 - y_1)(y_2 - y_3)} x_2 + \frac{(y - y_0)(y - y_1)(y - y_2)}{(y_3 - y_0)(y_3 - y_1)(y_3 - y_2)} x_3$$

Substituting the above values, we get,

$$x = \frac{(85 - 87.9)(85 - 81.3)(85 - 68.7)}{(94.8 - 87.9)(94.8 - 81.3)(94.8 - 68.7)} (2) + \frac{(85 - 94.8)(85 - 81.3)(85 - 68.7)}{(87.9 - 94.8)(87.9 - 81.3)(87.9 - 68.7)} (5) + \frac{(85 - 94.8)(85 - 87.9)(85 - 68.7)}{(81.3 - 94.8)(81.3 - 87.9)(81.3 - 68.7)} (8) + \frac{(85 - 94.8)(85 - 87.9)(85 - 81.3)}{(68.7 - 94.8)(68.7 - 87.9)(68.7 - 81.3)} (14)$$

$$x = 0.1438778 + 3.3798011 + 3.3010599 - 0.2331532 = 6.3038$$

Therefore the value of x when y = 6.3038

2. The following table gives the value of the elliptic integral

$$y(\theta) = \int_0^\theta \frac{d\theta}{\sqrt{1 - \frac{1}{2} \sin^2 \theta}} \quad \text{for}$$

certain values of  $\theta$ . Find  $\theta$  if  $y(\theta) = 0.3887$

$\theta$	$21^\circ$	$23^\circ$	$25^\circ$
$y(\theta)$	0.3706	0.4068	0.4433

Solution:



$$\theta = \frac{(y - y_1)(y - y_2)(y - y_3)}{(y_0 - y_1)(y_0 - y_2)(y_0 - y_3)}\theta_0 + \frac{(y - y_0)(y - y_2)(y - y_3)}{(y_1 - y_0)(y_1 - y_2)(y_1 - y_3)}\theta_1 + \frac{(y - y_0)(y - y_1)(y - y_3)}{(y_2 - y_0)(y_2 - y_1)(y_2 - y_3)}\theta_2 + \frac{(y - y_0)(y - y_1)(y - y_2)}{(y_3 - y_0)(y_3 - y_1)(y_3 - y_2)}\theta_3$$

$\theta$	$\theta_1=21^\circ$	$\theta_2=23^\circ$	$\theta_3=25^\circ$
$y(\theta)$	$y_1=0.3706$	$y_2=0.4068$	$y_3=0.4433$

we have

$$\theta = \frac{(0.3887 - 0.4068)(0.3887 - 0.4433)}{(0.3706 - 0.4068)(0.3706 - 0.4433)}(21) + \frac{(0.3887 - 0.3706)(0.3887 - 0.4433)}{(0.4068 - 0.3706)(0.4068 - 0.4433)}(23) + \frac{(0.3887 - 0.3706)(0.3887 - 0.4068)}{(0.3706 - 0.3706)(0.3706 - 0.4068)}(25)$$

$$\theta = 21.999^\circ$$

Therefore the value of  $\theta$  such that  $y(\theta) = 0.3887$  is  $\theta=21.999^\circ$

## **DIVIDED DIFFERENCES**

### **Problems based on Newton's Divided Difference Formula**

**2.2.1** Let the function  $y = f(x)$  take the values  $f(x_0), f(x_1), f(x_2), \dots, f(x_n)$  corresponding to the values  $x_0, x_1, x_2, \dots, x_n$  for the argument  $x$  where  $x_1 - x_0, x_2 - x_1, \dots, x_n - x_{n-1}$  need not be necessarily equal.

The first divided of  $f(x)$  for the argument  $x_0, x_1$  is defined as

$$\therefore f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \quad \dots(i)$$

Similarly  $f(x_1, x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$

$$f(x_2, x_3) = \frac{f(x_3) - f(x_2)}{x_3 - x_2} \text{ and so on.}$$

Thus, for defining a first divided difference, we need the functional values corresponding to two arguments.

The second divided difference of  $f(x)$  for three arguments  $x_0, x_1, x_2$  is defined as

$$\frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0} \quad \dots (ii)$$

Similarly  $f(x_1, x_2, x_3) = \frac{f(x_2, x_3) - f(x_1, x_2)}{x_3 - x_1}$

The third divided difference of  $f(x)$  for the four arguments  $x_0, x_1, x_2, x_3$  is defined as

$$f(x_0, x_1, x_2, x_3) = \frac{f(x_1, x_2, x_3) - f(x_0, x_1, x_2)}{x_3 - x_0} \quad \dots (iii)$$

The quantities in (i), (ii) and (iii) are called divided differences of orders 1, 2, 3 respectively.

- Using Newton's Divided Difference formula, find the value of  $f(8)$  and  $f(5)$  given the following data.

$x$	4	5	7	10	11	13
$f(x)$	48	100	294	900	1210	2028

**Solution:**

<b>x</b>	<b>x<sub>0</sub>=4</b>	<b>x<sub>1</sub>=5</b>	<b>x<sub>2</sub>=7</b>	<b>x<sub>3</sub>=10</b>	<b>x<sub>4</sub>=11</b>	<b>x<sub>5</sub>=13</b>
<b>f(x)</b>	<b>f(x<sub>0</sub>)=48</b>	<b>f(x<sub>1</sub>)=100</b>	<b>f(x<sub>2</sub>)=294</b>	<b>f(x<sub>3</sub>)=900</b>	<b>f(x<sub>4</sub>)=1210</b>	<b>f(x<sub>5</sub>)=2028</b>

The divided difference table for the given data is given below.  
Newton's divided difference formula,

$$f(x) = f(x_0) + (x-x_0)f(x_0,x_1) + (x-x_0)(x-x_1)f(x_0,x_1,x_2) + (x-x_0)(x-x_1)(x-x_2)f(x_0,x_1,x_2,x_3) + \dots$$

<b>x</b>	<b>f(x)</b>	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
4	48				
		$\frac{100-48}{5-4} = 52$			
5	100		$\frac{97-52}{7-4} = 15$		
		$\frac{294-100}{7-5} = 97$		$\frac{100-48}{5-4} = 52$	
7	294		$\frac{292-97}{10-5} = 21$		0
		$\frac{900-294}{10-7} = 202$		$\frac{100-48}{5-4} = 52$	
10	900		$\frac{310-202}{11-7} = 27$		0
		$\frac{1210-900}{11-10} = 310$		$\frac{100-48}{5-4} = 52$	
11	1210		$\frac{409-310}{13-10} = 33$		
		$\frac{2028-1210}{13-11} = 409$			
13	2028				

Using divided differences and the given data in (1),

$$f(x) = 48 + 52(x-4) + 15(x-4)(x-5) + (x-4)(x-5)(x-7)$$

when  $x=8$ ,

$$f(8) = 48 + 208 + 180 + 12 = 448$$

Therefore  $f(8) = 448$

when  $x=15$ ,

$$f(15) = 48 + 572 + 1650 + 880 = 3150$$

Therefore  $f(15) = 3150$

- Use Newton's divided difference formula, to fit a polynomial to the data

and find y when x=1

<b>x</b>	<b>-1</b>	<b>0</b>	<b>2</b>	<b>3</b>
<b>y</b>	<b>8</b>	<b>3</b>	<b>1</b>	<b>12</b>

Solution:

<b>x</b>	<b>x<sub>0</sub>=-1</b>	<b>x<sub>1</sub>=0</b>	<b>x<sub>2</sub>=2</b>	<b>x<sub>3</sub>=3</b>
<b>y</b>	<b>y<sub>0</sub>=8</b>	<b>y<sub>1</sub>=3</b>	<b>y<sub>2</sub>=1</b>	<b>y<sub>3</sub>=12</b>

The divided difference table for the given data as follows.

<b>x</b>	<b>y</b>	<b>Δy</b>	<b>Δ<sup>2</sup>y</b>	<b>Δ<sup>3</sup>f(x)</b>
-1	-8			
		$\frac{3+8}{0+1} = 11$		
0	3		$\frac{-1-11}{2+1} = -4$	
		$\frac{1-3}{2-0} = -1$		$\frac{4+4}{3+1} = 2$
2	1		$\frac{11+1}{2+1} = 4$	
		$\frac{12-1}{3-2} = 11$		
3	12			

By Newton's divided difference formula,

$$f(x) = f(x_0) + (x-x_0)f(x_0,x_1) + (x-x_0)(x-x_1)f(x_0,x_1,x_2) + (x-x_0)(x-x_1)(x-x_2)f(x_0,x_1,x_2,x_3) + \dots$$

Using these we get,

$$f(x) = -8+(x+1)11+(x+1)x(-4)+(x+1)x(x-2)2$$

$$= -8+11x+11-4x^2-4x+2x^3-2x^2-4x$$

$$= 2x^3-6x^2+3x+3$$

$$y = 2x^3-6x^2+3x+3$$

$$y(1) = 2-6+3+3 = 2$$

$$\text{Answer: } y(1)=2$$

**Interpolating with a Cubic Spline**

**Definition: Cubic Spline**

A cubic polynomial approximating the curve in every subinterval is called is called satisfying the following properties.

1.  $F(x_i)=f_i$  for  $i = 0, 1, 2, \dots, n$ .
2. On each interval  $[x_{i-1}, x_i]$ ,  $1 \leq i \leq n$ ,  $F(x)$  is a third degree polynomial.
3.  $F(x)$ ,  $F'(x)$  and  $F''(x)$  are continuous on the interval  $[x_0, x_n]$ .

The second derivatives at the end points of the given range are denoted as  $M_0$  and  $M_n$  respectively.

**Natural cubic spline**

A cubic spline  $F(x)$  with end conditions  $M_0=0$  and  $M_n=0$  where  $f''(x_i)=M_i$  in the interval  $[x_0, x_n]$  i.e.,  $f''(x_0) = 0$  and  $f''(x_n) = 0$  is called a natural cubic spline.

**Fitting a natural cubic spline for the given data:**

In case of natural cubic spline the derivatives  $f''(x_0) = 0$  and  $f''(x_n) = 0$ . i.e.,  $M_0=0$  and  $M_n=0$

Suppose that the values of  $x$  are equally spaced with a spacing  $h$ .

The cubic spline approximation in the subinterval  $(x_{i-1}, x_i)$  is given by

$$F_i(x) = \frac{1}{h} \left[ \frac{(x_i - x)^3}{6} M_{i-1} + \frac{(x - x_{i-1})^3}{6} M_i \right] + \frac{x_i - x}{h} \left( y_{i-1} - \frac{h^2}{6} M_{i-1} \right) + \frac{x - x_{i-1}}{h} \left( y_i - \frac{h^2}{6} M_i \right)$$

$$M_{i-1}+4M_i+M_{i+1} = \frac{6}{h^2} [y_{i-1}+y_{i+1}-2y_i] \text{ where } i = 1, 2, \dots, n- 1 \text{ with } M_0=0 \text{ and } M_n=0.$$

**Problems based on Cubic spline**

1. Obtain the cubic spline approximation for the function  $y = f(x)$  from the following data, given that  $y_0''=y_3''=0$ .

<b>x</b>	<b>-1</b>	<b>0</b>	<b>1</b>	<b>2</b>
<b>y</b>	<b>-1</b>	<b>1</b>	<b>3</b>	<b>35</b>

Solution:

<b>x</b>	<b><math>x_0=-1</math></b>	<b><math>x_1=0</math></b>	<b><math>x_2=1</math></b>	<b><math>x_3=2</math></b>
<b>y</b>	<b><math>y_0=-1</math></b>	<b><math>y_1=1</math></b>	<b><math>y_2=3</math></b>	<b><math>y_3=35</math></b>

The values of  $x$  are equally spaced with  $h = 1$ .

Therefore we have

$$M_{i-1}+4M_i+M_{i+1} = 6 [y_{i-1}+y_{i+1}-2y_i] \text{ where } i = 1, 2, \dots, n- 1$$

Further  $M_0=0$  and  $M_3=0$

$$\therefore M_0+4M_1+M_2 = 6 (y_0+y_2-2y_1)$$

$$\Rightarrow 4M_1+M_2 = 0 \text{ -----(1)}$$

and

$$M_1 + 4M_2 + M_3 = 6(y_1 + y_3 - 2y_2) \\ = 6(1 - 6 + 35) = 180$$

$$\therefore M_1 + 4M_2 = 180 \text{ -----(2)}$$

Solving (1) and (2),  $M_1 = -12$   $M_2 = 48$

The cubic spline in  $(x_{i-1}, x_i)$  is given by

$$y = \left[ \frac{(x_i - x)^3}{6} M_{i-1} + \frac{(x - x_{i-1})^3}{6} M_i \right] + \left( \frac{x_i - x}{1} \right) \left( y_{i-1} - \frac{1}{6} M_{i-1} \right) + \left( \frac{x - x_{i-1}}{1} \right) \left( y_i - \frac{1}{6} M_i \right) \text{ -----(3)}$$

where  $i = 1, 2, \dots, n - 1$ .

In the interval  $-1 \leq x \leq 0$ , i.e.,  $x_0 \leq x \leq x_1 (i=1)$  the cubic spline is given by

$$y = \left[ \frac{(x_1 - x)^3}{6} M_0 + \frac{(x - x_0)^3}{6} M_1 \right] + \left( \frac{x_1 - x}{1} \right) \left( y_0 - \frac{1}{6} M_0 \right) + \left( \frac{x - x_0}{1} \right) \left( y_1 - \frac{1}{6} M_1 \right)$$

$$\Rightarrow y = \frac{1}{6} [(x+1)^3(-12)] + (-x)(-1) + (x+1)(1+2)$$

$$= (-2)(x^3 + 3x^2 + 3x + 1) + x + 3x + 3$$

$$y = -2x^3 - 6x^2 - 2x + 1$$

In the interval  $0 \leq x \leq 1$ , i.e.,  $x_1 \leq x \leq x_2 (i=2)$  the cubic spline is given by

$$y = \left[ \frac{(x_2 - x)^3}{6} M_1 + \frac{(x - x_1)^3}{6} M_2 \right] + \left( \frac{x_2 - x}{1} \right) \left( y_1 - \frac{1}{6} M_1 \right) + \left( \frac{x - x_1}{1} \right) \left( y_2 - \frac{1}{6} M_2 \right)$$

$$\Rightarrow y = \frac{1}{6} [(1-x)^3(-12) + x^3(48)] + (1-x)(1+2) + (x-0)(3-4)$$

$$= (-2)(1-x)^3 + 8x^3 + 3 - 3x - x$$

$$\Rightarrow y = 2x^3 - 6x^2 + 6x - 2 + 8x^3 + 3 - 4x$$

$$\therefore y = 10x^3 - 6x^2 + 2x + 1$$

In the interval  $1 \leq x \leq 2$ , i.e.,  $x_2 \leq x \leq x_3 (i=3)$  the cubic spline is given by

$$y = \left[ \frac{(x_3 - x)^3}{6} M_2 + \frac{(x - x_2)^3}{6} M_3 \right] + \left( \frac{x_3 - x}{1} \right) \left( y_2 - \frac{1}{6} M_2 \right) + \left( \frac{x - x_2}{1} \right) \left( y_3 - \frac{1}{6} M_3 \right)$$

$$\Rightarrow y = \frac{1}{6} [(2-x)^3(48)] + (2-x)(3-8) + (x-1)(35)$$

$$= 8(8 - 12x + 6x^2 - x^3) + 5x - 10 + 35x - 35$$

$$\Rightarrow y = -8x^3 + 48x^2 - 56x + 19$$

$$\therefore y = -8x^3 + 48x^2 - 56x + 19$$

Hence the required cubic spline approximation for the given function is

$$y = \begin{cases} -2x^3 - 6x^2 - 2x + 1 & \text{for } -1 \leq x \leq 0 \\ 10x^3 - 6x^2 + 2x + 1 & \text{for } 0 \leq x \leq 1 \\ -8x^3 + 48x^2 - 56x + 19 & \text{for } 1 \leq x \leq 2 \end{cases}$$

2. Obtain the natural cubic spline which agrees with  $y(x)$  at the set of data points given below:

<b>x</b>	<b>2</b>	<b>3</b>	<b>4</b>
<b>y</b>	11	49	123

Hence find  $y(2.5)$

Solution:

x	2	3	4
y	11	49	123

The values of x are equally spaced with h = 1.

Therefore we have

$$M_{i-1} + 4M_i + M_{i+1} = 6 [y_{i-1} + y_{i+1} - 2y_i] \text{ where } i = 1, 2, \dots, n-1$$

Further  $M_0 = 0$  and  $M_2 = 0$

$$\therefore M_0 + 4M_1 + M_2 = 6 (y_0 + y_2 - 2y_1)$$

$$\Rightarrow 4M_1 = 6(11 - 98 + 123)$$

$$\therefore M_1 = 54$$

The cubic spline in  $(x_{i-1}, x_i)$  is given by

$$y = \left[ \frac{(x_i - x)^3}{6} M_{i-1} + \frac{(x - x_{i-1})^3}{6} M_i \right] + \left( \frac{x_i - x}{1} \right) \left( y_{i-1} - \frac{1}{6} M_{i-1} \right) + \left( \frac{x - x_{i-1}}{1} \right) \left( y_i - \frac{1}{6} M_i \right) \text{-----(3)}$$

where  $i = 1, 2, \dots, n-1$ .

In the interval  $2 \leq x \leq 3$ , i.e.,  $x_0 \leq x \leq x_1 (i=1)$  the cubic spline is given by

$$y = \left[ \frac{(x_1 - x)^3}{6} M_0 + \frac{(x - x_0)^3}{6} M_1 \right] + \left( \frac{x_1 - x}{1} \right) \left( y_0 - \frac{1}{6} M_0 \right) + \left( \frac{x - x_0}{1} \right) \left( y_1 - \frac{1}{6} M_1 \right)$$

$$\Rightarrow y = \frac{1}{6} [(x-2)^3(54)] + (3-x)(11) + (x-2)(49-9)$$

$$= 9(x^3 - 6x^2 + 12x - 8) + 33 - 11x + 40x - 80$$

$$y = 9x^3 - 54x^2 + 137x - 119$$

In the interval  $3 \leq x \leq 4$ , i.e.,  $x_1 \leq x \leq x_2 (i=2)$  the cubic spline is given by

$$y = \left[ \frac{(x_2 - x)^3}{6} M_1 + \frac{(x - x_1)^3}{6} M_2 \right] + \left( \frac{x_2 - x}{1} \right) \left( y_1 - \frac{1}{6} M_1 \right) + \left( \frac{x - x_1}{1} \right) \left( y_2 - \frac{1}{6} M_2 \right)$$

$$\Rightarrow y = \frac{1}{6} [(4-x)^3(54)] + (4-x)(40) + (x-3)(123)$$

$$= 9(64 - 48x + 12x^2 - x^3) + 160 - 40x + 123x - 369$$

$$\Rightarrow y = -9x^3 + 108x^2 + 349x + 367$$

Hence the required cubic spline approximation for the given function is

$$y = \begin{cases} 9x^3 - 54x^2 + 137x - 119 & \text{for } 2 \leq x \leq 3 \\ -9x^3 + 108x^2 - 349x + 367 & \text{for } 3 \leq x \leq 4 \end{cases}$$

## Newton's Forward and Backward Difference Formulas

### Introduction:

If a function  $y=f(x)$  is not known explicitly the value of y can obtained when a set of values of  $(x_i, y_i) i = 1, 2, 3, \dots, n$  are known by using the methods based on the principles of finite differences, provided the function  $y=f(x)$  is continuous. Here *the values of x being equally spaced*, i.e.,  $x_n = x_0 + nh, n = 0, 1, 2, \dots, n$

### Forward Differences

If  $y_0, y_1, y_2, \dots, y_n$  denote the set of values of  $y$ , then the first forward differences of  $y = f(x)$  are defined by

$\Delta y_0 = y_1 - y_0; \Delta y_1 = y_2 - y_1; \dots; \Delta y_{n-1} = y_n - y_{n-1}$   
 where  $\Delta$  is called the forward difference operator.

**Forward Difference table**

<b>x</b>	<b>y</b>	$\Delta$	$\Delta^2$	$\Delta^3$	$\Delta^4$	$\Delta^5$
$x_0$	$y_0$					
		$\Delta y_0$				
$x_1$	$y_1$		$\Delta^2 y_0$			
		$\Delta y_1$		$\Delta^3 y_0$		
$x_2$	$y_2$		$\Delta^2 y_1$		$\Delta^4 y_0$	
		$\Delta y_2$		$\Delta^3 y_1$		$\Delta^5 y_0$
$x_3$	$y_3$		$\Delta^2 y_2$		$\Delta^4 y_1$	
		$\Delta y_3$		$\Delta^3 y_2$		
$x_4$	$y_4$		$\Delta^2 y_3$			
		$\Delta y_4$				
$x_5$	$y_5$					

**Formula**

$$y(x_0 + nh) = y_0 + n\Delta y_0 + \frac{n(n-1)}{2!} \Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!} \Delta^3 y_0 + \dots$$



Problems based on Newton's forward interpolation formula

1. Using Newton's Forward interpolation formula, find  $f(1.5)$  from the following data

x	0	1	2	3	4
f(x)	858.3	869.6	880.9	829.3	903.6

Solution:

x	$x_0=0$	$x_1=1$	$x_2=2$	$x_3=3$	$x_4=4$
f(x)	858.3	869.6	880.9	829.3	903.6

*Difference Table*

To find y for x = 1.5	<b>x</b>	<b>y</b>	$\Delta y$	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
	0	858.3				
			869.6-858.3=11.3			
	1	869.6		11.3-11.3=0		
			880.9-869.6=11.3		0.1-0=0.1	
	2	880.9		11.4-11.3=0.1		-0.2-0.1=-0.3
			892.3-880.9=11.4		-0.1-0.1=-0.2	
By Newt on's forw	3	892.3		11.3-11.4=-0.1		
			903.6-892.3=11.3			
	4	903.6				

ard interpolation formula,

$$y(x_0 + nh) = y_0 + n\Delta y_0 + \frac{n(n-1)}{2!} \Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!} \Delta^3 y_0 + \dots$$

$$\Rightarrow y(1.5) = 869.6 + (0.5)(11.3) + \frac{(0.5)(0.5-1)(0.5-2)}{2} (0) + \frac{(0.5)(0.5-1)(0.5-2)}{2} (0.1) + \frac{(0.5)(0.5-1)(0.5-2)(0.5-3)}{6} (-0.3)$$

$$\Rightarrow y(1.5) = 869.6 + (0.5)(11.3) + \frac{(0.5)(0.5)(1.5)}{2} (0.1) + \frac{(0.5)(0.5)(1.5)(2.5)}{6} (0.3)$$

$$\Rightarrow y(1.5) = 869.6 + 5.65 + \frac{(0.0375)}{2} + \frac{(0.28125)}{6}$$

$$\Rightarrow y(1.5) = 869.6 + 5.65 + 0.01875 + 0.46875$$

$$\Rightarrow y(1.5) = 875.7375$$

2. Using Newton's forward interpolation, find the value of  $\log_{10}^{\pi}$ , given  $\log 3.141 = 0.4970679364$   $\log 3.142 = 0.4972061807$   $\log 3.143 = 0.4973443810$   
 $\log 3.144 = 0.49748253704$   $\log 3.145 = 0.4974825374$

Solution

<b>x</b>	<b>y=logx</b>	$\Delta y$	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
----------	---------------	------------	--------------	--------------	--------------

0	0.4970679364		
		0.1382443x10 <sup>-3</sup>	
1	0.4972061807	11.3-11.3=0	
		0.1382003x10 <sup>-3</sup>	0.1-0=0.1
2	0.4973443810	11.4-11.3=0.1	-0.2-0.1=-0.3
		0.1381564x10 <sup>-3</sup>	-0.1-0.1=-0.2
3	0.49748253704	11.3-11.4=-0.1	
		0.1381124x10 <sup>-3</sup>	
4	0.4974825374		

Here  $x_0=3.141$ ,  $h = 0.001$ ,  $y_0 = 0.4970679364$

The Newton's forward interpolation formula is

$$x_0+nh=\pi=3.1415926536$$

$$n = \frac{3.1415926536 - 3.141}{0.001} = 0.5926536$$

$$y(x_0 + nh) = y_0 + n\Delta y_0 + \frac{n(n-1)}{2!} \Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!} \Delta^3 y_0 + \dots$$

$$\Rightarrow y(\pi) = 0.4970679364 + (0.5926536)(0.000138244) + \frac{0.5926536(-0.4073464)(-0.440 \times 10^{-7})}{2}$$

$$\Rightarrow y(\pi) = 0.4970679364 + 0.0000819310 + 0.0000000053$$

$$\Rightarrow y(\pi) = 0.4971498727$$

3. From the following data, estimate the no. of persons earning weekly wages between 60 and 70 rupees.

Wages(in Rs.)	Below 40	40-60	60-80	80-100	100-120
No. of person(in thousands)	250	120	100	70	50

Solution

x	$x_0=40$	$x_1=60$	$x_2=80$	$x_4=100$	$x_5=120$
y	250	250+120=370	370+100=470	470+70=540	540+50=590

Here  $x_0=40$ ,  $h = 20$ ,  $y_0 = 250$

The Newton's forward interpolation formula is

$$x_0+nh=70$$

$$n = \frac{70 - 40}{20} = 1.5$$

wages x	Frequency(y)	$\Delta y$	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
Below	250				

	40			
$y(x_0 +$		120		
$\Rightarrow y(7$	Below 60	370	-20	
$\Rightarrow y(7$		100	-10	
$\Rightarrow y(7$	Below 80	470	-30	20
No		70	10	
. of	Below 100	540	-20	
per		50		
so	Below 120	590		
ns				
wh				
ose				
we				

ekly wages below 70 =423.5937

No. of persons whose weekly wages below 60 = 370

No. of persons whose weekly wages between 60 and 70 } No. of persons whose weekly wages below 70-  
 } = No. of persons whose weekly wages below 60

$$= 423.5937-370$$

-----

Newton's Backward Interpolation formula

The formula is used mainly to interpolate the values of y near the end of a set of values of x a short distance ahead (to the right) of y.

Formula:

$$y(x_n + nh) = y_n + n\nabla y_n + \frac{n(n+1)}{2!} \nabla^2 y_n + \frac{n(n+1)(n+2)}{3!} \nabla^3 y_n + \dots$$

Problems based on Newton's backward difference formula

1. The following data are taken from the steam table:

Temp:°C	140	150	160	170	180
Pressure kgf/cm <sup>2</sup>	3.685	4.854	6.302	8.076	10.225

Find the pressure at temperature t = 142°C and t = 175°C

Solution: We form the difference table:

t	p	Δp	Δ <sup>2</sup> p	Δ <sup>3</sup> p	Δ <sup>4</sup> p
140	3.685				
		1.169			
150	4.854		0.279		
		1.448		0.047	
160	6.302		0.326		0.002
		1.774		0.049	
170	8.076		0.375		
		2.149			
180	10.225				

$$+ \frac{0.2 \overbrace{(-0.8)} \overbrace{(-1.8)}}{6} \times 0.047 + \frac{0.2 \overbrace{(-0.8)} \overbrace{(-1.8)} \overbrace{(-2.8)}}{24} \times 0.002$$

$$= 3.685 + 0.2338 - 0.02332 + 0.002256 - 0.0000672$$

$$= 3.897668$$

$$= 3.898$$

$$P_4 (t = 175^\circ) = P_4 \left[ 180 + \left( -\frac{1}{2} \right) \times 10 \right], \text{ where } v = \frac{175 - 180}{10} = -0.5$$

$$= P_n + v\nabla P_n + \frac{v(v+1)}{2} \nabla^2 P_n + \dots$$

$$= 10.225 + (-0.5) \times 2.149 + \frac{(-0.5)(-0.5)}{2} \times 0.375$$

$$+ \frac{(-0.5)(-0.5)(-0.5)}{6} \times 0.049 + \frac{(-0.5)(-0.5)(-0.5)(-0.5)}{24} \times 0.002$$

$$= 10.225 - 1.0745 - 0.0046875 - 0.0030625 - 0.000078125$$

$$= 9.10048438 = 9.100$$

Unit-II  
Part-A

1. State the Lagrange's formula to find  $y(x)$  if three sets of values  $(x_0, y_0)$ ,  $(x_1, y_1)$  and  $(x_2, y_2)$  are given.

Ans: 
$$y(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2$$

2. Explain the inverse interpolation and the use of Lagrange's interpolation formula for inverse interpolation.

Ans: Lagrange's interpolation formula is a relation between two variables  $x$  and  $y$  in which either  $x$  or  $y$  is taken as independent variable. Replacing  $x$  by  $y$  and  $y$  by  $x$  in Lagrange's formula, we can use the resulting formula for finding  $x$  for a given  $y$ .

3. If  $f(3)=5$  and  $f(5)=3$  What is the form of  $f(x)$  by Lagrange's formula?

Ans: 
$$y = \frac{x-5}{3-5} (5) + \frac{x-3}{5-3} (3) \Rightarrow y = -\frac{5}{2} (x-5) + \frac{3}{2} (x-3)$$

$$\Rightarrow y = 8-x \Rightarrow f(x) = 8-x \Rightarrow f(x) = 8-x$$

4. State any two properties of divided difference.

Ans: The divided difference are symmetrical in all their arguments.

The divided differences of sum or difference of two functions is equal to the sum or difference of the corresponding separate divided differences.

5. Find the divided difference for the data

$x$	2	5	10	<u>Ans:</u>	$x$	$y$	$\Delta y$	$\Delta^2 y$
$y$	5	29	109		2	5	$\frac{29-5}{3}=8$	$\frac{16-8}{10-2}=1$
					5	29	$\frac{109-29}{10-5}=16$	
					10	109		

6. State the Newton's divided difference interpolation formula.

Ans:  $f(x) = f(x_0) + (x-x_0)f(x_0, x_1) + (x-x_0)(x-x_1)f(x_0, x_1, x_2) + \dots$   
 $+ (x-x_0)(x-x_1)\dots(x-x_{n-1})f(x_0, x_1, x_2, \dots, x_n)$

7. Using divided difference, show that  $f(x, x) = f'(x)$  through limiting process.

Ans:  $f(x, x+h) = \frac{f(x+h) - f(x)}{x+h-x} = \frac{f(x+h) - f(x)}{h}$

Taking limit as  $h \rightarrow 0$ ,  $f(x, x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = f'(x)$

8. Show that  $f(x_0, x_1) = f(x_1, x_0) = \frac{f(x_0) - f(x_1)}{x_0 - x_1}$

Ans:  $f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(x_0) - f(x_1)}{x_0 - x_1} = f(x_1, x_0)$

Again  $f(x_0, x_1) = \frac{f(x_0) - f(x_1)}{x_0 - x_1} = \frac{f(x_0)}{x_0 - x_1} + \frac{f(x_1)}{x_1 - x_0} = f(x_1, x_0)$

9. Fit a polynomial of least degree to the following data,

$x$	1	2	4	<u>Ans:</u>	$x_0=1$	$x_1=2$	$x_2=4$
$y$	5	10	26		$f(x_0)=5$	$f(x_1)=10$	$f(x_2)=26$

$x$	$y$	$f(x)$	$f'(x)$
1	5		
2	10	5	1
4	26	8	

By divided difference formula

$$y = 5 + (x-1)5 + (x-1)(x-2) = x^2 + 2x + 2.$$

10. If  $f(x) = \frac{1}{x^2}$ , find  $f(a, b)$  and  $f(2, 3)$ .

Ans:  $f(a) = \frac{1}{a^2}$ ,  $f(b) = \frac{1}{b^2}$ ,  $f(a, b) = \frac{f(b) - f(a)}{b - a} = \frac{\frac{1}{b^2} - \frac{1}{a^2}}{b - a}$

$$\Rightarrow f(a, b) = -\frac{(a+b)}{a^2 b^2} \quad \therefore f(2, 3) = \frac{-5}{(4)(9)} = \frac{-5}{36}$$

11. State the condition required for a natural cubic spline.

Ans: A cubic spline  $g(x)$  fits to each of the points is continuous and is continuous in slope and curvature such that  $S_0 = g_0''(x_0) = 0$  and  $S_n = g_{n-1}''(x_n) = 0$  is called a natural cubic spline.

12. What are the  $n^{\text{th}}$  divided difference of polynomial of  $n^{\text{th}}$  degree?

Ans: The  $n^{\text{th}}$  divided difference of an  $n^{\text{th}}$  degree polynomial are constant.

13. When will we use Newton's forward interpolation formula?

Ans: Newton's forward interpolation formula is used when

interpolation is required near the beginning of the table value and for extrapolation at a short distance from the initial value  $x_0$ .

13. State the Newton's backward interpolation is used to interpolate the values of  $y = f(x)$  nearer to the end of a set of table values and for extrapolation closer to the right of  $y_n$ .

14. Given  $f(0) = -1$ ,  $f(1) = 1$  and  $f(2) = 4$ . Find the Newton interpolating polynomial

Ans:

$y_0 = -1$	$\Delta y_0 = 1 - (-1)$	$\Delta y_1 = 4 - 1 = 3$	$\Delta^2 y_0 = 3 - 2 = 1$
$y_1 = 1$	$= y_1 - y_0$	$= y_2 - y_1$	
$y_2 = 4$			

$$n = \frac{x - x_0}{h} = x \quad f(x) = -1 + 2x + \frac{x(x-1)}{2} = \frac{1}{2}(x^2 + 3x - 2)$$

15. Find the sixth term in the sequence 8, 12, 19, 29, 42, ...

Ans:

$$\begin{aligned}
 y_5 &= E^5 y_0 \\
 &= (1 + \Delta)^5 y_0 \\
 &= 1 + 5\Delta + 10\Delta^2 y_0 \\
 &= 8 + 20 + 30 = 58.
 \end{aligned}$$

16. Given  $f(0) = -1$ ,  $f(1) = 1$  and  $f(2) = 4$ , find the roots of polynomial equation  $f(x) = 0$ .

Ans:

$x$	$y$	$\Delta y$	$\Delta^2 y$	$h = x$
0	-1			$f(x) = -1 + 2x + \frac{x(x-1)}{2}$ (1)
1	1	2		$= \frac{1}{2}(x^2 + 3x - 2)$
2	4	3	1	$f(x) = 0 \Rightarrow x = \frac{-3 \pm \sqrt{17}}{2}$



Part - B

1. Using Lagrange's formula to calculate  $f(3)$  from the following data

$x$	0	1	2	4	5	6
$y$	1	14	15	5	6	19

2. Using Lagrange's formula, fit a polynomial to the data

$x$	0	1	3	4
$y$	-12	0	6	12

3. Using Lagrange's interpolation find the polynomial through  $(0,0)$ ,  $(1,1)$  and  $(2,2)$

4. Find the cubic spline approx. for the function  $y=f(x)$  from the following data, given that  $y_0'' = y_3'' = 0$ .

$x$	-1	0	1	2
$y$	-1	1	3	35

5. Using Newton's backward difference formula to construct an interpolating polynomial of degree 3 for the data.

$$f(-0.75) = -0.07181250 \quad f(-0.5) = -0.024750 \quad f(-0.25) = 0.33493750$$

$$f(0) = 1.10100. \text{ Hence find } f(-1/3)$$

6. Using Lagrange's interpolation formula find  $f(x)$  from the following data

$x$	1	2	3	4	7
$f(x)$	2	4	8	16	128

7. From the following data, estimate the no. of persons earning weekly between 60 and 70 rupees

wages below 40	40-60	60-80	80-100	100-120	
No. of persons	250	370	470	540	590

8. Find the cubic polynomial following table using Newton's divided difference formula and hence find  $f(4)$

$x$	0	1	2	5
$f(x)$	2	3	12	147

9. For the given values evaluate  $f(9)$  using Lagrange's formula

$x$	5	7	11	13	17
$y$	150	392	1452	2366	5202

10. Fit the cubic spline for the data

$x$	1	2	3
$y$	-6	-1	16

Hence evaluate  $y(1.5)$  given that  $y_0'' = 0, y_2'' = 0$

11. Obtain the root of  $f(x) = 0$  by Lagrange's inverse interpolation given that  $f(30) = -30, f(34) = -13, f(38) = 3, f(42) = 18$ .

12. The following values of  $x$  &  $y$  are given

$x$	1	2	3	4
$y$	1	2	5	11

find the cubic spline and evaluate  $y(1.5)$

13. From the following table of half-yearly premium for policies maturing at different ages, estimate the premium for policies maturing at age 46

Age ( $x$ )	45	50	55	60	65
Premium ( $y$ )	114.84	96.16	83.32	74.48	68.48

14. Using Newton's Divided difference formula find  $f(x)$  &  $f'(0)$  from the following data.

$x$	1	2	7	8
$y$	1	5	5	4

# UNIT III

## NUMERICAL DIFFERENTIATION AND INTEGRATION

*Differentiation using interpolation formulae*

*Numerical integration by trapezoidal rule*

*Simpson's 1/3 and 3/8 rules*

*Romberg's method*

*Two and Three point Gaussian quadrature formulas*

*Double integrals using trapezoidal and simpsons's rules.*

**Numerical Differentiation**

- Differentiation using Forward Interpolation formula(for equal interval)
- Differentiation using Backward Interpolation formula(for equal interval)
- Differentiation using Stirling's(Central Difference) Formula(for equal interval)
- Maximum and Minimum
- Differentiation using divided difference(for unequal interval)

**Forward difference formula to compute the derivative**

Newton's forward interpolation formula is

$$f(x_0 + rh) = y_0 + \Delta y_0 + \frac{r(r-1)}{2!} \Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!} \Delta^3 y_0 + \frac{r(r-1)(r-2)(r-3)}{4!} \Delta^4 y_0 + \dots$$

(Here using of r for n is only for convenience)

Differentiating w.r.t. we get,

$$hf'(x_0 + rh) = \Delta y_0 + \frac{2r-1}{2} \Delta^2 y_0 + \frac{3r^2 - 6r + 2}{6} \Delta^3 y_0 + \frac{2r^3 - 9r^2 + 11r - 3}{12} \Delta^4 y_0 + \dots \quad \dots$$

(1)

$$h^2 f''(x_0 + rh) = \Delta^2 y_0 + (r-1) \Delta^3 y_0 + \frac{6r^2 - 18r + 11}{12} \Delta^4 y_0 + \dots$$

... (2)

$$h^3 f'''(x_0 + rh) = \Delta^3 y_0 + \frac{2r-3}{2} \Delta^4 y_0 + \dots$$

...(3)

Similarly we can find the remaining derivatives.

If we want to find the derivatives at a point  $x = x_0$ , then  $x_0 + rh = x_0$

i.e.,  $r=0$ .

Hence on substituting this value of  $r=0$  in the above formula (1), (2) and (3), we get

$$f'(x_0) = \frac{1}{h} \left[ \Delta y_0 - \frac{1}{2} \Delta^2 y_0 + \frac{1}{3} \Delta^3 y_0 - \frac{1}{4} \Delta^4 y_0 \dots \right]$$

$$f''(x_0) = \frac{1}{h^2} \left[ \Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12} \Delta^4 y_0 - \dots \right]$$

$$f'''(x_0) = \frac{1}{h^3} \left[ \Delta^3 y_0 + \frac{3}{2} \Delta^4 y_0 + \dots \right] \text{ and so on.}$$

Note: If the x value is nearer to the starting of the given table we use Forward Interpolation formula

**Backward difference formula to compute the derivatives**

Newton's backward difference formula is

$$f(x_n + rh) = y_n + \nabla y_n + \frac{r(r+1)}{2!} \nabla^2 y_n + \frac{r(r+1)(r+2)}{3!} \nabla^3 y_n + \frac{r(r+1)(r+2)(r+3)}{4!} \nabla^4 y_n + \dots$$

$$f'(x_n + rh) = \frac{1}{h} \left[ \nabla y_n + \frac{r+1}{2} \nabla^2 y_n + \frac{2r^2 + 6r + 2}{6} \nabla^3 y_n + \frac{2r^3 + 6r^2 + 11r + 3}{12} \nabla^4 y_n + \dots \right]$$

$$h^2 f''(x_n + rh) = \nabla^2 y_n + (r+1) \nabla^3 y_n + \frac{6r^2 + 18r + 11}{12} \nabla^4 y_n + \dots$$

$$h^3 f'''(x_n + rh) = \nabla^3 y_n + \frac{2r+3}{2} \nabla^4 y_n + \dots$$

Similarly we can find the remaining derivatives.

At the point  $x = x_n$ , i.e.,  $x_n + rh = x_n$ , we have  $r=0$ .

$$f'(x_n) = \frac{1}{h} \left[ \nabla y_n - \frac{1}{2} \nabla^2 y_n + \frac{1}{3} \nabla^3 y_n - \frac{1}{4} \nabla^4 y_n + \dots \right]$$

$$f''(x_n) = \frac{1}{h^2} \left[ \nabla^2 y_n + \nabla^3 y_n + \frac{11}{12} \nabla^4 y_n - \dots \right]$$

$$f'''(x_n) = \frac{1}{h^3} \left[ \nabla^3 y_n + \frac{3}{2} \nabla^4 y_n + \dots \right] \text{ and so on.}$$

Note: If the x value is nearer to the end of the given table we use backward Interpolation formula

### Central difference formula for computing the derivatives

We know that Stirling's central difference formula is

$$f(x_0 + rh) = y_0 + r \left( \frac{\Delta y_0 + \Delta y_{-1}}{2} \right) + \frac{r^2}{2!} \nabla^2 y_{-1} + \frac{r(r^2-1)}{3!} \left( \frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} \right) + \frac{r^2(r^2-1)}{4!} \nabla^4 y_{-2} + \frac{r(r^2-1)(r^2-4)}{5!} (\Delta^5 y_{-2} + \Delta^5 y_{-3}) + \dots$$

(Here using of r for n is only for convenience)

$$f(x_0 + rh) = y_0 + r \left( \frac{\Delta y_0 + \Delta y_{-1}}{2} \right) + \frac{r^2}{2!} \nabla^2 y_{-1}$$

$$\text{i.e., } + \frac{r^3 - r}{3!} \left( \frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} \right) + \frac{r^4 - r^2}{4!} \nabla^4 y_{-2}$$

$$+ \frac{r^5 - 5r^3 + 4r}{5!} (\Delta^5 y_{-2} + \Delta^5 y_{-3}) + \dots$$

Differentiating (1) w.r.t. 'r' we get,

$$hy'(x_0 + rh) = \left( \frac{\Delta y_0 + \Delta y_{-1}}{2} \right) + \Delta^2 y_{-1} + \left( \frac{3r^2 - 1}{12} \right) (\Delta^3 y_{-1} + \Delta^3 y_{-2}) + \left( \frac{2r^3 - r}{12} \right) \Delta^4 y_{-2} + \left( \frac{5r^4 - 15r + 4}{5!} (\Delta^5 y_{-2} + \Delta^5 y_{-3}) \right) + \dots$$

$$\therefore y'(x_0 + rh) = \frac{1}{h} \left[ \left( \frac{\Delta y_0 + \Delta y_{-1}}{2} \right) + \Delta^2 y_{-1} + \left( \frac{3r^2 - 1}{12} \right) (\Delta^3 y_{-1} + \Delta^3 y_{-2}) + \left( \frac{2r^3 - r}{12} \right) \Delta^4 y_{-2} + \left( \frac{r^4 - 15r^2 + 4}{5!} (\Delta^5 y_{-2} + \Delta^5 y_{-3}) \right) + \dots \right]$$

...(2)

Differentiating (2) w.r.t. 'r', we get,

$$y'' \left( x_0 + rh \right) = \frac{1}{h^2} \left[ \Delta^2 y_{-1} + \frac{r}{2} \left( \Delta^3 y_{-1} + \Delta^3 y_{-2} \right) + \left( \frac{6r^2 - 1}{12} \right) \Delta^4 y_{-2} + \left( \frac{20r^3 - 30r}{5!} \right) \left( \Delta^5 y_{-2} + \Delta^5 y_{-3} \right) + \dots \right] \text{-----(3)}$$

Similarly we can find the remaining derivatives.

If we want to find the derivative at a point  $x = x_0$ , then  $x_0 + rh = x_0$

i.e.,  $r=0$

Substituting  $r=0$  in (2) and (3) we get,

$$y' \left( x_0 \right) = \frac{1}{h} \left[ \left( \frac{\Delta y_0 + \Delta y_{-1}}{2} \right) - \frac{1}{12} \left( \Delta^3 y_{-1} + \Delta^3 y_{-2} \right) + \frac{1}{30} \left( \Delta^5 y_{-2} + \Delta^5 y_{-3} + \dots \right) \right]$$

$$y'' \left( x_0 \right) = \frac{1}{h^2} \left[ \Delta^2 y_{-1} - \frac{1}{12} \Delta^4 y_{-2} + \dots \right]$$

Similarly we can find the remaining derivatives.

**Note:**

✍ If the  $x$  value is middle of the given table we use central Difference formula

**Maximum and Minimum:**

**Steps**

- Write the Newton's Forward difference formula  $y(x)$ ,  $\frac{dy}{dx}$
- Write the forward difference table
- Find  $\frac{dy}{dx}$
- Put  $\frac{dy}{dx} = 0$  and find the value of  $x$
- Find  $\frac{d^2y}{dx^2}$
- For every value of  $x$  find  $\frac{d^2y}{dx^2}$
- If  $\frac{d^2y}{dx^2} < 0$ ,  $y$  is maximum at that  $x$ (maximum point)
- If  $\frac{d^2y}{dx^2} > 0$ ,  $y$  is minimum at that  $x$ (minimum point)
- To find the maximum and minimum value substitute the maximum and minimum points in  $y(x)$  formula respectively.

**Problems based on Differentiation using Interpolation formula**

1. Find the first, second and third derivatives of the function tabulated below at the point  $x=1.5$

x	1.5	2.0	2.5	3.0	3.5	4.0
f(x)	3.375	7.0	13.625	24.0	38.875	59.0

**Solution**

The difference table is as follows:

x	y=f(x)	$\Delta y$	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$1.5x_0$	$3.375y_0$	$3.625 \left( \Delta y_0 \right)$			
2.0	7.0		$3.0 \left( \Delta^2 y_0 \right)$		
		6.625		$0.75 \left( \Delta^3 y_0 \right)$	
2.5	13.625		3.75		0
		10.375		0.75	
3.0	24.0		4.50		0
		14.875		0.75	
3.5	38.875		5.25		
		20.125			
4.0	59.0				

Here we have to find the derivative at the point  $x=1.5$  which is the initial value of the table. Therefore by Newton's forward difference formula for derivatives at  $x=x_0$ , we have

$$f'(x_0) = \frac{1}{h} \left[ \Delta y_0 - \frac{1}{2} \Delta^2 y_0 + \frac{1}{3} \Delta^3 y_0 - \dots \right]$$

Here  $x_0=1.5$ ,  $h=0.5$

$$\therefore f'(1.5) = \frac{1}{0.5} \left[ 3.625 - \frac{1}{2} (3.0) + \frac{1}{3} (0.75) - \dots \right]$$

$$f'(1.5) = 4.75$$

At the point  $x = x_0$ ,

$$f''(x_0) = \frac{1}{h^2} \left[ \Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12} \Delta^4 y_0 - \dots \right]$$

Here  $x_0=1.5$ ,  $h=0.5$

$$\therefore f''(1.5) = \frac{1}{(0.5)^2} [1.0 - 0.75]$$

$$f''(1.5) = 9.0$$

At the point  $x = x_0$ ,

$$f'''(x_0) = \frac{1}{h^3} \left[ \Delta^3 y_0 + \frac{3}{2} \Delta^4 y_0 \right]$$

$$f'''(1.5) = \frac{1}{(0.5)^3} (0.75) = 6.0$$

$$f'''(1.5) = 6.0$$

-----2.  
Compute  $f'(0)$  and  $f'(4)$  from the data

x	0	1	2	3	4
y	1	2.718	7.381	20.086	54.598

**Solution:**

x	y=f(x)	$\Delta y$	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$0x_0$	$1y_0$				
		$1.718 (\Delta y_0)$			
1	2.718		$2.945 (\Delta^2 y_0)$		
		4.663		$5.097 (\Delta^3 y_0)$	
2	7.381		8.042		$8.668 (\Delta^4 y_0)$
		12.705		$13.765 (\Delta^3 y_n)$	$(\Delta^4 y_n)$
3	20.086		$21.807 (\Delta^2 y_n)$		
		$34.512 (\Delta y_n)$			
4	54.598				
$x_n$	$y_n$				

Here we have to find  $f'(0)$  .ie.x=0 which is the starting of the given table. So we use the forward interpolation formula.



$$f' \left( x_0 \right) \approx \frac{1}{h} \left[ \Delta y_0 - \frac{1}{2} \Delta^2 y_0 + \frac{1}{3} \Delta^3 y_0 - \frac{1}{4} \Delta^4 y_0 \dots \right]$$

$$f' \left( 1 \right) \approx \frac{1}{1} \left[ 1.718 - \frac{1}{2} 2.945 + \frac{1}{3} 5.097 - \frac{1}{4} 8.668 \right] = -0.2225$$

Here we have to find  $f'(4)$  .ie.x=4 which is the end of the given table. So we use the backward interpolation formula.

$$f'' \left( x_n \right) \approx \frac{1}{h^2} \left[ \nabla^2 y_n + \nabla^3 y_n + \frac{11}{12} \nabla^4 y_n - \dots \right]$$

$$f'' \left( 4 \right) \approx \frac{1}{1^2} \left[ 21.807 + 13.765 + \frac{11}{12} 8.668 \right] = 43.5177$$

2. Find  $\frac{dy}{dx}$  and  $\frac{d^2y}{dx^2}$  at  $x = 51$  from the following data.

x	50	60	70	80	90
y	19.96	36.65	58.81	77.21	94.61

Solution:

Here  $h=10$ . To find the derivatives of  $y$  at  $x=51$  we use Forward difference formula taking the origin at  $x_0=50$ .

We have  $r = \frac{x - x_0}{h} = \frac{51 - 50}{10} = 0.1$

$\therefore$  at  $x=51$ ,  $r=0.1$

$$\left(\frac{dy}{dx}\right)_{x=51} = \left(\frac{dy}{dx}\right)_{r=0.1} = \frac{1}{h} \left[ \Delta y_0 + \frac{2r-1}{2} \Delta^2 y_0 + \frac{3r^2-6r+2}{6} \Delta^3 y_0 + \frac{2r^3-9r^2+11r-3}{12} \Delta^4 y_0 + \dots \right]$$

The difference table is given by

x	y=f(x)	$\Delta y$	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
<b>50</b> 0	<b>19.96</b> 0				
		<b>16.69</b> $\Delta y_0$			
<b>60</b>	36.65		<b>5.47</b> $\Delta^2 y_0$		
		22.1 6		<b>-9.23</b> $\Delta^3 y_0$	
<b>70</b>	58.81		-3.76		<b>11.09</b> $\Delta^4 y_0$
		18.4 0		<b>2.76</b>	
<b>80</b>	77.21		<b>-1.00</b>		
		<b>17.4</b> 0			

90	94.61				
----	-------	--	--	--	--

$$\left(\frac{dy}{dx}\right)_{x=51} = \left(\frac{dy}{dx}\right)_{r=0.1} = \frac{1}{10} \left[ 16.69 + \frac{0.2-1}{2}(5.47) + \left[ \frac{3(0.1)^2 - 6(0.1) + 2}{6} \right] (-9.23) + \left[ \frac{2(0.1)^3 - 9(0.1)^2 + 11(0.1) - 3}{12} \right] (1.99) \right]$$

$$= \frac{1}{10} [6.69 - 2.188 - 2.1998 - 1.9863] = 1.0316$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=51} = \left(\frac{d^2y}{dx^2}\right)_{r=0.1} = \frac{1}{h^2} \left[ \Delta^2 y_0 + (-1) \Delta^3 y_0 + \frac{6r^2 - 18r + 11}{12} \Delta^4 y_0 + \dots \right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=51} = \left(\frac{d^2y}{dx^2}\right)_{r=0.1} = \frac{1}{100} \left[ 5.47 + (-1) (-9.23) + \frac{6(0.1)^2 - 18(0.1) + 11}{12} (11.99) \right]$$

$$= \frac{1}{100} [5.47 + 8.307 + 9.2523] = 0.2303$$

3. Find the maximum and minimum value of y tabulated below.

x	-2	-1	0	1	2	3	4
y	2	-0.25	0	-0.25	2	15.75	56

**Solution:**

$$\frac{dy}{dx} = \frac{1}{h} \left[ \Delta y_0 + \frac{2r-1}{2} \Delta^2 y_0 + \frac{3r^2-6r+2}{6} \Delta^3 y_0 + \frac{2r^3-9r^2+11r-3}{12} \Delta^4 y_0 + \dots \right]$$

x	y=f(x)	Δy	Δ <sup>2</sup> y	Δ <sup>3</sup> y	Δ <sup>4</sup> y	Δ <sup>5</sup> y
-2	2					
		-2.25				
-1	-0.25		2.5			
		0.25		-3		
0	0		-0.5		6	
		-0.25		3		0
1	-0.25		2.5		6	
		2.25		9		0
2	2		11.5		6	
		13.75		15		
3	15.75		26.5			
		40.25				
4	56					

Choosing  $x_0=0, r = \frac{x-0}{1} = x$

$$\frac{dy}{dx} = \frac{1}{1} \left[ -0.25 + \frac{2x-1}{2} (2.5) + \frac{3x^2-6x+2}{6} (9) + \frac{2x^3-9x^2+11x-3}{12} (6) \right]$$

$$= \frac{1}{1} [0.25 + 2.5x - 1.25 + 4.5x^2 - 9x + 3 + x^3 - 4.5x^2 + 5.5x - 1.5]$$

$$\frac{dy}{dx} = x^3 - x$$

$$\text{Now } \frac{dy}{dx} = 0 \Rightarrow x^3 - x = 0$$

$$\Rightarrow x = 0, x = 1, x = -1.$$

$$\frac{d^2y}{dx^2} = 3x^2 - 1$$

$$\text{at } x=0 \frac{d^2y}{dx^2} = -ve \quad \text{at } x=1 \frac{d^2y}{dx^2} = +ve \quad \text{at } x=-1 \frac{d^2y}{dx^2} = +ve$$

∴ y is maximum at x=0, minimum at x=1 and -1

$$\therefore y(x) = \left[ y_0 + x\Delta y_0 + \frac{x(x-1)}{2!} \Delta^2 y_0 + \dots \right]$$

Maximum value = y(0) = 0, Minimum value = y(1) = -0.25.

4. Consider the following table of data

x	0.2	0.4	0.6	0.8	1.0
f(x)	0.9798652	0.9177710	0.8080348	0.6386093	0.3843735

Find f'(0.25), f'(0.6) and f'(0.95).

**Solution:**

Here h=0.2

∴ 0.25 is nearer to the starting of the given table. So we use Newton's forward interpolation formula to evaluate f'(0.25)

∴ 0.95 is nearer to the ending of the given table. So we use Newton's backward interpolation formula to evaluate f'(0.95)

∴ 0.6 is middle point of the given table. So we use Central Difference formula to evaluate f'(0.6)

The difference table

x	y=f(x)	Δy	Δ <sup>2</sup> y	Δ <sup>3</sup> y	Δ <sup>4</sup> y
0.2x <sub>0</sub>	0.9798652y <sub>0</sub>				
		<b>-0.0620942</b> ↙ y <sub>0</sub>			
0.4	0.9177710		<b>0.047642</b> ↘ y <sub>0</sub>		
		-0.1097362		<b>-0.0120473</b> ↘ y <sub>0</sub>	
0.6	0.8080348		-0.0596893		↘ y <sub>0</sub>
					<b>0.01310985</b> ↘ y <sub>n</sub>
		-0.1694255		-	

				<b>0.02515715</b> ( $\Delta^3 y_n$ )	
0.8	0.6386093		- <b>0.08484645</b> ( $\Delta^2 y_n$ )		
		- <b>0.25427195</b> ( $\Delta y_n$ )			
1.0	0.3843735				
$x_n$	$y_n$				

**To find  $f'(0.25)$**

**Newton's forward interpolation formula for derivative**

$$hf'(\xi_0 + rh) \cong \Delta y_0 + \frac{2r-1}{2} \Delta^2 y_0 + \frac{3r^2-6r+2}{6} \Delta^3 y_0 + \frac{2r^3-9r^2+11r-3}{12} \Delta^4 y_0 + \dots$$

$$f'(0.25) \cong \frac{1}{0.2} \left[ \begin{aligned} & -0.0620942 + \frac{2(0.25)-1}{2} (-0.047642) \\ & + \frac{3(0.25)^2 - 6(0.25) + 2}{6} (-0.0120473) \\ & + \frac{2(0.25)^3 - 9(0.25)^2 + 11(0.25) - 3}{12} (-0.01310985) \end{aligned} \right]$$

= -0.2536 (correct to four decimal places)

**To find  $f'(0.95)$**

x	y=f(x)	$\Delta y$	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.2 $x_0$	0.9798652 ( $y_{-2}$ )				
		<b>-0.0620942</b> ( $\Delta y_{-2}$ )			
0.4	0.9177710 ( $y_{-1}$ )		<b>0.047642</b> ( $\Delta^2 y_{-2}$ )		
		-0.1097362 ( $\Delta y_{-1}$ )		<b>-0.0120473</b> ( $\Delta^3 y_{-2}$ )	
0.6	0.8080348 ( $y_0$ )		-0.0596893 ( $\Delta^2 y_{-1}$ )		<b>-0.01310985</b> ( $\Delta^4 y_{-2}$ )
		-0.1694255 ( $\Delta y_0$ )		<b>-0.02515715</b> ( $\Delta^3 y_{-1}$ )	
0.8	0.6386093 ( $y_1$ )		<b>-0.08484645</b> ( $\Delta^2 y_0$ )		
		<b>-0.25427195</b> ( $\Delta y_1$ )			
1.0	0.3843735 ( $y_2$ )				

**Newton's backward interpolation formula for derivative**

$$f'(x_n + rh) = \frac{1}{h} \left[ \nabla y_n + \frac{r+1}{2} \nabla^2 y_n + \frac{2r^2 + 6r + 2}{6} \nabla^3 y_n + \frac{2r^3 + 6r^2 + 11r + 3}{12} \nabla^4 y_n + \dots \right]$$

$$r = \frac{x - x_n}{h} = \frac{0.95 - 1}{0.2} = -0.25$$

$$f'(0.95) = \frac{1}{0.2} \left[ \begin{aligned} & -0.25427195 + \frac{2(-0.25)+1}{2} (-0.08484645) \\ & + \frac{3(-0.25)^2 + 6(-0.25) + 2}{6} (0.02515715) \\ & + \frac{2(-0.25)^3 + 9(-0.25)^2 + 11(-0.25) + 3}{12} (-0.01310985) \end{aligned} \right]$$

$f'(0.95) = -1.71604$

**To find  $f'(0.6)$**

**Central Difference formula (Stirling's Formula)**

$$f'(x_0 + rh) = \frac{1}{h} \left[ \left( \frac{\Delta y_0 + \Delta y_{-1}}{2} \right) + r \Delta^2 y_{-1} + \left( \frac{3r^2 - 1}{12} \right) (\Delta^3 y_{-1} + \Delta^3 y_{-2}) + \left( \frac{2r^3 - r}{12} \right) \Delta^4 y_{-2} + \left( \frac{r^4 - 15r^2 + 4}{5!} \right) (\Delta^5 y_{-2} + \Delta^5 y_{-3}) + \dots \right]$$

$$f'(0.6) = \frac{1}{0.2} \left[ \begin{aligned} & \left( \frac{-0.1694255 - 0.1097362}{2} \right) + (0.2)(-0.00596893) \\ & + \left( \frac{3(0.2)^2 - 1}{12} \right) (0.02515715 - 0.0120473) + \left( \frac{2(0.2)^3 - 0.2}{12} \right) (-0.01310985) \dots \end{aligned} \right]$$

$f'(0.6) = -0.74295$  (correct to 5 decimal places)

5. Given the following data, find  $y'(6)$ ,  $y'(5)$  and the maximum value of  $y$

x	0	2	3	4	7	9
y	4	26	58	112	466	922

**Solution:**

Since the intervals are not equal, we will use Newton's divided difference formula.

Divided Difference Table

x	y=f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	4				
		11=f(x <sub>0</sub> ,x <sub>1</sub> )			
2	26		7= f(x <sub>0</sub> ,x <sub>1</sub> ,x <sub>2</sub> )		
		32		1= f(x <sub>0</sub> ,x <sub>1</sub> ,x <sub>2</sub> ,x <sub>3</sub> )	
3	58		11		0= f(x <sub>0</sub> ,x <sub>1</sub> ,x <sub>2</sub> ,x <sub>3</sub> , x <sub>4</sub> )
		54		1	
4	112		16		0
		118		1	
7	466		22		
		228			
9	922				

By Newton's Divided Difference formula,

$$\begin{aligned} y &= f(x) = f(x_0) + (x-x_0)f(x_0, x_1) + (x-x_0)(x-x_1)f(x_0, x_1, x_2) + \dots \\ &= 4 + (x-0)11 + (x-0)(x-2)7 + (x-0)(x-2)(x-3)1 \\ &= x^3 + 2x^2 + 3x + 4 \end{aligned}$$

Therefore,  $f'(x) = 3x^2 + 4x + 3$   
 $f'(6) = 135$   
 $f'(5) = 98$ .

## Numerical Integration

### ★ Single integral

- Trapezoidal
- Simpson's one-third rule
- Simpson's three-eighth rule
- Romberg method
- Two and Three point Gaussian Quadrature Formulas

### ★ Double integral

- Trapezoidal rule
- Simpson's Rule

## Single Integral

### Trapezoidal Rule

$$\int_{x_0}^{x_n} f(x) dx = \frac{h}{2} \left[ \text{Sum of the first and last ordinates} + 2 \left( \text{Sum of the remaining ordinates} \right) \right]$$

$$\text{ie, } \int_{x_0}^{x_n} f(x) dx = \frac{h}{2} \left[ y_0 + y_n + 2(y_1 + y_2 + \dots + y_{n-1}) \right]$$

### Simpson's one third rule

$$\int_{x_0}^{x_n} f(x) dx = \frac{h}{3} \left[ \text{Sum of the first and last ordinates} + 2 \left( \text{Sum of the remaining odd ordinates} \right) + 4 \left( \text{Sum of the remaining even ordinates} \right) \right]$$

$$\text{ie, } \int_{x_0}^{x_n} f(x) dx = \frac{h}{3} \left[ y_0 + y_n + 2(y_1 + y_3 + \dots + y_{n-1}) + 4(y_2 + y_4 + \dots + y_{n-2}) \right]$$

### Simpson's three eight rule

$$\int_{x_0}^{x_n} f(x) dx = \frac{3h}{8} \left[ \text{Sum of the first and last ordinates} + 3 \left( \text{Sum of the remaining ordinates which are not divisible by 3} \right) + 2 \left( \text{Sum of the remaining ordinates which are divisible by 3} \right) \right]$$

$$\int_{x_0}^{x_n} f(x) dx = \frac{h}{8} \left[ y_0 + y_n + 3(y_1 + y_2 + y_4 + y_5 + \dots + y_{n-1}) + 2(y_3 + y_6 + \dots + y_{n-3}) \right]$$

Rule	Degree of y(x)	No.of intervals	Error	Order
Trapezoidal Rule	One	Any	$ E  < \frac{(b-a)h^2 M}{12}$	$h^2$
Simpson's one third rule	Two	Even	$ E  < \frac{(b-a)h^4 M}{180}$	$h^4$
Simpson's three eight rule	three	Multiple of 3	$ E  < \frac{3h^5}{8}$	$h^5$

### Two and Three point Gaussian Quadrature Formulas

#### Gaussian Two point formula

☞ If the Limit of the integral is -1 to 1 then we apply  $\int_{-1}^1 f(x)dx = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$ . This formula is exact for polynomials upto degree 3.

☞ If  $\int_a^b f(x)dx$  then  $x = \frac{b-a}{2}t + \frac{b+a}{2}$  and  $dx = \frac{b-a}{2}$  using these conditions convert  $\int_a^b f(x)dx$  into  $\int_{-1}^1 f(t)dt$  and then we apply the formula  $\int_{-1}^1 f(t)dt = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$

#### Gaussian Three point formula

☞ If the Limit of the integral is -1 to 1 then we apply  $\int_{-1}^1 f(x)dx = \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right] + \frac{8}{9}f(0)$ . This formula is exact for polynomials upto degree 5.

☞ Otherwise  $\int_a^b f(x)dx$  then  $x = \frac{b-a}{2}t + \frac{b+a}{2}$  and  $dx = \frac{b-a}{2}$  using these conditions convert  $\int_a^b f(x)dx$  into  $\int_{-1}^1 f(t)dt$  and then we apply the formula

$$\int_{-1}^1 f(t)dt = \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right] + \frac{8}{9}f(0)$$

### Problems based on single integrals

1. Using Trapezoidal rule evaluate  $\int_0^{\frac{\pi}{2}} \sqrt{\sin x} dx$



**Solution**

x	0	15	30	45	60	75	90
y	0	0.5087	0.7071	0.8408	0.9306	0.9828	1

**Trapezoidal rule**

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{2} [y_0 + y_n] + 2[y_1 + y_2 + \dots + y_{n-1}]$$

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin x} dx = \frac{12}{2} [0 + 1] + 2(0.5087 + 0.7071 + 0.8408 + 0.9306 + 0.9828)$$

$$= 1.17024$$

2. By dividing the range into ten equal parts, evaluate  $\int_0^{\pi} \sin x dx$  by Trapezoidal rule and Simpson's Rule. Verify your answer with integration

**Solution:**

Range =  $\pi - 0 = \pi$ . Hence  $h = \frac{\pi}{10}$

we tabulate below the values of y at different x's

x	0	$\frac{\pi}{10}$	$\frac{2\pi}{10}$	$\frac{3\pi}{10}$	$\frac{4\pi}{10}$	$\frac{5\pi}{10}$	$\frac{6\pi}{10}$	$\frac{7\pi}{10}$	$\frac{8\pi}{10}$	$\frac{9\pi}{10}$	$\pi$
sinx	0	.309	.5878	.809	.9511	1	.9511	.809	.5878	.309	0

**Trapezoidal rule**

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{2} [y_0 + y_n] + 2[y_1 + y_2 + \dots + y_{n-1}]$$

$$\int_0^{\pi} \sin x dx = \frac{10}{2} [0 + 0] + 2(0.3090 + 0.5878 + 0.8090 + 0.9306 + 0.9511 + 1 + 0.9511 + 0.9306 + 0.8090 + 0.3090)$$

= 1.9843 nearly

**Simpson's one third Rule:**

we use Simpson's one third rule only when the no.of intervals is even here the no of intervals =10(even)

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{2} [y_0 + y_n] + 2[y_1 + y_3 + \dots + y_{n-1}] + 4[y_2 + y_4 + \dots + y_{n-2}]$$

$$\int_0^{\frac{\pi}{2}} \sin x dx = \frac{10}{2} [0 + 0) + 2(0.9511 + 0.5878 + 0.9306 + 0.9511 + 0.9306) + 4(0.3090 + 0.8090 + 1 + 0.8090 + 0.3090)]$$

= 2.00091 nearly

We use Simpson's three eight rule only when the no.of intervals divisible by 3

Here the no of intervals =10 which is not divisible by 3.

So we cannot use this method.

**By actual integration,**

$$\int_0^{\pi} \sin x dx = \left[ -\cos x \right]_0^{\pi} = 2$$

Hence, Simpson's one third rule is more accurate than the Trapezoidal rule.

3. Evaluate (i)  $\int_{-1}^1 (3x^2 + 5x^4) dx$  and (ii)  $\int_0^1 (3x^2 + 5x^4) dx$  by Gaussian two and three point

formulas

**Solution:**

$$(i) \int_{-1}^1 (3x^2 + 5x^4) dx$$

**Gaussian two point formula**

Given interval is -1 and 1.

Hence we can apply

$$\int_{-1}^1 f(x) dx = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

$$\Rightarrow \int_{-1}^1 (3x^2 + 5x^4) dx = 3\left(\frac{-1}{\sqrt{3}}\right)^2 + 5\left(\frac{-1}{\sqrt{3}}\right)^4 + 3\left(\frac{1}{\sqrt{3}}\right)^2 + 5\left(\frac{1}{\sqrt{3}}\right)^4 = 3.112$$

**Gaussian two point formula**

Given interval is -1 and 1.

Hence we can apply

$$\int_{-1}^1 f(x) dx = \frac{5}{9} \left[ f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right) \right] + \frac{8}{9} f(0)$$

$$\begin{aligned} \int_{-1}^1 (3x^2 + 5x^4) dx &= \frac{5}{9} \left\{ \left[ 3\left(-\sqrt{\frac{3}{5}}\right)^2 + 5\left(-\sqrt{\frac{3}{5}}\right)^4 \right] + \left[ 3\left(\sqrt{\frac{3}{5}}\right)^2 + 5\left(\sqrt{\frac{3}{5}}\right)^4 \right] \right\} + \frac{8}{9} (0) \\ &= \frac{5}{9} \left\{ \left[ 3\left(\frac{3}{5}\right) + 5\left(\frac{9}{25}\right) \right] + \left[ 3\left(\frac{3}{5}\right) + 5\left(\frac{9}{25}\right) \right] \right\} \end{aligned}$$

$$\int_{-1}^1 (3x^2 + 5x^4) dx = 4$$

$$(ii) \int_0^1 (3x^2 + 5x^4) dx$$

**Gaussian two point formula**

Here the interval is 0 to 1. So we use the formula

$$x = \frac{b-a}{2}t + \frac{b+a}{2} \text{ and } dx = \frac{b-a}{2} dt$$

$$\text{i.e, } x = \frac{t+1}{2} \text{ and } dx = \frac{dt}{2}$$

$$\Rightarrow \int_{-1}^1 \left[ 3\left(\frac{t+1}{2}\right)^2 + 5\left(\frac{t+1}{2}\right)^4 \right] dt = 1.556$$

**Gaussian two point formula**

Here the interval is 0 to 1. So we use the formula

$$x = \frac{b-a}{2}t + \frac{b+a}{2} \text{ and } dx = \frac{b-a}{2} dt \quad \text{i.e, } x = \frac{t+1}{2} \text{ and } dx = \frac{dt}{2}$$

$$\Rightarrow \int_{-1}^1 \left[ 3\left(\frac{t+1}{2}\right)^2 + 5\left(\frac{t+1}{2}\right)^4 \right] \frac{dt}{2} = \frac{1}{2} \int_{-1}^1 \left[ 3\left(\frac{t+1}{2}\right)^2 + 5\left(\frac{t+1}{2}\right)^4 \right] dt$$

$$f(t) = \left[ 3\left(\frac{t+1}{2}\right)^2 + 5\left(\frac{t+1}{2}\right)^4 \right]$$

$$\int_{-1}^1 f(t) dt = \frac{5}{9} \left[ f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right) \right] + \frac{8}{9} f(0)$$

$$f\left(-\sqrt{\frac{3}{5}}\right) = \left[ 3\left(\frac{-0.7745+1}{2}\right)^2 + 5\left(\frac{-0.7745+1}{2}\right)^4 \right] = 0.038138$$

$$f\left(\sqrt{\frac{3}{5}}\right) = \left[ 3\left(\frac{0.7745+1}{2}\right)^2 + 5\left(\frac{0.7745+1}{2}\right)^4 \right] = 4.28$$

$$f(0) = \left[ 3\left(\frac{0+1}{2}\right)^2 + 5\left(\frac{0+1}{2}\right)^4 \right] = 1.0625$$

$$\int_{-1}^1 \left[ 3\left(\frac{t+1}{2}\right)^2 + 5\left(\frac{t+1}{2}\right)^4 \right] dt = \frac{5}{9} (0.038138 + 4.28) + \frac{8}{9} (1.0625)$$

$$= 4(\text{approximately})$$

$$\frac{1}{2} \int_{-1}^1 \left[ 3\left(\frac{t+1}{2}\right)^2 + 5\left(\frac{t+1}{2}\right)^4 \right] dt = \frac{4}{2} = 2$$

**Double Integral**  
**Trapezoidal Rule**

Evaluate  $\int_c^d \int_a^b f(x, y) dx dy$  where a, b, c, d are constants.

(D)	K L	(C)
J	M N	H
I	O P	G
(A)	E F	(B)

$$I = \frac{hk}{4} \left\{ [\text{sum of values in}] + 2(\text{sum of values in } \square) + 4[\text{sum of remaining values}] \right\}$$

**Simpson's Rule**

$$I = \frac{hk}{9} \left\{ \begin{aligned} &(\text{sum of the values of } f \text{ at four corners}) \\ &+ 2(\text{sum of the values of } f \text{ at the odd positions on the} \\ &\text{boundary except the corners}) \\ &+ 4(\text{sum of the values of } f \text{ at the even positions on the boundary}) \\ &+ \{4(\text{sum of the values of } f \text{ at the odd positions}) + \\ &8(\text{sum of the values of } f \text{ at the even positions}) \\ &\text{on the odd row } f \text{ of the matrix except boundary rows}\} + \\ &\{8(\text{sum of the values of } f \text{ at the odd positions}) + \\ &16(\text{sum of the values of } f \text{ at the even positions}) \\ &\text{on the even row } f \text{ of the matrix} \} \end{aligned} \right\}$$

**Problems based on Double integrals**

- Evaluate  $\int_1^{1.4} \int_{2.2}^{2.4} \frac{1}{xy} dx dy$  using Trapezoidal and Simpson's rule. Verify your result by actual integration.

**Solution:**

Divide the range of x and y into 4 equal parts

$$h = \frac{2.4 - 2}{4} = 0.1$$

$$k = \frac{1.4 - 1}{4} = 0.1$$

Get the values of  $f(x, y) = \frac{1}{xy}$  at nodal points

y\x	2	2.1	2.2	2.3	2.4
1	0.5	0.4762	0.4545	0.4348	0.4167
1.1	0.4545	0.4329	0.4132	0.3953	0.3788
1.2	0.4167	0.3968	0.3788	0.3623	0.3472
1.3	0.3846	0.3663	0.3497	0.3344	0.3205
1.4	0.3571	0.3401	0.3247	0.3106	0.2976

(i) Trapezoidal Rule

$$I = \frac{(0.1)(0.1)}{4} \left\{ \begin{array}{l} 0.5 + 0.4167 + 0.3571 + 0.2976 + 2 \left( \begin{array}{l} 0.4545 + 0.4167 + 0.3846 + 0.4762 + 0.4545 + \\ 0.4348 + 0.3788 + 0.3472 + 0.3205 + 0.3106 + \\ 0.3247 + 0.3401 \end{array} \right) \\ + 4 \left( \begin{array}{l} 0.4329 + 0.4132 + 0.3953 + 0.3968 + 0.3788 + 0.3623 + 0.3663 + 0.3497 + 0.3344 \end{array} \right) \end{array} \right\}$$

$$\mathbf{I=0.0614}$$

(ii) Simpson's Rule:

$$I = \frac{(0.1)(0.1)}{9} \left\{ \begin{array}{l} 0.5 + 0.4167 + 0.3571 + 0.2976 + 2(0.4167 + 0.4545 + 0.3472 + 0.3247) \\ + 4(0.3846 + 0.4545 + 0.4762 + 0.4348 + 0.3788 + 0.3205 + 0.3106 \\ + 0.3401 + 0.3788) + 8(0.3968 + 0.3623 + 0.3497 + 0.4132) + \\ 16(0.3663 + 0.3344 + 0.4329 + 0.3953) \end{array} \right\}$$

$$I = \frac{0.01(55.2116)}{9}$$

$$\mathbf{I=0.0613}$$

Unit-3

Part-A

1. State Newton's forward different formula to find  $\frac{dy}{dx}$  and  $\frac{d^2y}{dx^2}$  at  $x=x_0$

Ans:  $\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h} \left[ \Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \dots \right]$  and

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2} \left[ \Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12} \Delta^4 y_0 - \dots \right]$$

2. Write the formula to compute  $\frac{dy}{dx}$  and  $\frac{d^2y}{dx^2}$  at  $x=x_0+ph$  for a given data  $(x_i, y_i)$   $i=0, 1, \dots, n$ .

Ans:  $\frac{dy}{dx} = \frac{1}{h} \left[ \Delta y_0 + \frac{2p-1}{2} \Delta^2 y_0 + \frac{3p^2-6p+2}{6} \Delta^3 y_0 + \frac{2p^3-9p^2+11p-3}{12} \Delta^4 y_0 - \dots \right]$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2} \left[ \Delta^2 y_0 + (p-1) \Delta^3 y_0 + \frac{6p^2-18p+11}{12} \Delta^4 y_0 + \dots \right]$$

3. State Newton's backward interpolation formula to find  $\frac{dy}{dx}$  and  $\frac{d^2y}{dx^2}$  at  $x=x_n$ .

Ans:  $\left.\frac{dy}{dx}\right|_{x=x_n} = \frac{1}{h} \left[ \nabla y_n + \frac{\nabla^2 y_n}{2} + \frac{\nabla^3 y_n}{3} + \dots \right]$

$$\left.\frac{d^2y}{dx^2}\right|_{x=x_n} = \frac{1}{h^2} \left[ \nabla^2 y_n + \nabla^3 y_n + \frac{11}{12} \nabla^4 y_n + \dots \right]$$

4. Write the formula to compute  $\frac{dy}{dx}$  and  $\frac{d^2y}{dx^2}$  at  $x=x_n+ph$  for the data  $(x_i, y_i)$ ,  $i=0, 1, \dots, n$ .

Ans:  $\left.\frac{dy}{dx}\right|_x = \frac{1}{h} \left[ \nabla y_n + \frac{2p+1}{2} \nabla^2 y_n + \frac{3p^2+6p+2}{6} \nabla^3 y_n + \frac{2p^3+9p^2+11p+3}{12} \nabla^4 y_n + \dots \right]$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2} \left[ \nabla^2 y_n + (p+1) \nabla^3 y_n + \frac{6p^2+18p+11}{12} \nabla^4 y_n + \dots \right]$$

5. Find  $\frac{dy}{dx}$  at  $x=2$  from the following data .

x	2	3	4
y	26	58	112

Ans:  $\Delta y_0 = 32$     $\Delta y_1 = 54$     $\Delta^2 y_0 = 22$

$$\frac{dy}{dx} = 32 - \frac{1}{2} (22) = 21.$$

6. Find  $\frac{dy}{dx}$  at  $x=6$  from following table

x	2	4	6
y	3	11	27

Ans:  $\nabla y_n = 16$     $\nabla y_{n-1} = 8$     $\nabla^2 y_n = 16 - 8 = 8.$

$$\frac{dy}{dx} \Big|_{x=6} = \frac{1}{2} \left[ 16 + \frac{8}{2} \right] = 10.$$

7. A curve passing through the points (1,0), (2,1) and (4,5). Find the slope of the curve at  $x=3$

Ans:

x	y	$\Delta y$	$\Delta^2 y$
1	0		
2	1	1	
4	5	2	$\frac{1}{3}$

unequal intervals. so we use Newton's Divided difference formula

$$f(x) = f(x_0) + (x-x_0) f(x_0, x_1) + (x-x_0)(x-x_1) f(x_0, x_1, x_2)$$

$$= 0 + (x-1)(1) + (x-1)(x-2) \frac{1}{3} = x-1 + \frac{1}{3}(x^2 - 3x + 2)$$

$$f'(x) = 1 + \frac{2x}{3} - 1 = \frac{2x}{3}$$

Slope at  $x=3$  is  $\frac{2(3)}{3} = 2$

8. State the basic principle for deriving Simpson's  $\frac{1}{3}$  rule.

Ans: The curve passing through three consecutive points is replaced by a parabola.

9. State the order of error in Simpson's  $\frac{1}{3}$  rule.

Ans: Error in Simpson's  $\frac{1}{3}$  rule is of order  $h^4$ .

10. Using Simpson's rule, find  $\int_0^4 e^x dx$  given  $e^0 = 1$ ,  $e^1 = 2.72$ ,  $e^2 = 7.39$ ,  $e^3 = 20.09$  and  $e^4 = 54.6$

Ans: 
$$\int_0^4 e^x dx = \frac{1}{3} [(1 + 54.6) + 4(2.72 + 20.09) + 2(7.39)]$$
$$= 53.873$$

[ No. of intervals 4 = even. we use Simpson's  $\frac{1}{3}$  rule ]

11. A Curve passes through (2,8), (3,27), (4,64) and (5,125). Find the area of the curve between x axis and the lines  $x=2$  and  $x=5$  by trapezoidal rule.

Ans: 
$$\int_2^5 y dx = \frac{1}{2} [(8 + 125) + 2(27 + 64)] = 157.5 \text{ Sq. units}$$

12. Compute  $\int_1^2 \frac{dx}{x}$  using Simpson's rule with  $h=0.25$

Ans:

x	1	1.25	1.5	1.75	2
y	1	0.8	0.666	0.571	0.5

No. of intervals = 4 = even. we use Simpson's  $\frac{1}{3}$  rule.

$$\int_1^2 \frac{dx}{x} = \frac{0.25}{3} [(1 + 0.5) + 4(0.8 + 0.571) + 2(0.666)]$$
$$= 0.6931.$$



13. Use Simpson's  $3/8$  rule with  $h=0.5$  to evaluate

$$\int_0^1 \frac{dx}{1+x}$$

Ans:-

$x$	0	0.5	1
$y$	1	$\frac{2}{3}$	$\frac{1}{2}$

No. of intervals = 2 = not divisible by 3. So we cannot use Simpson's  $3/8$  rule

14. Evaluate  $\int_{-1}^1 |x| dx$  with two subintervals by Simpson's  $1/3$  rule and by Trapezoidal rule.

Ans:- By Simpson's rule,  $I = \frac{1}{3} [1+0+1] = \frac{2}{3}$ .

By Trapezoidal rule  $I = \frac{1}{2} [1+1] = 1$

15. If  $I_1 = 0.775$ ,  $I_2 = 0.7828$ , find  $I$  using Romberg's method.

Ans:- By Romberg's method  $I = I_2 + \left( \frac{I_2 - I_1}{3} \right) = 0.7802$ .

16. Find  $I = \int_{-1}^1 \frac{dx}{1+x^2}$  by two point Gaussian formula.

Ans:-

$$I = \frac{1}{1+1/3} + \frac{1}{1+1/3} = \frac{3}{2} = 1.5 \qquad I = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

17. Find  $I = \int_0^1 \frac{dx}{1+x}$  by Gaussian two point formula

Ans:-  $I = \int_{-1}^1 \frac{dt}{t+3}$  using  $x = \frac{1+t}{2}$  then  $I = \frac{1}{3+\frac{1}{\sqrt{3}}} + \frac{1}{3-\frac{1}{\sqrt{3}}} = 0.6923$

18. Evaluate  $\int_{-1}^1 \cos x dx$  using two point Gaussian formula.

Ans:-  $\int_{-1}^1 \cos x dx = 2 \cos\left(\frac{1}{\sqrt{3}}\right) = 1.6758$ .

Part-B

1. Find  $\frac{dy}{dx}$  at  $x=1.5$  given

$x$	1	2	3	4	5
$y$	77	78	127	248	375

2. Evaluate  $\int_1^2 \int_3^4 \frac{1}{(x+y)^2} dx dy$  taking  $h=k=0.5$  by both Trapezoidal rule and Simpson's rule.

3. Evaluate  $\int_3^7 \frac{dx}{1+x^2}$  using Gaussian quadrature with 3 points

4. Using the given data find  $f'(5)$

$x$	0	2	3	4	7	9
$f(x)$	4	26	58	112	466	922

5. Evaluate  $I = \int_0^1 \frac{dx}{1+x}$  by two and three point Gaussian formula.

6. Use Romberg's method to compute  $\int_0^1 \frac{1}{1+x^2} dx$  correct to 4 decimal places by taking  $h=0.5, 0.25$  and  $0.125$

7. Evaluate  $\int_0^1 \int_0^1 \frac{dx dy}{1+x+y}$  by using Trapezoidal rule, with step sizes  $h=k=0.5$

8. Evaluate  $\int_0^1 \int_0^2 \frac{2xy dx dy}{(1+x^2)(1+y^2)}$  Trapezoidal rule with  $h=k=0.25$ .

9. Consider the following data

$x$	0.2	0.4	0.6	0.8	1.0
$f(x)$	0.9798652	0.9177710	0.8080348	0.6386093	0.38437

Find  $f'(0.25)$  using Newton's forward difference approx.  $f'(0.6)$  using Stirling approx. and  $f'(0.95)$  using Newton's backward difference approx.

10. Using Trapezoidal rule  $\frac{3}{8}$  rule  $\int_{1.4}^{2.0} \int_{1.0}^{1.5} \ln(x+2y) dy dx$  choosing  $\Delta x=0.15$  &  $\Delta y=0.25$ .

11. The velocity  $v$  of a particle at a distance  $s$  from a point on its path is given by the table below

$s$	0	10	20	30	40	50	60
$v$	47	58	64	65	61	52	38

estimate the time taken to travel to metres by ~~choosing~~ using Simpson's  $\frac{1}{3}$  rule. Compare your answer with Simpson's  $\frac{2}{8}$  rule.

# UNIT IV

## INITIAL VALUE PROBLEMS FOR ORDINARY DIFFERENTIAL EQUATIONS

Taylor series method

Euler methods

Runge-Kutta method for solving first and second order equations

Milne's predictor and corrector method

Adam's predictor and corrector method.

**Introduction**

An ordinary differential equation of order n is a relation of the form  $F(x, y, y', y'', \dots, y^{(n)}) = 0$  where  $y = y(x)$  and  $y^{(r)} = \frac{d^r y}{dx^r}$ . The solution of this differential equation involves n constants and these constants are determined with the help of n conditions  $y, y', y'', \dots, y^{(n-1)}$  are prescribed at  $x = x_0$ , by

$$y(x_0) = y_0, y'(x_0) = y'_0, \dots, y^{(n-1)}(x_0) = y_0^{(n-1)}$$

These conditions are called the *initial conditions* because they depend only on  $x_0$ .

*The differential equation together with the initial conditions is called an initial value problem.*

**Taylor's Series**

**Point wise solution**

If  $y(x)$  is the solution of (1), then by Taylor series,

$$y(x) = y_0 + \frac{(x-x_0)}{1!} y'_0 + \frac{(x-x_0)^2}{2!} y''_0 + \frac{(x-x_0)^3}{3!} y'''_0 + \dots$$

Put  $x_1 = x_0 + h$  where h is the step-size, we have

$$y(x_1) = y_1 = y_0 + \frac{h}{1!} y'_0 + \frac{h^2}{2!} y''_0 + \frac{h^3}{3!} y'''_0 + \dots \quad (3)$$

once  $y_1$  has been calculated from (1),  $y'_1, y''_1, y'''_1$  can be calculated from

$$y' = f(x, y)$$

Expanding  $y(x)$  in a Taylor series about  $x = x_1$ , we get

$$y_2 = y_1 + \frac{h}{1!} y'_1 + \frac{h^2}{2!} y''_1 + \frac{h^3}{3!} y'''_1 + \dots$$

Where  $y_2 = y(x_2)$  and  $x_2 = x_1 + h$

The Taylor algorithm is given as follows

$$y_{m+1} = y_m + \frac{h}{1!} y'_m + \frac{h^2}{2!} y''_m + \frac{h^3}{3!} y'''_m + \dots$$

Where  $y_m^{(r)} = \frac{d^r y}{dx^r}$  at the point  $(x_m, y_m)$  where  $m = 0, 1, 2, \dots$

**Problems based on Taylor's Series**

Solve  $y' = y^2 + x; y(0) = 1$  using Taylor series method and compute  $y(0.1)$  and  $y(0.2)$

**Solution**

Here  $x_0 = 0, y_0 = 1$ .

Given $y' = y^2 + x$	;	$y'_0 = 1$	
$y'' = 2yy' + 1$	;	$y''_0 = 3$	
$y''' = 2yy'' + 2y'^2$	;	$y'''_0 = 8$	
$y^{(4)} = 6y'' + 2yy'''$	;	$y^{(4)}_0 = 34$	etc.,

To find

Here  $h = 0.1$

By Taylor algorithm

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \dots$$

$$\therefore y_1 = 1 + (0.1) + \frac{0.01}{2} + \frac{0.001}{6} + \frac{0.001}{24}$$

$$y_{(0.1)} = 1.1164$$

To find  $y_{(0.2)}$

$$x_2 = x_1 + h \text{ where } x_2 = 0.2$$

$$y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \dots$$

$$y_1' = 0.1 + (1.1164) = 1.3463$$

$$y_1'' = 1 + 2(1.1164)(1.3463) = 4.006$$

$$y_1''' = 2(1.1164)(4.006) + 2(1.3463)^2 \\ = 12.5696$$

$$y_{(0.2)} = 1.1164 + (0.1)(1.3463) + \frac{0.01}{2}(4.006) + \frac{0.0001}{6}(12.5696)$$

$$\Rightarrow y_{(0.2)} = 1.2732$$

2. Evaluate  $y_{(0.1)}$  and  $y_{(0.2)}$ , correct to four decimal places by Taylor series method, if  $y_{(0)}$  satisfies  $y' = xy + 1$ ,  $y_{(0)} = 1$

**Solution**

Here  $x_0 = 0$ ,  $y_0 = 1$

$$\begin{aligned} y' &= xy + 1 & ; & & y_0' &= 1 \\ y'' &= xy' + y & ; & & y_0'' &= 1 \\ y''' &= xy'' + 2y' & ; & & y_0''' &= 2 \\ y^{iv} &= xy''' + 3y'' & ; & & y_0^{iv} &= 3 \end{aligned}$$

To find  $y_{(0.1)}$

By Taylor algorithm

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \dots$$

Let  $x_1 = x_0 + h$  where  $h = 0.1$

$$\therefore x_1 = 0.1$$

$$\therefore y_1 = 1 + (0.1) + \frac{0.01}{2} + \frac{0.001}{3} + \frac{0.0001}{24}$$

$$= 1.1057$$

To find  $y_{(0.2)}$

Let  $y_2 = y_{(0.2)}$  where  $x_2 = x_1 + h$

$$\therefore x_2 = 0.2$$

By Taylor algorithm,

$$y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \dots$$

$$y_1' = 1 + (0.1)(1.1057) = 1.11057$$

$$y_1'' = (0.1)(1.11057) = 1.11057 = 1.216757$$

$$y_1''' = (0.1)(1.216757) + 2(0.11057) = 2.3428157$$

$$\therefore y_2 = 1.1057 + (0.1)(1.11057) + \frac{0.01}{2}(1.216757) + \frac{0.0001}{6}(2.3428157)$$

$$\Rightarrow y_2 = 1.2178$$

Hence  $y(0.1) = 1.1057$  and  $y(0.2) = 1.2178$ .

3. Solve by Taylor series method,  $y' = xy + y^2$ ,  $y(0) = 1$  at  $x = 0.1$  and  $0.2$ , correct to four decimal places.

**Solution**

Given  $x_0 = 0$ ,  $y_0 = 1$

$$y' = xy + y^2 \quad ; \quad y_0' = 1$$

$$y'' = y + xy' + 2yy' \quad ; \quad y_0'' = 3$$

$$y''' = 2y' + xy'' + 2(y')^2 + 2yy'' \quad ; \quad y_0''' = 10$$

$$y^{iv} = 3y'' + xy''' + 6y'y'' + 2yy''' \quad ; \quad y_0^{iv} = 47$$

To find  $y_1 = y(0.1)$

Let  $x_1 = x_0 + h$ . Here  $h = 0.1$

$$x_1 = x_0 + h \Rightarrow x_1 = 0.1$$

By Taylor algorithm

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \dots$$

$$\therefore y_1 = 1 + (0.1)(1) + \frac{0.01}{2}(3) + \frac{0.001}{6}(10) + \frac{0.0001}{24}(47)$$

$$= 1.1 + 0.015 + 0.0017 + 0.0002$$

$$\therefore y(0.1) = 1.1169$$

To find  $y(0.2)$

$$y_2 = y(0.2) \text{ where } x_2 = x_1 + h$$

Here  $x_1 = 0.1$ ,  $h = 0.1$ ,  $x_2 = 0.2$

By Taylor algorithm,

$$y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \dots$$

$$x_1 = 0.1, y_1 = 1.1169$$

$$y_1' = x_1 y_1 + y_1^2 \Rightarrow y_1' = 1.35925$$

$$y_1'' = y_1 + x_1 y_1' + 2y_1 y_1' \Rightarrow y_1' = 1.35925$$

$$= 1.1169 + 0.1(1.35925) + 2(1.1169)(1.35925) \\ = 4.2891$$

$$y_1''' = 2y_1' + x_1 y_1'' + 2(y_1')^2 + 2y_1 y_1'' \\ = 2.7185 + 0.42891 + 3.6951 + 6.5810 \\ = 16.4235$$

$$y_1^{iv} = 3y_1'' + x_1 y_1''' + 6y_1 y_1'' + 2y_1 y_1''' \\ = 212.8673 + 1.64235 + 34.9797 + 36.6868 \\ = 86.17615$$

$$\therefore y_2 = 1.1169 + 0.1(4.2891) + \frac{0.01}{2}(16.4235) + \frac{0.001}{6}(86.17615) + \frac{0.0001}{24}(6.17615)$$

$$= 1.1169 + 0.42891 + 0.0821175 + 0.01436167 + 0.000429167$$

$$\therefore y(0.2) = 1.2774$$

$$\text{Hence } y(0.1) = 1.1169 \text{ and } y(0.2) = 1.2774.$$

### Taylor series method for simultaneous first order differential equation

We can solve the equations of the form  $\frac{dy}{dx} = f(x, y, z)$ ;  $\frac{dz}{dx} = g(x, y, z)$  with initial conditions  $y(x_0) = y_0$  and  $z(x_0) = z_0$ .

The values of  $y$  and  $z$  at  $x_1 = x_0 + h$  are given by Taylor algorithm,

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \dots$$

$$z_1 = z_0 + \frac{h}{1!} z_0' + \frac{h^2}{2!} z_0'' + \frac{h^3}{3!} z_0''' + \dots$$

The derivatives on R.H.S of the above expressions are found at  $x = x_0$  using the given equations.

Similarly  $y_2$  and  $z_2$  corresponding to  $x_2 = x_1 + h$  are calculated by Taylor series method.

### Problems

1. Evaluate  $x(0.1)$  and  $y(0.1)$  given  $\frac{dx}{dt} = 1 + ty$ ;  $\frac{dy}{dt} = -tx$  given  $x = 0, y = 1$  at  $t = 0$  by Taylor series method.

### Solution

Given  $t_0 = 0, x_0 = 0, y_0 = 1$

$$x' = 1 + ty$$

$$y' = -tx$$

$$x'' = y + ty'$$

$$y'' = -x + tx'$$

$$x''' = 2y' + ty''$$

$$y''' = -x' + tx''$$

$$x^{iv} = 3y'' + ty'''$$

$$y^{iv} = -x'' + tx'''$$

Then

$$x_0' = 1$$

$$y_0' = 0$$

$$\begin{aligned}x_0'' &= 1 & y_0'' &= 0 \\x_0''' &= 0 & y_0''' &= -2 \\x_0^{iv} &= 0 & y_0^{iv} &= -3\end{aligned}$$

By Taylor algorithm, we have

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \dots$$

$$\therefore y(0.1) = 1 + (0.1) \left( \frac{-0.01}{2} \right) + \frac{0.001}{6} \left( -2 \right) + \frac{0.0001}{24} \left( -3 \right)$$

$$\therefore y(0.1) = 0.9997$$

$$x(0.1) = x_0 + \frac{h}{1!} x_0' + \frac{h^2}{2!} x_0'' + \frac{h^3}{3!} x_0''' + \dots$$

$$= 0 + (0.1) \left( \frac{-0.01}{2} \right) + \dots$$

$$= 0.105$$

2. Find  $x(0.2)$  and  $y(0.2)$  using Taylor series method given that  $\frac{dx}{dt} = xy + 2t$ ;  $\frac{dy}{dt} = 2ty + x$ , with initial conditions  $x = 1, y = 1$  at  $t = 0$ .

**Solution**

$$\text{Given } t_0 = 0, x_0 = 1, y_0 = -1$$

Taking  $h = 0.2, t_1 = t_0 + h \Rightarrow t_1 = 0.2$

$$\begin{aligned}x' &= xy + 2t & y' &= 2ty + x \\x'' &= xy' + x'y + 2 & y'' &= 2ty' + 2y + x' \\x''' &= xy'' + 2x'y + x''y & y''' &= 4y' + 2ty'' + x'' \\x_0' &= -1 & y_0' &= 1 \\x_0'' &= 4 & y_0'' &= -3 \\x_0''' &= -9 & y_0''' &= 8\end{aligned}$$

By Taylor algorithm, we have

$$x_1 = x_0 + \frac{h}{1!} x_0' + \frac{h^2}{2!} x_0'' + \frac{h^3}{3!} x_0''' + \dots$$

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \dots$$

$$\therefore x(0.2) = 1 + (0.2) \left( -1 \right) + \frac{2(0.2)^2}{2} - \frac{3(0.2)^3}{6} + \dots$$

$$\therefore x(0.2) = 0.796$$

$$y_1 = y(0.2) = -1 + 0.2 - (0.5)(0.04) + \frac{4}{3}(0.008)$$

$$= -0.8493$$

Hence  $x(0.2) = 0.796$  and  $y(0.2) = -0.8493$



**Taylor series method for second order differential equations**

Consider the differential equation  $\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right)$  with the initial conditions  $y(x_0) = y_0$  and  $y'(x_0) = y'_0$ , where  $y_0, y'_0$  are known values.

This equation can be reduced into a set of simultaneous equations, by putting  $y' = p$   
 $\therefore$  we have  $y' = p, y(x_0) = y_0$  (1)

And  $p' = f(x, y, p), p(x_0) = p_0 = y'_0$  (2)

Successively differentiating  $y''$ , the expression for  $y''', y^{(4)}$  etc., are known.

By Taylor series method, we find

$$y_1 = y_0 + \frac{h}{1!} y'_0 + \frac{h^2}{2!} y''_0 + \frac{h^3}{3!} y'''_0 + \dots$$

$$= y_0 + h p_0 + \frac{h^2}{2!} p'_0 + \frac{h^3}{3!} p''_0 + \dots$$

Also by Taylor series method, we have

$$p_1 = y'_1 = p_0 + \frac{h}{1!} p'_0 + \frac{h^2}{2!} p''_0 + \frac{h^3}{3!} p'''_0 + \dots$$

Then the values  $y''_1, y'''_1, y^{(4)}_1$  are found from  $y''_0, y'''_0$  etc

$$y_2 = y_1 + \frac{h}{1!} y'_1 + \frac{h^2}{2!} y''_1 + \frac{h^3}{3!} y'''_1 + \dots$$

Thus we calculate  $y_1, y_2, \dots$

**Problems**

1. Find  $y(0.2)$  and  $y'(0.4)$  given  $y'' = xy$  if  $y(0) = 1, y'(0) = 1$  by Taylor series method.

**Solution**

Given  $y'' = xy, x_0 = 0, y_0 = 1$  and  $y'(0) = 1$

Then we have

$$y''' = xy' + y$$

$$y^{(4)} = xy'' + 2y'$$

$$y^{(5)} = 3y'' + xy''' \text{ etc}$$

$$\therefore y''_0 = 0, y'''_0 = 1, y^{(4)}_0 = 2, y^{(5)}_0 = 0$$

By Taylor Algorithm,

$$y_1 = y_0 + \frac{h}{1!} y'_0 + \frac{h^2}{2!} y''_0 + \frac{h^3}{3!} y'''_0 + \dots$$

Taking,  $h = 0.2, x_1 = x_0 + h \Rightarrow x_1 = 0.2$

$$y_1 = y(0.2) = 1 + (0.2) + \frac{0.04}{2} + \frac{0.008}{6} + \frac{0.0016}{24}$$

$$\therefore y(0.2) = 1.2014$$

To find  $y'_1$

Set  $p = y'$ . Then  $p' = xy$

$$\therefore p'' = xy' + y; p''' = xy'' + 2y'; p^{iv} = 3y'' + xy'''$$

$$\therefore p_0 = 1, p_0' = 0, p_0'' = 1, p_0''' = 2, p_0^{iv} = 0$$

By Taylor Algorithm,

$$p_1 = p_0 + \frac{h}{1!} p_0' + \frac{h^2}{2!} p_0'' + \frac{h^3}{3!} p_0''' + \dots$$

$$\therefore p_1 = 1 + \left(0.2\right) \left(0\right) + \frac{0.04}{2} \left(1\right) + \frac{0.08}{6} \left(2\right)$$

$$\Rightarrow y_1' = 1 + 0.02 + 0.0027$$

$$= 1.0227$$

Let  $x_2 = x_1 + h$ . Since  $h = 0.2, x_1 = 0.2 \Rightarrow x_2 = 0.4$

By Taylor series method,

$$\therefore y \left(0.4\right) = y \left(0.2\right) + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \dots$$

$$y'' = xy \Rightarrow y_1'' = x_1 y_1 = \left(0.2\right) \left(1.0227\right) = 0.20454$$

$$y''' = xy' + y \Rightarrow y_1''' = \left(0.2\right) \left(1.0227\right) + 1.0227 = 1.40594$$

$$y^{iv} = 2y' + xy'' \Rightarrow y_1^{iv} = \left(0.2\right) \left(1.0227\right) + 2 \left(1.0227\right) = 0.00048 + 2.0454 = 2.04588$$

$$\therefore y_2 = 1.0227 + \left(0.2\right) \left(1.0227\right) + \frac{0.04}{2} \left(0.20454\right) + \frac{0.008}{6} \left(1.40594\right) + \frac{0.0016}{24} \left(2.04588\right)$$

$$\Rightarrow y \left(0.4\right) = 1.4084$$

Hence  $y \left(0.2\right) = 1.0227$  and  $y \left(0.4\right) = 1.4084$

2. Find  $y$  at  $x = 1.1, 1.2$  given  $y'' = x^3 - y^2 y'$ ,  $y \left(1\right) = 1$  and  $y' \left(1\right) = 1$ , correct to four decimal places using Taylor series method.

**Solution**

Given  $x_0 = 1, y_0 = 1; y_0' = 1$

$$y'' = x^3 - y^2 y' \Rightarrow y_0'' = 0$$

$$y''' = 3x^2 - 2y y_1' - y^2 y'' \Rightarrow y_0''' = 1$$

$$y^{iv} = 6x - 2y^3 - 4yy' y'' - 2yy' y''' - y^2 y''''$$

$$\Rightarrow y_0^{iv} = 6 - 2 - 0 - 0 - 1 = 3$$

$$\therefore y_0' = 1, y_0'' = 0, y_0''' = 1, y_0^{iv} = 3$$

By Taylor series method,

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \dots$$

Taking,  $h = 0.1, x_0 = 1, x_1 = x_0 + h = 1.1$

We have

$$\therefore y \left(1.1\right) = 1 + \left(0.1\right) \left(1\right) + \frac{0.01}{2} \left(0\right) + \frac{0.001}{6} \left(1\right) + \frac{0.0001}{24} \left(3\right)$$

$$\therefore y \left(1.1\right) = 1.1002$$

To find  $y_1'$

Set  $p = y'$ . Then  $p' = x^3 - y^2 y'$ ,  $p_0 = y_0' = 1$

$$p_0^1 = 1 - 1 = 0, p_0^2 = 1, p_0^3 = 3 \text{ etc.},$$

$$\therefore p_1 = p_0 + \frac{h}{1!} p_0^1 + \frac{h^2}{2!} p_0^2 + \frac{h^3}{3!} p_0^3 + \dots$$

$$\Rightarrow p_1 = 1 + 0.1 + \frac{0.01}{2} + \frac{0.001}{6}$$

$$\therefore p_1 = 1.0053 \Rightarrow y_1' = 1.0053$$

$$y'' = x^3 - y^2 y' \Rightarrow y_1'' = 1 - (1.002)(1.0053) = 0.11415$$

$$y''' = 3xy^2 - 2y y' y'' - y^2 y'' \Rightarrow y_1''' = 3(1)(1.002)^2 - 2(1.002)(1.0053)(0.11415) - (1.002)^2(0.11415) = 1.268$$

By Taylor Algorithm,

$$y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \dots$$

Where  $y_2 = y(x_2)$ ,  $x_2 = 1.2$

$$= 1.1002 + 0.1(1.0053) + \frac{0.01}{2}(0.11415) + \frac{0.001}{6}(1.268)$$

$$\Rightarrow y(1.2) = 1.2015$$

Hence  $y(1) = 1.1002$  and  $y(1.2) = 1.2015$

### Euler method

Let  $y_1 = y(x_1)$ , where  $x_1 = x_0 + h$

Then  $y_1 = y(x_0 + h)$ . Then by Taylor's series,

$$y_1 = y(x_0) + \frac{h}{1!} y'(x_0) + \frac{h^2}{2!} y''(x_0) + \dots \quad (1)$$

Neglecting the terms with  $h^2$  and higher powers of  $h$ , we get from (1),

$$y_1 = y_0 + hf(x_0, y_0) \quad (2)$$

Expression (2) gives an approximate value of  $y$  at  $x_1 = x_0 + h$ .

Similarly, we get  $y_2 = y_1 + hf(x_1, y_1)$  for  $x_2 = x_1 + h$ .

$$\therefore \text{for any } m, y_{m+1} = y_m + hf(x_m, y_m), m = 0, 1, 2, \dots \quad (3)$$

In Euler's method, we use (3) to compute successively  $y_1, y_2, \dots$  etc., with an Error  $= O(h^2)$

### Modified Euler method

The algorithm presented already in Modified Euler method in unit IV is sometimes referred as Improved Euler Method.

Therefore a different algorithm for Modified Euler method to solve

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0 \text{ is explained with illustrations.}$$

**Explanation: Modified Euler Method**

$$y_{n+1} = y_n + h \left[ f \left( x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n) \right) \right]$$

$$(or) \ y_{n+1} = y_n + h \left[ f \left( x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n) \right) \right]$$

Where  $y_{n+1} = y_n + h$  and h is the step - size

**Problems based on Euler's Method**

1, Using Euler's method, compute y in the range  $0 \leq x \leq 0.5$ , if y satisfies  $\frac{dy}{dx} = 3x + y^2, y(0) = 1$ .

**Solution**

Here  $f(x, y) = 3x + y^2, x_0 = 0, y_0 = 1$

By Euler's, method

$$y_{n+1} = y_n + hf(x_n, y_n), n = 0, 1, 2, \dots$$

Choosing  $h = 0.1$ , we compute the values of y using (1)

$$y_{n+1} = y_n + hf(x_n, y_n), n = 0, 1, 2, \dots$$

$$y_{0.1} = y_1 = y_0 + hf(x_0, y_0) = 1 + 0.1(0 + 1^2) = 1.1$$

$$y_{0.2} = y_2 = y_1 + hf(x_1, y_1) = 1.1 + 0.1(0.3 + 1.1^2) = 1.251$$

$$y_{0.3} = y_3 = y_2 + hf(x_2, y_2) = 1.251 + 0.1(0.6 + 1.251^2) = 1.4675$$

$$y_{0.4} = y_4 = y_3 + hf(x_3, y_3) = 1.4675 + 0.1(0.9 + 1.4675^2) = 1.7728$$

$$y_{0.5} = y_5 = y_4 + hf(x_4, y_4) = 1.7728 + 0.1(1.2 + 1.7728^2) = 2.2071$$

2. Given  $\frac{dy}{dx} = \frac{y-x}{y+x}$ , with  $y = 1$  for  $x = 0$ . Find y approximately for  $x = 0.1$  by Euler's method in five steps.

**Solution**

Given  $y_0 = 1, x_0 = 0$ , choosing  $h = 0.002$ ,

$$x_i = x_0 + ih, i = 1, 2, 3, 4, 5$$

To find  $y_1, y_2, y_3, y_4$  and  $y_5$  where  $y_i = y(x_i)$

$$Here \ f(x, y) = \frac{y-x}{y+x}$$

Using  $y_{n+1} = y_n + hf(x_n, y_n), n = 0, 1, 2, \dots$

We get

$$y_1 = y_0 + hf \left( x_0, y_0 \right) = 1 + 0.02 \left[ \frac{1-0}{1+0} \right]$$

$$= 1.02$$

$$y_2 = y_1 + hf \left( x_1, y_1 \right) = 1.02 + 0.02 \left[ \frac{1.02 - 0.02}{1.02 + 0.02} \right]$$

$$= 1.0392$$

$$y_3 = y_2 + hf \left( x_2, y_2 \right) = 1.0392 + 0.02 \left[ \frac{1.0392 - 0.04}{1.0392 + 0.04} \right]$$

$$= 1.0577$$

$$y_4 = y_3 + hf \left( x_3, y_3 \right) = 1.0577 + 0.02 \left[ \frac{1.0577 - 0.06}{1.0577 + 0.06} \right]$$

$$= 1.0756$$

$$y_5 = y_4 + hf \left( x_4, y_4 \right) = 1.0756 + 0.02 \left[ \frac{1.0756 - 0.08}{1.0756 + 0.08} \right]$$

$$= 1.0928$$

Hence  $y = 1.0928$  and  $x = 0.1$

3. Compute  $y$  at  $x = 0.25$  by Modified Euler method given  $y' = 2xy$ ,  $y(0) = 1$

**Solution**

Here  $f(x, y) = 2xy$ ,  $x_0 = 0$ ,  $y_0 = 1$

Choose  $h = 0.25$ ,  $x_1 = x_0 + h = 0.25$

By Modified Euler method

$$y_1 = y_0 + h \left[ f \left( x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f \left( x_0, y_0 \right) \right) \right]$$

$$\therefore f \left( x_0, y_0 \right) = 2(0)(1) = 0$$

$$f \left( x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f \left( x_0, y_0 \right) \right) = f \left( 0.125, 1 \right) = 0.25$$

$$\therefore y_1 = 1 + 0.25(0.25) = 1.0625$$

Hence  $\therefore y(0.25) = 1.0625$

4. Using Modified Euler method, find  $y(0.1)$  and  $y(0.2)$  given  $\frac{dy}{dx} = x^2 + y^2$ ,  $y(0) = 1$ .

**Solution**

Here  $x_0 = 0$ ,  $y_0 = 1$ ,  $h = 0.1$

$$f \left( x, y \right) = x^2 + y^2, y_0 + \frac{h}{2} f \left( x_0, y_0 \right) = 1 + \frac{0.1}{2} (0 + 1) = 1.05$$

$$y_1 = y_0 + hf \left( x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f \left( x_0, y_0 \right) \right) = 1 + 0.1 f \left( 0.05, 1.05 \right)$$

$$\begin{aligned}
 &= 1 + 0.1(0.05) + (0.05) = 1.1105 \\
 y(0.1) &= 1.1105 \\
 f(x_1, y_1) &= f(0.1, 1.1105) = (0.1) + (1.1105) = 1.24321 \\
 y_1 + \frac{h}{2} f(x_1, y_1) &= 1.1105 + 0.05(1.24321) = 1.17266 \\
 \therefore y_2 &= y_1 + hf\left(x_1 + \frac{h}{2}, y_1 + \frac{h}{2} f(x_1, y_1)\right) \\
 &= 1.1105 + (0.1)f(0.15, 1.17266) \\
 &= 1.1105 + (0.1)(1.15) + (1.17266) \\
 \therefore y(0.2) &= y_2 = 1.2503 \\
 y(0.2) &= 1.2503
 \end{aligned}$$

### Fourth-order Runge-Kutta method

This method is commonly used for solving the initial value problem

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0$$

### Working Rule

The value of  $y_1 = y(x_1)$  where  $x_1 = x_0 + h$  where  $h$  is the step-size is obtained as follows.

We calculate successively.

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

Finally compute the increment

$$\Delta y = \frac{1}{6} [k_1 + 2k_2 + 2k_3 + k_4]$$

The approximate value of  $y_1$  is given by

$$y_1 = y_0 + \Delta y \Rightarrow y_1 = y_0 + \frac{1}{6} [k_1 + 2k_2 + 2k_3 + k_4]$$

**Error in R-K fourth order method** =  $O(h^5)$

In general, the algorithm can be written as

$$y_{m+1} = y_m + \frac{1}{6} [k_1 + 2k_2 + 2k_3 + k_4] \text{ where}$$

$$k_1 = hf(x_m, y_m)$$

$$k_2 = hf\left(x_m + \frac{h}{2}, y_m + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_m + \frac{h}{2}, y_m + \frac{k_2}{2}\right)$$

$$k_4 = hf\left(x_m + h, y_m + k_3\right) \quad \text{where } m = 0, 1, 2, \dots$$

### Runge-Kutta method for second order differential equations

Consider the second order differential equation  $\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right)$  with initial conditions  $y(x_0) = y_0$  and  $y'(x_0) = y'_0$ . This can be reduced to a system of simultaneous linear first order equations, by putting  $z = \frac{dy}{dx}$ . Then we have,

$$\frac{dy}{dx} = z \quad \text{with } y(x_0) = y_0$$

$$\frac{dz}{dx} = f(x, y, z), \quad \text{with } z(x_0) = y'(x_0) = y'_0$$

i.e.,  $\frac{dy}{dx} = g(x, y, z)$ , where  $g(x, y, z) = z$

and  $\frac{dz}{dx} = f(x, y, z)$  with initial conditions  $y(x_0) = y_0$  and

$$z(x_0) = z_0 \quad \text{where } z_0 = y'_0$$

Now, starting from  $(x_0, y_0, z_0)$ , the increments  $\Delta y$  and  $\Delta z$  in  $y$  and  $z$  are given by (h-step size)

$$k_1 = hg(x_0, y_0, z_0)$$

$$l_1 = hf(x_0, y_0, z_0)$$

$$k_2 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$l_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$k_3 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$l_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$k_4 = hg(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$l_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$\Delta y = \frac{1}{6} [k_1 + 2k_2 + 2k_3 + k_4]$$

$$\Delta z = \frac{1}{6} [l_1 + 2l_2 + 2l_3 + l_4]$$

Then for  $x_1 = x_0 + h$ , the values of  $y$  and  $z$  are  $y_1 = y_0 + \Delta y$  and  $z_1 = z_0 + \Delta z$  respectively.

By repeating the above algorithm the value of  $y$  at  $x_2 = x_1 + h$  can be found.

### Problems based on RK Method

1. The value of  $y$  at  $x = 0.2$  if  $y$  satisfies  $\frac{dy}{dx} = x^2y + x$ ,  $y(0) = 1$  taking  $h = 0.1$  using Runge-Kutta method of fourth order.

#### Solution

Here  $f(x, y) = x^2y + x$ ,  $x_0 = 0, y_0 = 1$ .

Let  $x_1 = x_0 + h$ , choosing  $h = 0.1, x_1 = 0.1$ .

Then by R-K fourth order method,

$$y_1 = y_0 + \frac{1}{6} [k_1 + 2k_2 + 2k_3 + k_4]$$

$$k_1 = hf(x_0, y_0) = 0.1 \times 0 = 0$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1 \times 0.05 = 0.00525$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1 \times 1.0026 = 0.00525$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1 \times 1.00525 = 0.011005$$

$$y_1 = 1 + \frac{1}{6} [0 + 0.00525 + 0.00525 + 0.011005] = 1.0053$$

$$y(0.1) = 1.0053$$

To find  $y_2 = y(0.2)$  where  $x_2 = x_1 + h$ . Then  $x_2 = 0.2$

$$y_2 = y_1 + \frac{1}{6} [k_1 + 2k_2 + 2k_3 + k_4]$$

$$k_1 = hf(x_1, y_1) = 0.1 \times 1.0053 = 0.010053$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.1 \times 1.15 + 0.15 = 0.0108$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.1 \times 1.15 + 0.15 = 0.013935$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = 0.1 \times 1.2 + 0.2 = 0.02258$$

$$\therefore y_2 = 1.0053 + \frac{1}{6} [0.010053 + 2 \times 0.0108 + 0.013935 + 0.02258] = 1.0227$$

$$\therefore y(0.2) = 1.0227$$

Hence  $y(0.2) = 1.0227$ .

2. Apply Runge-Kutta method to find an approximate value of  $y$  for  $x = 0.2$  in steps of 0.1 if

$$\frac{dy}{dx} = x + y^2, y(0) = 1, \text{ correct to four decimal places.}$$

**Solution**

Here  $f(x, y) = x + y^2, x_0 = 0, y_0 = 1$

We choose  $h = 0.1$

$$y_1 = y_0 + \frac{1}{6} [k_1 + 2k_2 + 2k_3 + k_4]$$

$$k_1 = hf(x_0, y_0) = 0.1 \times 1^2 = 0.1$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1 \times 0.05 + 0.05 = 0.1152$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1 \times 0.05 + 0.0576 = 0.1168$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1 \times 1 + 0.1168 = 0.1347$$



$$\therefore y_1 = 1 + \frac{1}{6} [1 + 2(0.1152 + 0.1168) + 0.1347]$$

$$\therefore y_1 = 1.1165$$

Hence  $y(0.1) = 1.1165$ .

3. Use Runge-Kutta method to find  $y$  when  $x=1.2$  in steps of 0.1, given that  $\frac{dy}{dx} = x^2 + y^2$  and  $y(1) = 1.5$ .

**Solution**

Given  $f(x, y) = x^2 + y^2, x_0 = 1, y_0 = 1.5$

Let  $x_1 = x_0 + h$ , we choose  $h = 0.1$

$$y_1 = y_0 + \frac{1}{6} [k_1 + 2k_2 + 2k_3 + k_4]$$

$$k_1 = hf(x_0, y_0) = 0.1 [1 + 2.25] = 0.325$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1 [1.05^2 + 2.6625^2] = 0.3866$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1 [1.05^2 + 2.8673^2] = 0.397$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1 [1.1^2 + 3.897^2] = 0.4809$$

$$\therefore y_1 = 1.5 + \frac{1}{6} [0.325 + 2(0.3866 + 0.397) + 0.4809] = 1.8955$$

$$\Rightarrow y_1 = 1.8955$$

To compute  $y(1.2)$ :

$y_2 = y(1.2)$  where  $x_2 = x_1 + h = 1.2$ , since  $h = 0.1$

$$y_2 = y_1 + \frac{1}{6} [k_1 + 2k_2 + 2k_3 + k_4]$$

$$k_1 = hf(x_1, y_1) = 0.1 [1.1^2 + 3.8955^2] = 0.4803$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.1 [1.15^2 + 4.1356^2] = 0.5883$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.1 [1.15^2 + 4.1897^2] = 0.6117$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = 0.1 [1.2^2 + 6.286^2] = 0.7726$$

$$\therefore y_2 = 1.8955 + \frac{1}{6} [0.4803 + 2(0.5883 + 0.6117) + 0.7726] = 2.5043$$

Hence  $y(1.2) = 2.5043$ .

4. Solve the equation  $\frac{dy}{dx} = xz + 1, \frac{dz}{dx} = -xy$  for  $x = 0.3$  and  $0.6$ . Given that  $y = 0, z = 1$  when  $x = 0$

**Solution**

Here  $f_1(x, y, z) = 1 + xz$ ,  $x_0 = 0$ ,  $y_0 = 0$ ,  $z_0 = 0$  and  $h = 0.3$

To find  $y_{0.3}$  and  $z_{0.3}$

$$k_1 = hf(x_0, y_0) = 0.3(1) = 0.3$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.3(1.15) = 0.3450$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.3(1.15 \times 0.9966) = 0.3448$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.3(1.3 \times 0.99224) = 0.3893$$

$$l_1 = hf(x_0, y_0, z_0) = 0.3(0) = 0$$

$$l_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right) = 0.3(0.15 \times 0.15) = -0.00675$$

$$l_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right) = 0.3(0.15 \times 0.1725) = -0.00776$$

$$l_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3) = 0.3(0.3 \times 0.3448) = -0.031036$$

$$y_{0.3} = y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 0 + \frac{1}{6}(0.3 + 2(0.3450 + 0.3448) + 0.3893)$$

$$\therefore y_{0.3} = 0.3448$$

$$z_{0.3} = z_1 = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

$$\Rightarrow z_{0.3} = 1 - \frac{1}{6}(0 + 2(0.00675 + 0.00776) + 0.031036)$$

$$\therefore z_{0.3} = 0.9899$$

To find  $y$  at  $x = 0.6$ , the starting values are  $x_1 = 0.3$ ,  $y_1 = 0.3448$ ,  $z_1 = 0.9899$  and  $h = 0.3$

$$k_1 = hf(x_0, y_0) = 0.3(1) + 0.3(0.9899) = 0.3891$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.3(1) + 0.45(0.9744) = 0.4315$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.3(1) + 0.45(0.9535) = 0.4287$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.3(1) + 0.6(0.9142) = 0.4645$$

$$l_1 = hf(x_0, y_0, z_0) = 0.3(1) - 0.3(0.3448) = -0.0310$$

$$l_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right) = 0.3(1) - 0.45(0.53935) = -0.0728$$

$$l_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right) = 0.3(1) - 0.45(0.56055) = -0.0757$$

$$l_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3) = 0.3(1) - 0.6(0.7735) = -0.1392$$

$$y_2 = y_1 + \frac{1}{6} [k_1 + 2k_2 + 2k_3 + k_4]$$

$$= 0.3448 + \frac{1}{6} [0.3891 + 2(0.4315 + 0.4287) + 0.4645]$$

$$y(0.6) = 0.7738$$

$$z_2 = z_1 + \frac{1}{6} [l_1 + 2l_2 + 2l_3 + l_4]$$

$$= 0.9899 - \frac{1}{6} [0.0310 + 2(0.0728 + 0.0757) + 0.1392]$$

$$\therefore z(0.3) = 0.9210$$

Using R-K method of fourth order solve  $y'' = x^2 - y^2$  for  $x=0.2$ , given that  $y=1$  and  $y'=0$  when  $x=0$ .

**Solution**

Let  $y' = z$  then  $y'' = z'$

Hence the given equation reduces to the form,

$$\frac{dy}{dx} = z \quad \text{and} \quad \frac{dz}{dx} = xz^2 - y^2$$

Given  $x_0 = 0, y_0 = 1, z_0 = 0$  and  $h = 0.2$

Take  $f_1(x, y, z) = z, f_2(x, y, z) = xz^2 - y^2$

$$k_1 = hf(x_0, y_0, z_0) = (0.2)(0) = 0$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.2)(0.1) = -0.02$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.2)(-0.999) = -0.01998$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = (0.3)(-0.1958) = -0.03916$$

$$l_1 = hf(x_0, y_0, z_0) = (0.2)(-1^2) = -0.2$$

$$l_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right) = (0.2)(0.1)(0.1) - (0.99)^2 = -0.1998$$

$$l_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right) = (0.2)(0.1)(0.0999) - (0.99)^2 = -0.1958$$

$$l_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3) = (0.2)(0.2)(-0.1958) - (0.98)^2 = -0.1905$$

$$y_1 = y_0 + \frac{1}{6} [k_1 + 2k_2 + 2k_3 + k_4]$$

$$= 1 - \frac{1}{6} [0.2 + 2(0.02 + 0.01998) + 0.03916]$$

$$\therefore y(0.2) = 0.9801$$

$$\text{Also } z(0.2) = 0 - \frac{1}{6} [0.2 + 2(0.1998 + 0.1958) + 0.1905] = -0.1969$$

## Multi-Step Methods (Predictor-Corrector Methods)

### Introduction

Predictor-corrector methods are methods which require function values at  $x_n, x_{n-1}, x_{n-2}, x_{n-3}$  for the computation of the function value at  $x_{n+1}$ . A predictor is used to find the value of  $y$  at  $x_{n+1}$  and then the corrector formula is used to improve the value of  $y_{n+1}$ .

The following two methods are discussed in this chapter

- (1) Milne's Method (2) Adam's Method

### Milne's Predictor-corrector method

Consider the initial value problem  $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$ . Assume that  $y_0 = y(x_0), y_1 = y(x_1), y_2 = y(x_2)$  and  $y_3 = y(x_3)$  where  $x_{i+1} = x_i + h, i = 0, 1, 2, 3$  are known, these are the starting values.

#### Milne's predictor formula

$$y_{4,p} = y_0 + \frac{4h}{3} [y_1' - y_2' + 2y_3'] \text{ and}$$

#### Milne's corrector formula

$$y_{4,c} = y_2 + \frac{h}{3} [y_2' + 4y_3' + y_4'] \text{ where } y_4' = f(x_4, y_{4,p})$$

### Problems based on Predictor-Corrector method

1. Using Milne's method, compute  $y(0.8)$  given that  $\frac{dy}{dx} = 1 + y^2, y(0) = 1, y(0.2) = 0.2027, y(0.4) = 0.4228$  and  $y(0.6) = 0.6841$

#### Solution

we have the following table of values

x	y	$y' = 1 + y^2$
0	0	1.0
0.2	0.2027	1.0411
0.4	0.4228	1.1787
0.6	0.6481	1.4681

$$\therefore x_0 = 0, x_1 = 0.2, x_2 = 0.4, x_3 = 0.6$$

$$y_0 = 0, y_1 = 0.2027, y_2 = 0.4228, y_3 = 0.6841$$

$$y_0' = 1, y_1' = 1.0411, y_2' = 1.1787, y_3' = 1.4681$$

To find  $y(0.8)$ :

$$x_4 = 0.8. \text{ Here } h = 0.2$$

By Milne's predictor formula,

$$y_{4,p} = y_0 + \frac{4h}{3} [y_1' - y_2' + 2y_3']$$

$$= 0 + \frac{0.8}{3} [1.0411 - 1.1787 + 2(1.4681)]$$

$$\therefore y_{4,p} = 1.0239$$

$$y'_4 = f(x_4, y_4) = 1 + (0.239)^2 \\ = 2.0480$$

By Milne's corrector formula,

$$y_{4,c} = y_2 + \frac{h}{3} [y'_2 + 4y'_3 + y'_4] \\ = 0.4228 + \frac{0.2}{3} [1.1787 + 4(1.4681) + 2.0480] \\ \therefore y(0.8) = 1.0294$$

2. Given  $y' = x^2 - y$ ,  $y(0) = 1$ ,  $y(0.1) = 0.9052$ ,  $y(0.2) = 0.8213$ , find  $y(0.3)$  by Taylor series method. Also find  $y(0.4)$  by Milne's method

**Solution**

$$\begin{aligned} \text{Given } x_0 &= 0, & y_0 &= 1 \\ x_1 &= 0.1, & y_1 &= 0.9052 \\ x_2 &= 0.2, & y_2 &= 0.8213 \\ x_3 &= 0.3, & y_3 &= y(x_3) \end{aligned}$$

By Taylor algorithm

$$y_3 = y_2 + hy'_2 + \frac{h^2}{2!} y''_2 + \dots$$

$$y' = x^2 - y \Rightarrow y'' = 2x - y'$$

$$y''' = 2 - y'', y^{iv} = y''' \text{ etc}$$

$$\therefore y'_2 = (0.2)^2 - 0.8213 = -0.7813$$

$$y''_2 = 2(0.2) + 0.7813 = 1.1813$$

$$y'''_2 = 2 - 1.1813 = 0.8187$$

$$y^{iv}_2 = -8187$$

$$\therefore y_3 = 0.8213 - (0.1)(0.7813) + \frac{0.01}{2} (1.1813) - \frac{0.001}{6} (0.8187) - \frac{0.0001}{24} (8187)$$

$$\therefore y(0.3) = 0.7492$$

$$\text{For } x_3 = 0.3, y_3 = 0.7492 \text{ and } y'_3 = 0.09 - 0.7492 = -0.6592$$

$$\text{Also } y'_0 = -1, y'_1 = (0.1)^2 - 0.9052 = -0.8952 \text{ and } y'_2 = -0.7813$$

By Milne's method

$$y_{4,p} = y_0 + \frac{4h}{3} [y'_1 - y'_2 + 2y'_3]$$

$$y_{4,p} = 1 - \frac{0.4}{3} [0.8952 - 0.7813 + 2(-0.6592)]$$

$$y_{4,p} = 0.6897$$

$$y'_4 = (0.4)^2 - 0.6897 = -0.5297$$

By Correctors formula,

$$y_{4,c} = y_2 + \frac{h}{3} [y_2' + 4y_3' + y_4']$$

$$y_{4,c} = 0.8213 - \frac{0.1}{3} [0.7813 + 4(0.6592) + 0.5297]$$

$$y_{4,c} = 0.6897 \Rightarrow y(0.4) = 0.6897$$

### Adam-Bash Forth Predictor-Corrector Method

Using Newton's backward difference interpolation formula, we derive a set of predictor and corrector formulae. This method also requires past four values to estimate the fifth value.

Adam's predictor formula

$$y_{n+1,p} = y_n + \frac{h}{24} [5y_n' - 59y_{n-1}' + 37y_{n-2}' - 9y_{n-3}']$$

Adam's corrector formula

$$y_{n+1,c} = y_n + \frac{h}{24} [9y_{n+1}' + 19y_n' - 5y_{n-1}' + y_{n-2}']$$

The errors in these formulae are

$$\frac{251}{720} h^5 f^{iv} \epsilon \text{ and } -\frac{19}{720} h^5 f^{iv} \epsilon \text{ respectively.}$$

In particular,

$$y_{4,p} = y_3 + \frac{h}{24} [5y_3' - 59y_2' + 37y_1' - 9y_0']$$

And

$$y_{4,c} = y_3 + \frac{h}{24} [y_4' + 19y_3' - 5y_2' + y_1']$$

Given  $y' = 1 + y^2$ ,  $y(0) = 0$ ,  $y(0.2) = 0.2027$ ,  $y(0.4) = 0.4228$ ,  $y(0.6) = 0.6841$ , estimate  $y(0.8)$  using Adam's method.

### Solution

Form the given data

$x_0 = 0,$	$y_0 = 0,$	$y_0' = 1$
$x_1 = 0.2$	$y_1 = 0.2027$	$y_1' = 1.0411$
$x_2 = 0.4$	$y_2 = 0.4228$	$y_2' = 1.1786$
$x_3 = 0.6$	$y_3 = 0.6841$	$y_3' = 1.4680$

To find  $y_4$  for  $x_4 = 0.8$ . Here  $h = 0.2$

$$y_{4,p} = y_3 + \frac{h}{24} [5y_3' - 59y_2' + 37y_1' - 9y_0']$$

$$y_{4,p} = 0.6841 + \frac{0.2}{24} [5(1.4680) - 59(1.1786) + 37(1.0411) - 9]$$

$$y_{4,p} = 1.0235$$

$$y_{4,c} = y_3 + \frac{h}{24} [y_4' + 19y_3' - 5y_2' + y_1']$$

$$y_4' = 1 + (0.0235)^2 = 2.0475$$

$$\therefore y_{4,c} = 0.6481 + \frac{0.2}{24} [2.0475 + 19(1.4680) - 5(1.1786) + 1.0411]$$

$$y_{4,c} = 1.0297$$

$$\therefore y_{0.8} = 1.0297$$

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Unit -IV  
Part-A

1. Write the fourth ~~down~~ order Taylor algorithm.

Ans: Let  $y' = f(x, y)$  with  $y(x_0) = y_0$ . Then the Taylor algorithm is given by

$$y(x_1) = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{(4)} + \dots \text{ Where } x_1 = x_0 + h$$

and  $y_0^{(r)} = \frac{d^r y}{dx^r}$  at  $(x_0, y_0)$

2. What are the merits and demerits of Taylor series method of solution?

Ans: It is powerful single step method. It is the best method if the expression for higher order derivatives are simpler

The major demerit of this method is evaluation of higher order derivatives becomes tedious for complicated algebraic expressions.

3. Given  $y' = x + y$ ,  $y(0) = 1$ . find  $y(0.1)$  by Taylor series method.

Ans:  $y' = x + y$      $y'' = 1 + y'$      $y''' = 2 + y''$

$x_0 = 0$      $y_0 = 1$  then  $y(0.1) = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \dots$

$y(0.1) = 1 + 0.1 + \frac{0.01}{2} (2) + \frac{0.001}{6} (2) = 1.1103.$

4. Find  $y(0.1)$  by Euler's method, given that  $\frac{dy}{dx} = 1 - y$   $y(0) = 0$

Ans:  $y_1 = y_0 + h f(x_0, y_0) = 0 + (0.1)(1.0) = 0.1 \Rightarrow y(0.1) = 0.1$



5. State algorithm for modified Euler's method, to solve

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

~~Ans:~~  $y_{n+1}^{(1)} = y_n + h f(x_n, y_n)$

$$y_{n+1} = y_n + \frac{h}{2} [f(x_n, y_n) + f(x_{n+1}, y_{n+1}^{(1)})]$$

Where  $n=0, 1, 2, \dots$  and  $x_{n+1} = x_n + h$ .

6. What are the distinguished properties of Rungekutta method?

Ans: These methods do not require the higher order derivatives and requires only the function values at different points.

To evaluate  $y_{n+1}$ , we need only  $y_n$  but not previous of  $y$ 's.

The solution by these methods agree with Taylor Series solution upto the terms of  $h^r$  where  $r$  is the order of R-k method.

7. Which is better Taylor Series method or R-k method? Why?

Ans: R-k method is better. since higher order derivatives of  $y$  are not required. Taylor series method involves use of higher order derivatives which may be difficult in case of complicated algebraic functions.

8. state the order of error in R-k method of fourth order

Ans: Error  $O(h^4)$ , where  $h$  is the interval of differencing.

9. Write the predictor-Error and Corrector-Error in Milne's Method.

Ans:

$$\text{Predictor-Error} = \frac{14}{45} h^5 f^{(5)}(\xi)$$

$$\text{Corrector-Error} = -\frac{h^5}{90} y^{(5)}(\xi)$$

10. Distinguish Single-step and Multi-step method.

Ans:

Single-step methods: To find  $y_{n+1}$  the information at  $y_n$  is enough.

Multi-step methods: To find  $y_{n+1}$ , the past four values  $y_{n-3}, y_{n-2}, y_{n-1}, y_n$  are needed.



Part-B

1. Solve  $\frac{dy}{dx} = y - x^2$ ,  $y(0) = 1$ , find (i)  $y(0.1)$  and  $y(0.2)$  by R-k method (ii)  $y(0.3)$  by Euler's method (iii)  $y(0.4)$  by Milne's predictor corrector method.
2. Solve  $y'' - 0.1(1-y^2)y' + y = 0$  subject to  $y(0) = 0$ ,  $y'(0) = 1$  using 4<sup>th</sup> order R-k method (i) Find  $y(0.2)$  and  $y'(0.2)$  use step size  $h = 0.2$ .
3. Use R-k method to obtain an approx. solution to the differential equation  $\frac{dy}{dx} = y - x + 5$  at the points  $x = 2.1, 2.2, 2.3$  with initial condition.
4. Tabulate  $y(0.1)$ ,  $y(0.2)$  and  $y(0.3)$  using Taylor series method given that  $y' = y^2 + x$  and  $y(0) = 1$ .
5. Solve the initial value problem  $\frac{dy}{dx} = x - y^2$ ,  $y(0) = 1$  to find  $y(0.4)$  by Adam's method.  $h = 0.1$
6. Using Adam's method find  $y(0.4)$  given that  $y' = xy/9$ ,  $y(0) = 1$ ,  $y(0.1) = 1.01$ ,  $y(0.2) = 1.022$ ,  $y(0.3) = 1.033$ .
7. Using Taylor's series method solve  $\frac{dy}{dx} = xy + y^2$ ,  $y(0) = 1$  and at  $x = 0.1, 0.2$  and  $0.3$  continue the solution at  $x = 0.4$  by Milne's predictor-corrector formula.
8. Consider the initial value problem  $\frac{dy}{dx} = y - x^2 + 1$ ,  $y(0) = 0.5$  find (i)  $f(0.2)$  using the modified Euler method (ii) 4<sup>th</sup> order R-k method find  $y(0.4)$  and  $y(0.6)$  (iii) find  $y(0.8)$  using Adam's predictor corrector formula. find  $y'$ .

**UNIT V**  
**BOUNDARY VALUE PROBLEMS IN**  
**ORDINARY AND**  
**PARTIALDIFFERENTIAL EQUATIONS**

*Finite difference solution of second order ordinary differential equation*

*Finite difference solution of one dimensional heat equation by explicit and implicit methods*

*One dimensional wave equation*

*Two dimensional Laplace equation*

*Poisson equation*

- **Solution of Boundary Value Problems in ODE**
- **Solution of One Dimensional Heat Equation**
- **Solution of One Dimensional Wave Equation**
- **Solution of Laplace Equation**
- **Solution of Poisson Equation**

There are number of methods for solving second order boundary value problems.

- Finite Difference Method
- Shooting Method

### **Finite difference solution of second order ordinary differential equation**

Let us consider

$$y''(x)+f(x)y'(x)+g(x)y(x)=0$$

with boundary conditions  $y(x_0)=a$  and  $y(x_n)=b$

#### **Formula**

$$y''(x) = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} \quad \text{and} \quad y'(x) = \frac{y_{i+1} - y_{i-1}}{2h}$$

### **Steps for solving the second order ODE**

- Put  $y''(x)$ ,  $y'(x)$ ,  $y(x)$  by  $y_i''$ ,  $y_i'$ ,  $y_i$  respectively
- substitute the above formula
- Form an equation
- Substitute the boundary conditions
- Solve the equations by putting  $i=1, 2, 3$
- Write the solution of the intermediate values of boundary conditions

### **Problems based on Finite difference method for ODE**

1. Solve by finite difference method, the boundary value problem  $y''(x) - y(x)=2$  where

$$y(0)=0 \text{ and } y(1)=1, \text{ taking } h=\frac{1}{4}$$

#### **Solution:**

Given  $y''(x) - y(x)=2$

Step1:

Put  $y''(x)$ ,  $y(x)$  by  $y_i''$ ,  $y_i$  respectively

$$\text{i.e, } y_i'' - y_i = 2 \text{-----(1)}$$

Step2:

substitute the formula  $y''(x) = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}$

∴(1) becomes  $\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} - y_i = 2$

$\Rightarrow y_{i-1} - (2 + h^2)y_i + y_{i+1} = 2 \left[ \frac{1}{16} \right]$

$\Rightarrow y_{i-1} - (2 + h^2)y_i + y_{i+1} = \left[ \frac{1}{8} \right]$

$\Rightarrow y_{i-1} - \frac{33}{16} y_i + y_{i+1} = \left[ \frac{1}{8} \right]$

$\Rightarrow y_{i-1} - 2.0625 y_i + y_{i+1} = 0.125$  -----(2)

**Step3:**

The boundary conditions are  $y_0=0, y_4=1$

$h = \frac{1}{4}$  gives

$x_0=0$	$x_1 = \frac{1}{4}$	$x_2 = \frac{1}{2}$	$x_3 = \frac{3}{4}$	$x_4=1$
$y_0=0$	$y_1=?$	$y_2=?$	$y_3=?$	$y_4=1$

**Step4**

Put  $i=1, 2, 3$  we get the following equations

$-2.0625 y_1 + y_2 = 0.125$  -----(3)

$y_1 - 2.0625 y_2 + y_3 = 0.125$  -----(4)

$y_2 - 2.0625 y_3 = -0.875$  -----(5)

Solving these three equations we get

<b><math>y_1=0.0451</math></b>	<b><math>y_2=0.2183</math></b>	<b><math>y_3=0.5301</math></b>
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2. Solve the equation  $y''(x) - x y(x) = 0$  where  $x_i=0, \frac{1}{3}, \frac{2}{3}$ , given that  $y(0)+y'(0)=1$  and  $y(1)=1$ .

**Solution:**

**Step1:**

Put  $y''(x), y(x), x$  by  $y_i'', y_i, x_i$  respectively

i.e,  $y_i'' - x_i y_i = 0$  -----(1)

**Step2:**

substitute the formula  $y''(x) = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}$

∴(1) becomes  $\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} - x_i y_i = 0$

$$\Rightarrow y_{i-1} - \left(2 + \frac{1}{9}x_i\right)y_i + y_{i+1} = 0 \text{-----(2)}$$

**Step3:**

The conditions are  $y(0)+y'(0)=1$  and  $y(1)=1$

$x_0=0$	$x_1=\frac{1}{3}$	$x_2=\frac{2}{3}$	$x_3=1$
$y_0=?$	$y_1=?$	$y_2=?$	$y_3=1$

Putting  $i=0, 1, 2$  we get

$$y_{-1} - 2y_0 + y_1 = 0 \text{-----(3)}$$

$$y_0 - \frac{55}{27}y_1 + y_2 = 0 \text{-----(4)}$$

$$y_1 - \frac{56}{27}y_2 + y_3 = 0 \text{-----(5)}$$

**Step4:**

To solve the above equations we need the values of  $y_{-1}$  and  $y_3$   
we have  $y_3=1$

So now we have to find the value of  $y_{-1}$  by using the condition  $y(0)+y'(0)=1$

$$y(0)+y'(0)=1 \Rightarrow y_0 + y'_0 = 1$$

$$\Rightarrow y_0 + \frac{y_1 - y_{-1}}{2h} = 1 \text{ [put } i=0 \text{ in } y'_i = \frac{y_{i+1} - y_{i-1}}{2h} \text{ ]}$$

put  $h=\frac{1}{3}$  we get

$$y_{-1} = \frac{2y_0 + 3y_1 - 2}{3} \text{-----(6)}$$

substitute (6) in (3),

$$-2y_0 + 3y_1 = 1 \text{-----(7)}$$

$$y_0 - \frac{55}{27}y_1 + y_2 = 0 \text{-----(8)}$$

$$y_1 - \frac{56}{27}y_2 + 1 = 0 \text{-----(9)}$$

Solving the above three equations we get

$y_0=-0.9880$	$y_1=-0.3253$	$y_2=0.3253$
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### **Classification of second order PDE**

The most general linear pde of second order can be written as

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + Fu = 0 \text{-----(1)}$$

where A, B, C, D, E, F are in general functions of x and y.

Equation (1) is said to be

$$\rightarrow \text{Elliptic} \quad \text{if } B^2 - 4AC < 0$$



- hyperbolic if  $B^2-4AC>0$   
 → Parabolic if  $B^2-4AC=0$

**Note:**

✍ The same differential equation may be elliptic in one region, parabolic in another region and hyperbolic in some other region.

**Examples**

<b>Elliptic</b>	<b>Parabolic</b>	<b>Hyperbolic</b>
1. $u_{xx}+u_{yy}=0$ (Laplace Equation)	$u_{xx}=\frac{1}{\alpha^2} u_t$ (One dimensional heat equation)	$u_{xx}=\frac{1}{\alpha^2} u_{tt}$ (One dimensional wave equation)
2. $u_{xx}+u_{yy}=f(x, y)$ (Poisson Equation)		

**Problems based on the classification of 2<sup>nd</sup> order PDE**

1. Classify  $u_{xx}+4u_{xy}+(x^2+4y^2)+u_{yy}=\sin(x+y)$

**Solution:**

$A = 1, B = 4, C = x^2+4y^2$   
 $B^2-4AC=4[4-x^2-4y^2]$

The equation is elliptic if  $4-x^2-4y^2<0 \Rightarrow \frac{x^2}{4} + \frac{y^2}{1} > 1$  (outside the ellipse  $\frac{x^2}{4} + \frac{y^2}{1} = 1$ )

Parabolic  $4-x^2-4y^2=0 \Rightarrow \frac{x^2}{4} + \frac{y^2}{1} = 1$  (on the ellipse  $\frac{x^2}{4} + \frac{y^2}{1} = 1$ )

Hyperbolic  $4-x^2-4y^2>0 \Rightarrow \frac{x^2}{4} + \frac{y^2}{1} < 1$  (inside the ellipse  $\frac{x^2}{4} + \frac{y^2}{1} = 1$ )

2. Classify  $(x+1)u_{xx}-2(x+2)u_{xy}+(x+3)u_{yy}=0$

**Solution:**

$A=(x+1), B=-2(x+2), C=(x+3)$   
 $B^2-4AC=4>0$   
 $\therefore$  hyperbolic.

**One Dimensional Heat Equation**

One dimensional heat equation is  $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$

**Classification:** Parabolic

There are two types for finding the Finite difference solution of one dimensional heat flow equation

- (i) **Bender-Schmidt Method(Explicit Method valid for  $0 < \lambda \leq \frac{1}{2}$ )**
- (ii) **Crank-Nicholson Method(Implicit Method)**

**Bender-Schmidt Method(Explicit Method)**

Coefficient of  $\frac{\partial u}{\partial t} = a$ ,  $x_i = x_0 + ih$  and  $t_j = t_0 + jk$

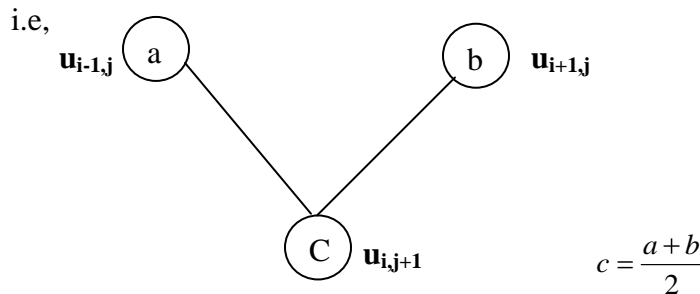
**Case 1: (If h is given and k is not given)**

Assume that  $\lambda = \frac{1}{2}$

Find k, using the formula  $k = \lambda ah^2$

Using the given (i), (ii) and (iii) conditions fill the first row, first and last column of the table  
Next using the Bender-Schmidt recurrence relation,

$$u_{i,j+1} = \frac{1}{2} (u_{i-1,j} + u_{i+1,j})$$



**Case 2: (If h and k are given)**

Find  $\lambda$ , using the formula  $k = \lambda ah^2$

Using the given (i), (ii) and (iii) conditions fill the first row, first and last column of the table  
Next using the Bender-Schmidt recurrence relation,

$$u_{i,j+1} = \lambda (u_{i-1,j} + u_{i+1,j}) + (1 - 2\lambda)u_{i,j}$$

Using the formula and tabulate the remaining rows.

**Examples**

1. Solve  $u_{xx} = 2u_t$  when  $u(0, t) = u(4, t) = 0$  and with initial condition  $u(x, 0) = x(4-x)$  upto  $t=5$  sec, assuming  $\Delta x = h = 1$ .

Solution:

Given:  $a =$  Coefficient of  $\frac{\partial u}{\partial t}$

$\Rightarrow a = 2$  ;  $h = 1$  and  $k$  is not given.

Assume that  $\lambda = \frac{1}{2} \Rightarrow k = \lambda ah^2 = \frac{1}{2} \times 2 \times 1 = 1$

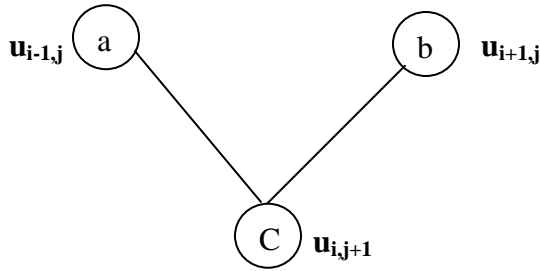
By the given conditions  $x = 0, 1, 2, 3, 4$  and  $t = 0, 1, 2, 3, 4, 5$

- Condition(i)  $u(0, t)=0 \Rightarrow u(0, 0)=0, u(0, 1)=0, u(0, 2)=0, u(0, 3)=0, u(0, 4)=0, u(0, 5)=0$   
 Condition(ii)  $u(4, t)=0 \Rightarrow u(4, 0)=0, u(4, 1)=0, u(4, 2)=0, u(4, 3)=0, u(4, 4)=0, u(4, 5)=0$   
 Condition(iii)  $u(x, 0)=x(4-x) \Rightarrow u(1,0)=1(4-1)=3,$   
 $u(2,0)=2(4-2)=4,$   
 $u(3,0)=3(4-3)=3,$

Bender-Schmidt recurrence relation,

$$u_{i,j+1} = \frac{1}{2} (u_{i-1,j} + u_{i+1,j})$$

i.e,



$$c = \frac{a+b}{2}$$

$x_i = x_0 + ih$  and  $t_j = t_0 + jk$

i \ j		0	1	2	3	4
		x=0	x=1	x=2	x=3	x=4
0	t=0	0	3	4	3	0
1	t=1	0	2	3	2	0
2	t=2	0	1.5	2	1.5	0
3	t=3	0	1	1.5	1	0
4	t=4	0	0.75	1	0.75	0
5	t=5	0	0.5	0.75	0.5	0

2. Find the values of the function satisfying the pde  $4u_{xx}=u_t$  and the boundary conditions  $u(0, t) = u(8,t)=0$  and  $u(x, 0)=4x - \frac{x^2}{2}$  for points  $x=0,1, 2, 3, 4, 5, 6, 7, 8, t=\frac{j}{8}, j=0,1, 2, 3, 4, 5$

Solution:

Given:  $a = \text{Coefficient of } \frac{\partial u}{\partial t}$

$\Rightarrow a = 1/4$ ; Here  $h$  and  $k$  is not given. Take  $h=1$

Assume that  $\lambda = 1/2 \Rightarrow k = \lambda h^2 = \frac{1}{2} \times \frac{1}{4} \times 1 = \frac{1}{8}$

By the given conditions  $x=0,1, 2, 3, 4, 5, 6, 7, 8$  and  $t=0, \frac{1}{8}, \frac{2}{8}, \frac{3}{8}, \frac{4}{8}, \frac{5}{8}$

Condition(i)  $u(0, t)=0 \forall t=0, \frac{1}{8}, \frac{2}{8}, \frac{3}{8}, \frac{4}{8}, \frac{5}{8}$

Condition(ii)  $u(8, t)=0 \forall t=0, \frac{1}{8}, \frac{2}{8}, \frac{3}{8}, \frac{4}{8}, \frac{5}{8}$

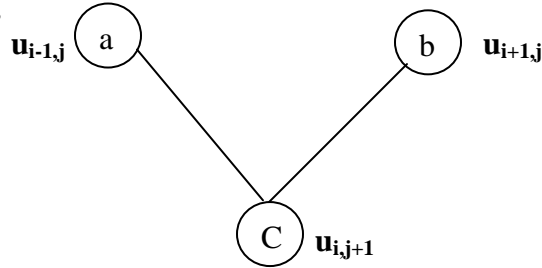
Condition(iii)  $u(x, 0)=4x - \frac{x^2}{2} \Rightarrow u(1,0) = 4 - \frac{1}{2} = 3.5, u(2,0) = 8 - \frac{4}{2} = 6, u(3,0) = 12 - \frac{9}{2} = 7.5,$

$$u(4,0) = 16 - \frac{16}{2} = 8, \quad u(5,0) = 20 - \frac{25}{2} = 7.5, \quad u(6,0) = 24 - \frac{36}{2} = 6, \quad u(7,0) = 28 - \frac{49}{2} = 3.5.$$

Bender-Schmidt recurrence relation,

$$u_{i,j+1} = \frac{1}{2} (u_{i-1,j} + u_{i+1,j})$$

i.e.,



$$x_i = x_0 + ih \text{ and } t_j = t_0 + jk$$

i	j	0	1	2	3	4	5	6	7	8
		x=0	x=1	x=2	x=3	x=4	x=5	x=6	x=7	x=8
0	t=0	0	3.5	6	7.5	8	7.5	6	3.5	0
1	t=1/8	0	3	5.5	7	7.5	7	5.5	3	0
2	t=2/8	0	2.75	5	6.5	7	6.5	5	2.75	0
3	t=3/8	0	2.5	4.625	6	6.5	6	4.625	2.5	0
4	t=4/8	0	2.3125	4.25	5.5625	6	5.5625	4.25	2.3125	0
5	t=5/8	0	2.125	3.9875	5.125	5.5625	5.125	3.9875	2.125	0

3. Solve  $u_{xx} = u_t$  when  $u(0, t) = 0, u(4, t) = 0$  with initial boundary condition  $u(x, 0) = x(4-x)$ ,  
 $h = 1, k = \frac{1}{4}$  upto  $t = 1$  sec.

Solution:

Given:  $a =$  Coefficient of  $\frac{\partial u}{\partial t}$

$$\Rightarrow a = 1; \text{ Here } h \text{ and } k \text{ are given. Take } h = 1, k = \frac{1}{4}$$

$$\therefore \lambda = \frac{kh^2}{a} = \frac{1}{4}$$

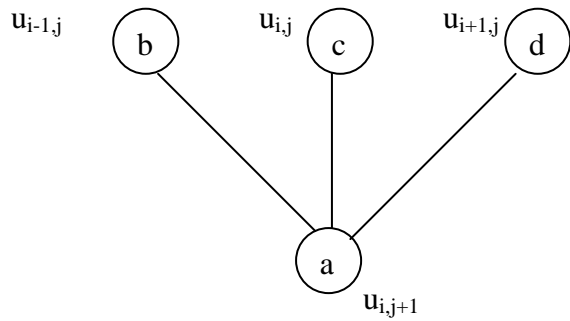
This is in the interval  $0 < \lambda \leq \frac{1}{2}$

So we use Bender-Schmidt method

Formula:

$$u_{i,j+1} = \frac{1}{2} (u_{i-1,j} + \lambda u_{i+1,j} + (1 - 2\lambda)u_{i,j})$$

$$\text{put } \lambda = \frac{1}{4} \Rightarrow u_{i,j+1} = \left[ \frac{1}{4} (u_{i-1,j} + u_{i+1,j}) + \frac{1}{2} u_{i,j} \right]$$



$$a = \frac{1}{4} (b + d) + \frac{1}{2} c$$

<b>i</b>		<b>0</b>	<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>
		x=0	x=1	x=2	x=3	x=4
<b>0</b>	t=0	0	3	4	3	0
<b>1</b>	t=1/4	0	2.5	3.5	2.5	0
<b>2</b>	t=1/2	0	2.125	3	2.125	0
<b>3</b>	t=3/4	0	1.8125	2.5625	1.8125	0
<b>4</b>	t=1	0	1.5468	2.1875	1.5468	0

→boundary condition

→boundary condition

↓  
boundary condition

↓  
boundary condition

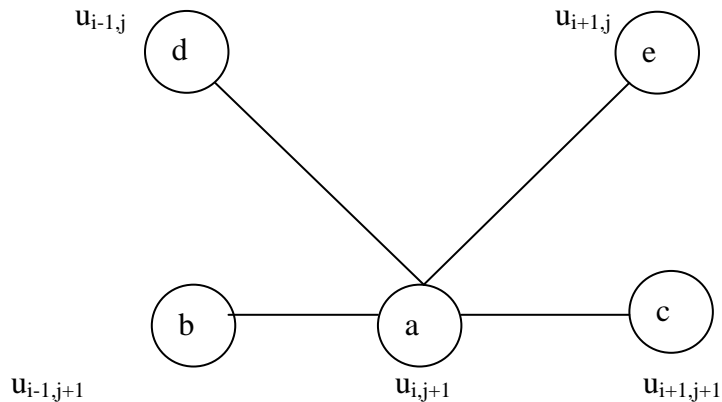
**Crank-Nicholson's Method(Implicit Method)**

**Formula**

**Case(1) (h is given and k is not given)**

Assume  $\lambda=1$  where  $\lambda = \frac{k}{ah^2}$  find k using this formula

$$u_{i,j+1} = \frac{1}{4} (u_{i+1,j+1} + u_{i-1,j+1} + u_{i-1,j} + u_{i+1,j})$$



$$a = \frac{1}{4}(b + c + d + e)$$

using the formula and tabulate the values

**Case(2) (h and k are given)**

Find the value of  $\lambda$  using the formula  $\lambda = \frac{k}{ah^2}$

**Formula:**

$$(2\lambda + 2)u_{i,j+1} - \lambda u_{i-1,j+1} - \lambda u_{i+1,j+1} = \lambda u_{i-1,j} + \lambda u_{i+1,j} + (2 - 2\lambda)u_{i,j}$$

Using the formula and tabulate the values.

**Problems based on Crank- Nicholson Method**

1. Solve  $u_{xx}=2u_t$  when  $u(0, t) = u(4,t)=0$  and with initial condition  $u(x, 0)=x(4-x)$  assuming  $\Delta x=h=1$  and compute u for one time step.

**Solution:**

Here h is given and k is not given.

$$\therefore \lambda=1. \quad k=\lambda ah^2=2$$

**Formula**

$$u_{i,j+1} = \frac{1}{4} (u_{i+1,j+1} + u_{i-1,j+1} + u_{i-1,j} + u_{i+1,j})$$

we have to find  $u_{11}, u_{21}, u_{31}$   
 put  $j=0$ ,

$$u_{i,1} = \frac{1}{4} (u_{i+1,1} + u_{i-1,1} + u_{i-1,0} + u_{i+1,0})$$

put  $i=1, 2, 3$  we get the following equations

**$4u_{11}-4u_{21}=4$ ;  $-u_{11}+4u_{21}-u_{31}=6$ ;  $0u_{11}-u_{21}+4u_{31}=4$**  respectively.

We solve these equations we get the solution

**$u_{11}=1.571, u_{21}=2.2857, u_{31}=1.571$**

i		0	1	2	3	4	
		x=0	x=1	x=2	x=3	x=4	
j	0	t=0	0	3	4	3	0
	1	t=2	0	1.571	2.2857	1.571	0

**One dimensional wave equation**

One dimensional wave equation is  $u_{xx} = \frac{1}{a^2} u_{tt}$

There are two types

**Type1:**

Subject to the boundary conditions  $u(0, t) = 0, u(1, t) = 0$  and the initial condition  $u(x,0)=f(x)$  and  $u_t(x, 0)=0$ .

**Type2:**

Subject to the boundary conditions  $u(0, t) = 0, u(1, t) = 0$  and the initial condition  $u(x,0)=0$  and  $u_t(x, 0)=f(x)$

**Formula**

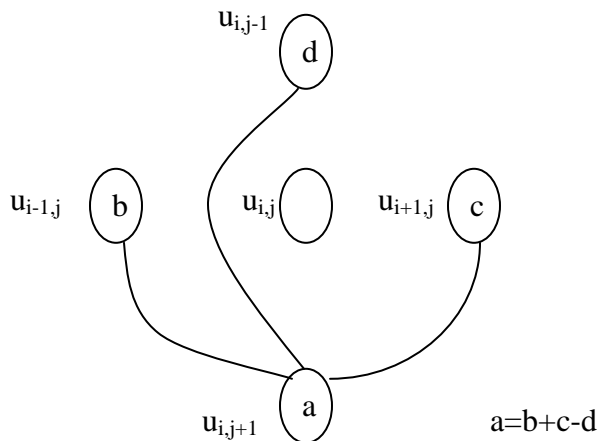
$$u_{i,j+1} = 2(-\lambda^2 a^2 u_{i,j} + \lambda^2 a^2 (u_{i-1,j} + u_{i+1,j})) u_{i,j-1} \dots\dots\dots(1)$$

where  $\lambda = \frac{k}{h}$ .

This formula is the explicit scheme for the wave equation.

If  $k = \frac{h}{a}$  the equation (1) takes the form

$$u_{i,j+1} = (u_{i-1,j} + u_{i+1,j}) - u_{i,j-1}$$



and  $u_t(x, 0) = 0 \Rightarrow u_{i,1} = \frac{1}{2} (u_{i-1,0} + u_{i+1,0})$  which gives the values of  $u$  at the first time step in terms of the values of  $u$  at time  $t = 0$ .  
This is used to solve the type 1

**Suppose the problem is given in type2**

We write  $u_t(x, 0) = f(x)$  as  $u_t(x, 0) = f(ih)$  and substitute in the formula

$$\frac{u_{i,j+1} - u_{i,j-1}}{2k} = f(ih) \text{ for } j=0$$

And then find the value of  $u_{11}, u_{21}, u_{31}$ .

The values of Remaining rows are same as type1

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**Problems based on one dimensional wave equation**

1. Find the nodal values of the equation  $16u_{xx} = u_{tt}$  taking  $\Delta x = 1$  given that  $u(0, t) = 0$ ,  $u(5, t) = 0$ ,  $u_t(x, 0) = 0$  and  $u(x, 0) = x^2(5-x)$  and upto one half of the period of vibration.

**Solution:**

One dimensional wave equation is  $u_{xx} = \frac{1}{\alpha^2} u_{tt} \Rightarrow \alpha = 4$ .

$u(5, t) = 0 \Rightarrow l = 5$

Period of vibration is  $= \frac{2l}{a} = \frac{5}{2} = 2.5$

Half period of vibration = 1.25

$\Delta x = 1 \Rightarrow h = 1$ . Since  $k$  is not given we choose  $k = \frac{h}{a} \Rightarrow k = \frac{1}{4}$



**Formula:**

$$u_{i,j+1} = \frac{1}{2}(u_{i+1,j} + u_{i-1,j} - u_{i,j-1}) \text{-----(1)}$$

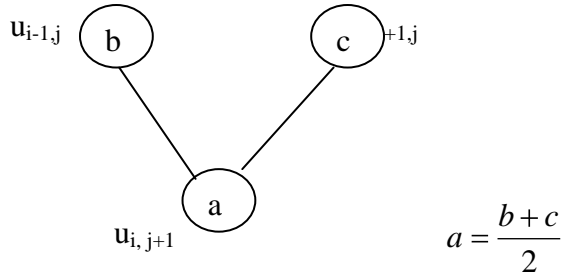
Conditions

$$u(0,t)=0 \text{ for } t=0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4} \quad u(5,t)=0 \text{ for } t=0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}$$

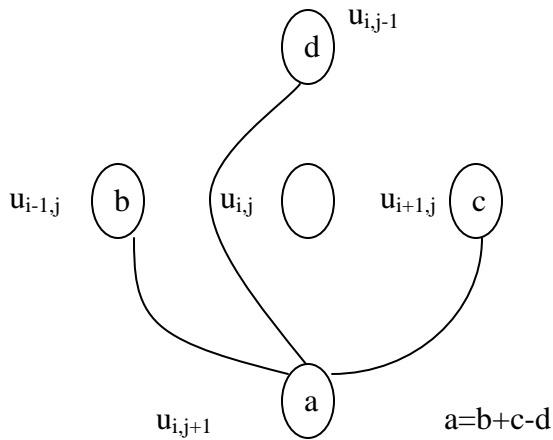
$$u(x,0) = x^2(5-x) \Rightarrow u(1,0)=4, \quad u(2,0)=12, u(3,0)=18, u(4,0)=16$$

$$u_t(x,0) = 0 \Rightarrow \frac{u_{i,j+1} - u_{i,j-1}}{2k} = 0 \text{ for } j=0$$

$\Rightarrow u_{i,1} = u_{i,-1} \Rightarrow$  we find the values of the second using this formula



The remaining rows obtained by the following formula



The values of u at the time steps

<b>i</b>		<b>0</b>	<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>
		x=0	x=1	x=2	x=3	x=4	x=5
<b>0</b>	t=0	0	4	12	18	16	0
<b>1</b>	t= $\frac{1}{4}$	0	6	11	14	9	0
<b>2</b>	t= $\frac{1}{2}$	0	7	8	2	-2	0
<b>3</b>	t= $\frac{3}{4}$	0	2	-2	-8	-7	0
<b>4</b>	t=1	0	-9	-14	-11	-10	0
<b>5</b>	t= $\frac{5}{4}$	0	-16	-22	-32	-18	0

2. Solve  $u_{xx}=u_{tt}, 0 < x < 2, t > 0$  subject  $u(x,0)=0, u_t(x, 0)=100(2x-x^2), u(0, t) =0, u(2,t)=0$ .  
 Choosing  $h=1/2$ , compute  $u$  for 4 time steps.

**Solution**

Here  $h=1/2$  and  $a = 1$ . We choose  $k = \frac{h}{a} \Rightarrow k = \frac{1}{2}$

The simplest explicit scheme is given by

$$u_{i,j+1} = \frac{1}{2}(u_{i+1,j} + u_{i-1,j} - u_{i,j-1}) \text{-----(1)}$$

The boundary conditions are

$$u(0,t)=0 \text{ for } t=0, \frac{1}{2}, 1, \frac{3}{2}, 2$$

$$u(2,t)=0 \text{ for } t=0, \frac{1}{2}, 1, \frac{3}{2}, 2$$

$$u(x,0) = 0 \text{ for } x =0, \frac{1}{2}, 1, \frac{3}{2}, 2$$

$$u_t(x, 0)=100(2x-x^2) \Rightarrow \frac{u_{i,j+1} - u_{i,j-1}}{2k} = 100(2ih - i^2h^2) \text{ for } j=0$$

Here  $h=k=1/2 \Rightarrow u_{i,1} - u_{i,-1} = 100\left(i - \frac{i^2}{4}\right)$  for  $i=1, 2, 3$ .

$$\Rightarrow u_{i,-1} = u_{i,1} - 100\left(i - \frac{i^2}{4}\right) \text{ for } i=1, 2, 3. \text{-----(2)}$$

From (1) we get  $u_{i,1} = \frac{1}{2}(u_{i+1,0} + u_{i-1,0} - u_{i,-1})$

Sub (2) in (1) we get

$$u_{i,1} = \frac{u_{i+1,0} + u_{i-1,0}}{2} + 50\left(i - \frac{i^2}{4}\right) \text{ for } i=1, 2, 3.$$

But  $u_{i,0} = 0, \forall i$

$$\therefore u_{i,1} = 50\left(i - \frac{i^2}{4}\right) \text{ for } i=1, 2, 3$$

So we have  $u_{11}=37.5, u_{21}=50, u_{31}=37.5$

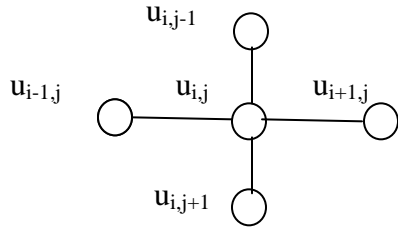
<b>i</b>		<b>0</b>	<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>
		x=0	x= $\frac{1}{2}$	x=1	x= $\frac{3}{2}$	x=2
<b>0</b>	t=0	0	0	0	0	0
<b>1</b>	t= $\frac{1}{2}$	0	37.5	50	37.5	0
<b>2</b>	t=1	0	50	75	50	0
<b>3</b>	t= $\frac{3}{2}$	0	37.5	50	37.5	0
<b>4</b>	t=2	0	0	0	0	0

**Two dimensional Laplace Equation**

Laplace equation is  $\nabla^2 u = u_{xx} + u_{yy} = 0$

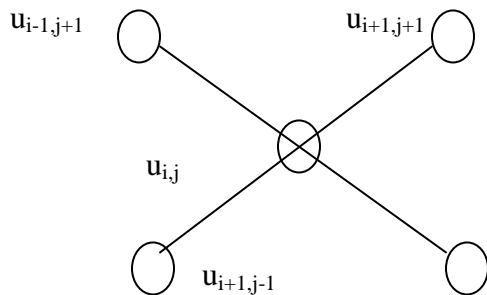
**Standard Five Point Formula(SFPF)**

$$u_{i,j} = \frac{1}{4} (u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1})$$



**Diagonal Five Point Formula(DFPF)**

$$u_{i,j} = \frac{1}{4} (u_{i-1,j+1} + u_{i+1,j+1} + u_{i-1,j-1} + u_{i+1,j-1})$$



$u_{i-1,j-1}$

**Liebmann's Iteration Process**

	b13	b12	b11	b10	b9	
b14		u1	u2	u3		b8
b15		u4	u5	u6		b7
b16		u7	u8	u9		b6
	b1	b2	b3	b4	b5	

We compute the initial values of  $u_1, u_2, \dots, u_9$  by using SFPF and DFPF. First we compute  $u_5$  by SFPF.

$$u_5 = \frac{1}{4} [b_7 + b_{15} + b_{11} + b_3]$$

We compute  $u_1, u_3, u_7, u_9$  by using DFPF

$$u_1 = \frac{1}{4} [b_1 + u_5 + b_3 + b_{15}] \quad u_3 = \frac{1}{4} [b_5 + b_5 + b_3 + b_{11}]$$

$$u_7 = \frac{1}{4} [b_{13} + u_5 + b_{15} + b_{11}] \quad u_9 = \frac{1}{4} [b_7 + b_{11} + u_5 + b_9]$$

Finally we compute  $u_2, u_4, u_6, u_8$  using SFPP

$$u_2 = \frac{1}{4} [b_3 + u_5 + u_1 + u_3] \quad u_4 = \frac{1}{4} [b_1 + u_7 + b_{15} + u_5]$$

$$u_6 = \frac{1}{4} [b_3 + u_9 + u_5 + b_7] \quad u_8 = \frac{1}{4} [b_{11} + u_5 + u_7 + u_9]$$

These initial values are called rough values.

After that we use only SFPP method

Once all the values are computed their accuracy can be improved by Gauss-Seidal method.

The Gauss-Seidal formula is given by

$$u_{i,j}^{(n+1)} = \frac{1}{4} [u_{i-1,j}^{(n+1)} + u_{i+1,j}^n + u_{i,j-1}^n + u_{i,j+1}^{(n+1)}]$$

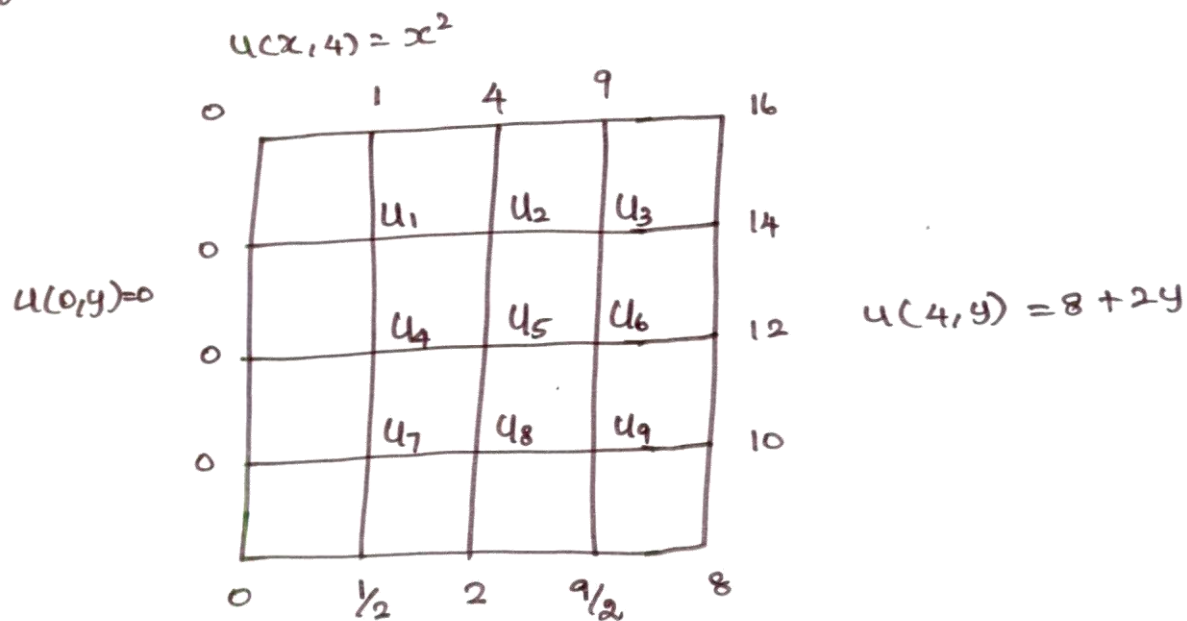
## Problems based on Laplace Equation

1. Solve  $u_{xx} + u_{yy} = 0$  in  $0 \leq x \leq 4$ ,  $0 \leq y \leq 4$

Given that  $u(0, y) = 0$ ,  $u(4, y) = 8 + 2y$ ,  $u(x, 0) = \frac{x^2}{2}$  and  $u(x, 4) = 2$  taking  $h = k = 1$ . Obtain the result to one decimal.

Solution:-

Given  $0 \leq x \leq 4$ ,  $0 \leq y \leq 4$  and  $h = k = 1$ . Therefore the region of  $u(x, y)$  can be divided into 16 squares as shown in fig.



Rough Values:

$$u_5 = \frac{0 + 12 + 4 + 2}{4} = 4.5 \quad [\text{SFPP}]$$

$$u_1 = \frac{0 + 4 + 4.5 + 0}{4} = 2.125 \quad [\text{DFPP}]$$

$$u_3 = \frac{4 + 16 + 12 + 4.5}{4} = 9.125 \quad [\text{DFPP}]$$

$$u_7 = \frac{0 + 4 + 9 + 0}{4} = 3.25 \quad [\text{DFPP}]$$

$$U_1 = \frac{2+4.5+8+12}{4} = 6.625 \text{ [DFPF]}. \quad \text{www.Vidarthiplus.in}$$

$$U_2 = \frac{2.125+4+4.5+9.125}{4} = 4.9375 \text{ [SFPP]}$$

$$U_4 = \frac{0+2.125+4.5+1.625}{4} = 2.0625 \text{ [SFPP]}$$

$$U_6 = \frac{4.5+9.125+12+6.625}{4} = 8.0625$$

$$U_8 = \frac{1.625+4.5+6.625+2}{4} = 3.6875$$

Liebmann iteration formula

$$U_{i,j}^{(n+1)} = \frac{1}{4} [U_{i-1,j}^{(n+1)} + U_{i,j+1}^{(n+1)} + U_{i+1,j}^{(n)} + U_{i,j-1}^{(n)}]$$

Iteration 1 :

$$U_1^{(1)} = \frac{1}{4} [0+1+2.0625+4.9375] = 2$$

$$U_2^{(1)} = \frac{1}{4} [2+4+4.5+9.125] = 4.906 \quad U_3^{(1)} = \frac{1}{4} [4.906+9+8.0625+12] = 8.992$$

$$U_4^{(1)} = \frac{1}{4} [0+2+1.625+4.5] = 2.031 \quad U_5^{(1)} = \frac{1}{4} [2.031+4.906+3.6875+8.0625] = 4.671$$

$$U_6^{(1)} = \frac{1}{4} [4.671+8.992+6.625+12] = 8.072 \quad U_7^{(1)} = \frac{1}{4} [2.031+0.5+3.6875] = 1.554$$

$$U_8^{(1)} = 3.712 \quad U_9^{(1)} = \frac{1}{4} [3.712+8.072+4.5+10] = 6.571.$$

Second Iteration

$$U_1^{(2)} = 1.984 \quad U_2^{(2)} = 4.912 \quad U_3^{(2)} = 8.996 \quad U_4^{(2)} = 2.05 \quad U_5^{(2)} = 4.687 \quad U_6^{(2)} = 8.063$$

$$U_7^{(2)} = 1.565 \quad U_8^{(2)} = 3.706 \quad U_9^{(2)} = 6.567 \quad \text{www.Vidarthiplus.in}$$



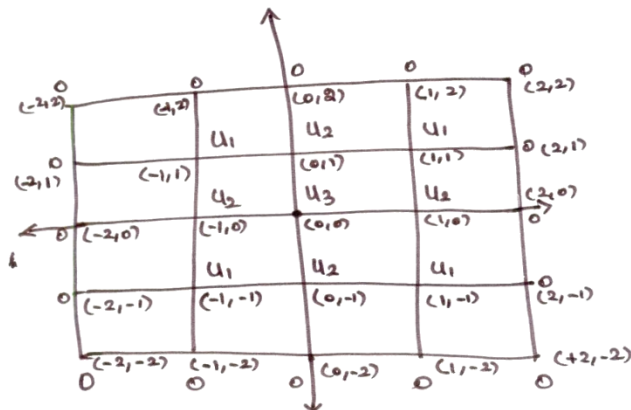
Iteration 3:  $u_1^{(3)} = 1.990$   $u_2^{(3)} = 4.918$   $u_3^{(3)} = 8.995$   $u_4^{(3)} = 2.06$   $u_5^{(3)} = 4.187$   
 $u_6^{(3)} = 8.062$   $u_7^{(3)} = 1.567$   $u_8^{(3)} = 3.705$   $u_9^{(3)} = 6.566$ .  
 $\therefore u_9 \approx 1.567, u_8 \approx 3.705, u_7 \approx 6.56, u_6 \approx 8.06, u_5 \approx 4.68, u_4 \approx 2.06$   
 $u_3 \approx 8.99, u_2 \approx 4.91, u_1 \approx 1.99.$

Poisson Equation:

Formula:  $u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = h^2 f(ih, jh)$

Problem:

Solve the Poisson's equation  $\nabla^2 u = 8x^2 y^2$  for the square mesh of the region with  $u(x,y) = 0$  on the boundary and mesh length = 1



Given  $h=1$ . From the standard five point formula we get

$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = 8i^2 j^2$$

For  $u_1$  ( $i=-1, j=-1$ )  $\textcircled{1} \Rightarrow 2u_2 - 4u_1 = 8 \Rightarrow u_2 - 2u_1 = 4 \text{ --- } \textcircled{2}$

For  $u_2$  ( $i=1, j=0$ )  $\textcircled{1} \Rightarrow 2u_1 + u_3 - 4u_2 = 0 \text{ --- } \textcircled{3}$

For  $u_3$  ( $i=0, j=0$ )  $\textcircled{1} \Rightarrow u_2 - u_3 = 0 \text{ --- } \textcircled{4}$ .

From (4) we get  $u_2 = u_3$  Solving (2), (3) and (4) we get

$$u_1 = -3, u_2 = -2, u_3 = -2.$$



Unit-5

Part-A

1. State the finite approx. for  $y'$  and  $y''$  with error terms

Ans:  $y_i = y(x_i)$  and  $x_{i+1} = x_i + h$ ,  $i = 0, 1, \dots, n$ .

Then  $y'_i = \frac{y_{i+1} - y_{i-1}}{2h}$  Error =  $O(h^2)$

$y''_i = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}$  Error =  $O(h^2)$

2. Solve  $xy'' + y' = 0$ ,  $y(1) = 1$ ,  $y(2) = 2$  with  $h = 0.5$

Ans:

Finite difference scheme

$$4x_i (y_{i+1} - 2y_i + y_{i-1}) + y_i = 0$$

$$\text{For } i=1, \quad 4x_1 (y_2 - 2y_1 + y_0) + y_1 = 0$$

$$\Rightarrow 11y_1 = 18 \Rightarrow y_1 = \frac{18}{11} = 1.6364$$

3. Give the finite difference scheme to solve  $u_{xx} + u_{yy} = 0$  numerically.

Ans: For square mesh of sides  $\Delta x = h = \Delta y$ ,

$$u_{i,j} = \frac{1}{4} [u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}]$$

Where  $u_{i,j} = u(x_i, y_j)$

4. What is the purpose of Liebmann's process?

Ans: The purpose of Liebmann's process is to find the solution of Laplace equation  $u_{xx} + u_{yy} = 0$  by iteration over a square with boundary values.

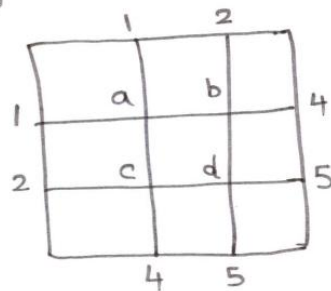
5. Express  $u_{xx} + u_{yy} = f(x, y)$  in finite difference scheme

Ans: For a square mesh with interval of differencing  $h$ , we have

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{ij} = h^2 f(ih, jh).$$

6. For the following mesh, in solving  $\nabla^2 u = 0$  find one set of rough values of  $u$  at interior mesh points

Ans:-



By Symmetry  $b = c$ . Assume  $b = 3$  [ $\because$   $b$  is at  $1/3$  distance from  $u = 5$ ]

$$\Rightarrow b = 2 + \frac{1}{3}[5 - 2] = 3$$

Rough values:  $a = \frac{1}{4}[1 + 1 + 2b] = 2$ ;  $b = 3$  and  $d = \frac{1}{4}[5 + 5 + 2b] = 4$

7. Write the one-dimensional heat flow equation in finite difference scheme.

Ans: One dimensional heat equation is  $\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha^2} \frac{\partial u}{\partial t}$ .

Using finite difference scheme to solve the eqn

we get  $u_{i,j+1} = \lambda [u_{i-1,j} + u_{i+1,j}] + (1 - 2\lambda)u_{i,j}$

Where  $\lambda = \frac{k\alpha^2}{h^2}$ .

8. State Bender-Schmidt finite difference explicit scheme to solve  $\frac{\partial y}{\partial t} = \alpha^2 \frac{\partial^2 y}{\partial x^2}$ .

Ans:-  $U_{i,j+1} = \frac{1}{2} [U_{i-1,j} + U_{i+1,j}]$

---

9. State Crank-Nicholson scheme for solving one dimensional heat equation.

Ans:- Heat equation  $\frac{\partial u}{\partial t} = c \frac{\partial^2 u}{\partial x^2}$

Implicit scheme:

$$-\lambda U_{i+1,j+1} + 2(1+\lambda)U_{i,j+1} - \lambda U_{i-1,j+1} = -\lambda U_{i+1,j} + 2(1-\lambda)U_{i,j} + \lambda U_{i-1,j}$$

When  $\lambda = \frac{kc}{h^2}$ .

---

10. In solving the wave equation  $u_{tt} = a^2 u_{xx}$  how will you express the initial condition  $u_t(x,0) = 0$ ? Indicate the final result also.

Ans:-

Using forward difference,  $\frac{U_{i,j+1} - U_{i,j}}{k} = 0$  for  $j=0$

$\Rightarrow U_{i,1} = U_{i,0}$  and using central differences,

$\frac{U_{i,j+1} - U_{i,j-1}}{2k} = 0 \Rightarrow U_{i,1} = U_{i,-1}$  and

$U_{i,1} = \frac{U_{i-1,0} + U_{i+1,0}}{2}$ .

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Part-B

1. Solve  $u_{xx} + u_{yy} = 0$  in the square region bounded by  $x=0$ ,  $x=4$ ,  $y=0$ ,  $y=4$  and with boundary conditions  $u(0,y)=0$ ,  $u(4,y)=8+2y$ ,  $u(x,0)=\frac{x^2}{2}$  and  $u(x,4)=x^2$  taking  $h=k=1$  by Liebmann's method. Obtain the values of  $u$  at the nine interior mesh points by always correcting the computed value to two places of decimals.

2. Using Bender-Schmidt recurrence method, solve numerically the parabolic equation  $2u_{xx} = u_t$  subject to boundary and initial conditions (i)  $u(0,t)=0$ ,  $t > 0$  (ii)  $u(12,t)=0$ ,  $t > 0$  (iii)  $u(x,0) = 3x(12-x)$ ,  $0 \leq x \leq 12$ . Assuming  $h=2$ , find the values of  $u$  upto  $t=5$  properly choosing the step size in the time duration.

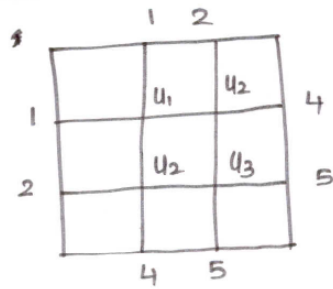
3. Solve  $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$  for  $0 < x < 2$ ,  $t > 0$ ;  $u(0,t) = u(2,t) = 0$ ,  $t > 0$  and  $u(x,0) = \sin \frac{\pi x}{2}$ ,  $0 < x < 2$  using  $\Delta x = 0.5$  and  $\Delta t = 0.25$  for one time step by Crank-Nicholson implicit finite difference method.

4. Solve  $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$ ;  $0 < x < 1$ ,  $t > 0$  given  $u(x,0) = 0$ ,  $\frac{\partial u}{\partial t}(x,0) = 0$ ,  $u(0,t) = 0$  and  $u(1,t) = 100 \sin \pi t$ . Compute  $u(x,t)$  for 4 time steps.

5. Solve the equation  $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$  subject to the conditions  $u(x,0) = \sin \pi x$ ,  $0 \leq x \leq 1$ ;  $u(0,t) = u(1,t) = 0$  using Crank-Nicholson method, taking  $h = \frac{1}{3}$ ,  $k = \frac{1}{36}$  (Do one time step)

6. Solve  $u_{tt} = 4u_{xx}$  with boundary conditions  $u(0,t) = 0 = u(4,t)$ ,  $u_t(x,0) = 0$  and  $u(x,0) = x(4-x)$ .

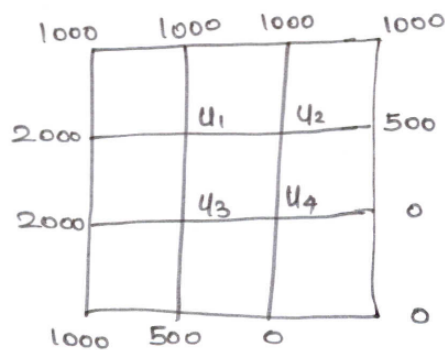
7. Using Liebmann method, solve the equation  $u_{xx} + u_{yy} = 0$  for the following square mesh with boundary values as shown in fig. Iterate until the maximum difference between successive values at any point is less than 0.001



8. Solve  $xy'' + y = 0$ ,  $y(1) = 1$ ,  $y(2) = 2$  with  $h = 0.25$  by finite difference method.

9. Solve  $y'' = xy$ ,  $y(0) = -1$  and  $y(1) = 2$  finite difference method taking the no. of subintervals  $n = 2$ .

10. Given the values of  $u(x, y)$  on the boundary of the square given in the following figure. Evaluate the function  $u(x, y)$  satisfying Laplace equation  $\nabla^2 u = 0$  at the pivotal points.



Reg. No. : 

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**R 3421**

B.E./B.Tech. DEGREE EXAMINATION, NOVEMBER/DECEMBER 2007.

Sixth Semester

Computer Science and Engineering

MA 1011/MA 1251 — NUMERICAL METHODS

(Common to Chemical Engineering, Information Technology, Electronics and  
Communication Engineering and Mechanical Engineering)

(Regulation 2004)

Time : Three hours

Maximum : 100 marks

Answer ALL questions.

PART A — (10 × 2 = 20 marks)

1. On what type of equations Newton's method can be applicable?
2. By Gauss elimination method solve  $x + y = 2$  and  $2x + 3y = 5$ .
3. What is the nature of  $n^{\text{th}}$  divided differences of a polynomial of  $n^{\text{th}}$  degree?
4. Find the second divided differences with arguments  $a, b, c$  if  $f(x) = \frac{1}{x}$ .
5. State Simpson's  $\frac{1}{3}$  rule formula to evaluate  $\int_a^b f(x) dx$ .
6. Write down the formula to calculate errors in quadrature formulae.
7. Explain the terms initial and boundary value problems.

8. Using Euler's method find  $y(0.2)$  given that  $y' = x + y$ ,  $y(0) = 1$ .
9. Write down the diagonal five point formula in Laplace equation.
10. What is the order of error in solving Laplace and Poisson's equations by using finite difference method?

PART B — (5 × 16 = 80 marks)

11. (a) (i) Use Newton's method to find the real root of  $3x - \cos x - 1 = 0$ . (8)
- (ii) Apply Gauss Jordan method to solve the equations  $x + y + z = 9$ ,  
 $2x - 3y + 4z = 13$ ,  $3x + 4y + 5z = 40$ . (8)

Or

- (b) (i) Solve by Jacobi iteration method correct to two decimal places.  
 $10x + y - z = 11.19$ ,  $x + 10y + z = 28.08$ ,  $-x + y + 10z = 35.61$ . (8)
- (ii) Obtain by power method the numerically largest eigenvalue of the  
matrix  $\begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix}$ . (8)

12. (a) (i) Using Newton's divided difference interpolation, find the polynomial of the given data (8)

$$x: -1 \quad 0 \quad 1 \quad 3$$

$$f: 2 \quad 1 \quad 0 \quad -1$$

- (ii) Find the cubic spline interpolation. (8)

$$x: 1 \quad 2 \quad 3 \quad 4 \quad 5$$

$$f: 1 \quad 0 \quad 1 \quad 0 \quad 1$$

Or

- (b) (i) From the given table, the values of  $y$  are consecutive terms of a series of which 23.6 is the 6<sup>th</sup> term. Find the first and tenth terms of the series. (8)

$$x: 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9$$

$$y: 4.8 \quad 8.4 \quad 14.5 \quad 23.6 \quad 36.2 \quad 52.8 \quad 73.9$$



(ii) Given the values (8)

$x:$	5	7	11	13	17
$f(x):$	150	392	1452	2366	5202

Evaluate  $f(9)$  using Newton's divided difference formula.

13. (a) (i) Using the given data find  $f'(5)$  (8)

$x:$	0	2	3	4	7	9
$f(x):$	4	26	58	112	466	922

(ii) Evaluate  $I = \int_0^1 \frac{dx}{1+x}$  by two and three point Gaussian formulae. (8)

Or

(b) (i) Use Romberg's method to compute  $\int_0^1 \frac{1}{1+x^2} dx$  correct to 4 decimal places by taking  $h = 0.5, 0.25$  and  $0.125$ . (8)

(ii) Evaluate  $\int_0^1 \int_0^1 \frac{dx dy}{1+x+y}$  by using Trapezoidal rule, with step sizes  $h = k = 0.5$ . (8)

14. (a) Using Runge-Kutta method of fourth order, solve  $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$  given that  $y(0) = 1$  at  $x = 0.2$  and  $x = 0.4$ . (16)

Or

(b) Given  $y' = x(x^2 + y^2)e^{-x}$ ,  $y(0) = 1$  find  $y$  at  $x = 0.1, 0.2$  and  $0.3$  by Taylor's series method and compute  $y(0.4)$  by Milne's method. (16)

15. (a) Solve  $\nabla^2 u = -10(x^2 + y^2 + 10)$  over the square mesh with sides  $x = 0, y = 0, x = 3, y = 3$  with  $u = 0$  on the boundary and mesh length 1 unit. (16)

Or

(b) Solve by Crank-Nicholson method the equation  $u_{xx} = u_t$ , subject to  $u(x, 0) = 0, u(0, t) = 0$  and  $u(1, t) = t$  for two time steps. (16)

**C 3276**

B.E./B.Tech. DEGREE EXAMINATION, MAY/JUNE 2007.

Sixth Semester

(Regulation 2004)

Computer Science and Engineering

MA 1251/MA 1011 — NUMERICAL METHODS

(Common to Chemical Engineering/Information Technology/Electronics and  
Communication Engineering/Mechanical Engineering)

Time : Three hours

Maximum : 100 marks

Answer **ALL** questions.

PART A — (10 × 2 = 20 marks)

1. State the formula for the method of false position to determine a root of  $f(x) = 0$ .
2. State the sufficient condition on  $\phi(x)$  for the convergence of an iterative method for  $f(x) = 0$  written as  $x = \phi(x)$ .
3. Show that the divided differences are symmetrical in their arguments.
4. State Newton's backward difference interpolation formula.
5. State Romberg's method integration formula to find the value of  $I = \int_a^b f(x) dx$ , using  $h$  and  $\frac{h}{2}$ .
6. Write down the Simpson's  $-\frac{3}{8}$  rule of integration given  $(n + 1)$  data.
7. State the Taylor series formula to find  $y(x_1)$  for solving  $\frac{dy}{dx} = f(x, y)$ ,  $y(x_0) = y_0$ .

8. State Milne's predictor-corrector formula.
9. Name at least two numerical methods that are used to solve one dimensional diffusion equation.
10. Write down Laplace equation and its finite difference analogue and the standard five-point formula.

PART B — (5 × 16 = 80 marks)

11. (a) (i) Prove the quadratic convergence of Newton-Raphson method. Find a positive root of  $f(x) = x^3 - 5x + 3 = 0$ , using this method.
- (ii) Solve the following system by Gauss-Seidal method :  
$$28x + 4y - z = 32$$
$$x + 3y + 10z = 24$$
$$2x + 17y + 4z = 35.$$

Or

- (b) (i) Find the inverse of the matrix by Gauss-Jordan method :

$$A = \begin{pmatrix} 4 & 1 & 2 \\ 2 & 3 & -1 \\ 1 & -2 & 2 \end{pmatrix}$$

- (ii) Find the dominant Eigen value and the corresponding Eigen vector

of the matrix  $A = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ .

12. (a) (i) If  $f(0) = 0, f(1) = 0, f(2) = -12, f(4) = 0, f(5) = 600$  and  $f(7) = 7308$ , find a polynomial that satisfies this data using Newton's divided difference interpolation formula. Hence, find  $f(6)$ .
- (ii) Given the following table, find  $f(2.5)$  using cubic spline functions :

$i$	0	1	2	3
$x_i$	1	2	3	4
$f(x_i)$	0.5	0.3333	0.25	0.2

Or

- (b) (i) Find the Lagrange's polynomial of degree 3 to fit the data :

$y(0) = -12, y(1) = 0, y(3) = 6$  and  $y(4) = 12$ . Hence, find  $y(2)$ .

- (ii) Find a polynomial of degree two for the data by Newton's forward difference method :

$x$	0	1	2	3	4	5	6	7
$y$	1	2	4	7	11	16	22	29

13. (a) (i) Find  $f'(6)$  and the maximum value of  $y = f(x)$  given the data :

$x$	0	2	3	4	7	9
$f(x)$	4	26	58	112	466	992

- (ii) Evaluate  $\int_0^1 \frac{dx}{1+x^2}$  using Romberg method by taking  $h = 0.5, 0.25$  and  $0.125$  successively.

Or

- (b) (i) Evaluate  $I = \int_0^1 \frac{dx}{1+x}$  using three point Gauss-quadrature formula.

- (ii) Use Trapezoidal rule to evaluate  $I = \int_1^2 \int_1^2 \frac{dxdy}{x+y}$  taking 4 subintervals.

14. (a) (i) Solve  $\frac{dy}{dx} = \log_{10}(x+y), y(0) = 2$  by Euler's modified method and find the values of  $y(0.2), y(0.4)$  and  $y(0.6)$ , taking  $h = 0.2$ .

- (ii) Given  $y'' + xy' + y = 0, y(0) = 1, y'(0) = 0$ , find the value of  $y(0.1)$  by Runge-Kutta method of fourth order.

Or

- (b) (i) Solve  $\frac{dy}{dx} = xy + y^2, y(0) = 1$ , using Milne's predictor-corrector formulae and find  $y(0.4)$ . Use Taylor series method to find  $y(0.1), y(0.2)$  and  $y(0.3)$ .

- (ii) Compute  $y(0.2)$ , given  $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}, y(0) = 1$ , by Runge-Kutta method of fourth order, taking  $h = 0.2$ .

15. (a) (i) Solve the boundary value problem  $y'' = xy$ , subject to the conditions  $y(0) + y'(0) = 1$ ,  $y(1) = 1$ , taking  $h = \frac{1}{3}$ , by finite difference method.
- (ii) Using Bender-Schmitt formula, solve  $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ ,  $u(0, t) = 0$ ,  $u(5, t) = 0$ ,  $u(x, 0) = x^2(25 - x^2)$ . Assume  $\Delta x = 1$ . Find  $u(x, t)$  up to  $t = 5$ .

Or

- (b) (i) Use Crank-Nicholson scheme to solve  $\frac{\partial^2 u}{\partial x^2} = 16 \frac{\partial u}{\partial t}$ ,  $0 < x < 1$  and  $t > 0$  given  $u(x, 0) = 0$ ,  $u(0, t) = 0$  and  $u(1, t) = 100t$ . Compute  $u(x, t)$  for one time step, taking  $\Delta x = \frac{1}{4}$ .
- (ii) Evaluate  $u(x, t)$  at the pivotal points of the equation  $16 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$ ,  $u(0, t) = 0$ ,  $u(5, t) = 0$ ,  $\frac{\partial u}{\partial t}(x, 0) = 0$  and  $u(x, 0) = x^2(5 - x)$  taking  $\Delta x = 1$  and up to  $t = 1.25$ .

**T 3321**

B.E./B.Tech. DEGREE EXAMINATION, APRIL/MAY 2008.

Sixth Semester

(Regulation 2004)

Computer Science and Engineering

MA 1011/MA 1251 — NUMERICAL METHODS

(Common to Chemical Engineering, Information Technology, Electronics and  
Communication Engineering and Mechanical Engineering/Automobile Engineering)

Time : Three hours

Maximum : 100 marks

Answer ALL questions.

PART A — (10 × 2 = 20 marks)

1. By Newton's method, find an iterative formula to find  $\sqrt{N}$  (where  $N$  is a positive number).
2. Why Gauss Seidel iteration is a method of successive corrections?
3. Define forward, backward, central differences and divided difference.
4. Evaluate  $\Delta^{10}(1-x)(1-2x)(1-3x)\dots(1-10x)$ , by taking  $h=1$ .
5. Show that the divided difference operator  $\Delta$  is linear.
6. In numerical integration, what should be the number of intervals to apply Simpson's one-third rule and Simpson's three-eighths rule.
7. Explain one-step methods and multistep methods.
8. State the Adams-Bashforth predictor-corrector formula.

9. For what points of  $x$  and  $y$ , the equation  $x \cdot f_{xx} + y \cdot f_{yy} = 0$ ,  $x > 0$ ,  $y > 0$  is elliptic.
10. Define difference quotient of a function  $y(x)$ .

PART B — (5 × 16 = 80 marks)

11. (a) (i) Solve  $u_{xx} + u_{yy} = 0$  over the square mesh of side 4 units; satisfying the following conditions.

$$u(x, 0) = 3x \quad \text{for } 0 \leq x \leq 4$$

$$u(x, 4) = x^2 \quad \text{for } 0 \leq x \leq 4$$

$$u(0, y) = 0 \quad \text{for } 0 \leq y \leq 4; \quad u(4, y) = 12 + y \quad \text{for } 0 \leq y \leq 4. \quad (10)$$

- (ii) Solve  $\frac{\partial^2 u}{\partial x^2} - 2 \cdot \frac{\partial u}{\partial t} = 0$  given  $u(0, t) = 0$ ,  $u(4, t) = 0$ ,  $u(x, 0) = x(4 - x)$ .

$$\text{Assume } h = 1. \text{ Find the values of } u \text{ upto } t = 5. \quad (6)$$

Or

- (b) Solve  $y_{tt} = 4y_{xx}$  subject to the conditions  $y(0, t) = 0$ ,  $y(2, t) = 0$ ,  $y(x, 0) = x(2 - x)$ ,  $\frac{\partial y}{\partial t}(x, 0) = 0$ . Do 4 steps. Find values upto 2 decimal accuracy.

12. (a) (i) Apply Gauss-Jordan method to find the solution of the following system.

$$10x + y + z = 12; \quad 2x + 10y + z = 13; \quad x + y + 5z = 7. \quad (10)$$

- (ii) Solve for a positive root of  $x - \cos x = 0$  by Regula Falsi method. (6)

Or

- (b) (i) Find the numerically largest eigen value of  $A = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}$  and

$$\text{the corresponding eigen vector by power method.} \quad (10)$$

- (ii) Find the root between 1 and 2 of  $2x^3 - 3x - 6 = 0$  by Newton-Raphson method correct to five decimal places. (6)

13. (a) (i) Using Lagrange's interpolation formula, find  $y(10)$  from the following table : (10)

$x$ :	5	6	9	11
$y$ :	12	13	14	16

- (ii) Find the sixth term of the sequence 8, 12, 19, 29, 42. (6)

Or

- (b) (i) From the following table of half-yearly premium for policies maturing at different ages, estimate the premium for policies maturing at age 46. (10)

Age $x$ :	45	50	55	60	65
Premium $y$ :	114.84	96.16	83.32	74.48	68.48

- (ii) Form the divided difference table for the following data : (6)

$x$ :	-2	0	3	5	7	8
$y = f(x)$ :	-792	108	-72	48	-144	-252

14. (a) (i) Evaluate  $\int_0^1 \frac{dx}{1+x^2}$  using Romberg's method. Hence obtain an approximate value of  $\pi$ . (10)

- (ii) A river is 80 meters wide. The depth ' $d$ ' in meters at a distance  $x$  meters from one bank is given by the following table. Calculate the area of cross-section of the river using Simpson's  $\frac{1}{3}$  rule. (6)

$x$ :	0	10	20	30	40	50	60	70	80
$d$ :	0	4	7	9	12	15	14	8	3

Or

- (b) (i) The table given below reveals the velocity  $v$  of a body during the time ' $t$ ' specified. Find its acceleration at  $t=1.1$ . (10)

$t$ :	1.0	1.1	1.2	1.3	1.4
$v$ :	43.1	47.7	52.1	56.4	60.8

- (ii) Evaluate the integral of

$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$  between the limits  $x=0$  to  $0.8$  using two point Gaussian quadrature formula. (6)



15. (a) (i) Given  $y' + xy^2 + y = 0$ ,  $y(0) = 1$ , find the value of  $y(0.2)$  by using Runge-Kutta method of fourth order. (10)
- (ii) Using Taylor series method, find correct to four decimal places, the value of  $y(0.1)$ , given  $\frac{dy}{dx} = x^2 + y^2$  and  $y(0) = 1$ . (6)

Or

- (b) (i) Using Milne's method find  $y(4.4)$  given  $5xy' + y^2 - 2 = 0$  given  $y(4) = 1$ ;  $y(4.1) = 1.0049$ ;  $y(4.2) = 1.0097$  and  $y(4.3) = 1.0143$ . (10)
- (ii) Using Euler's method, solve numerically the equation  $y' = x + y$ ,  $y(0) = 1$ , for  $x = 0.0$  (0.2) (1.0). Check your answer with the exact solution. (6)