Instructions

(1) This question paper consists of two parts: Part A and Part B and carries a total of 100 Marks.
(2) There is no negative marking.
(3) Candidates are asked to fill in the required fields on the sheet attached to the answer book.
(4) Part A carries 20 multiple choice questions of 2 marks each. Answer all questions in Part A.
(5) Answers to Part A are to be marked in the OMR sheet provided.
(6) For each question, darken the appropriate bubble to indicate your answer.
(7) Use only HB pencils for bubbling answers.
(8) Mark only one bubble per question. If you mark more than one bubble, the question will be evaluated as incorrect.
(9) If you wish to change your answer, please erase the existing mark completely before marking the other bubble.
(10) Part B has 12 questions. Answer any 6 in this part. Each question in this part carries 10 marks.
(11) Answers to Part B are to be written in the separate answer book provided.
(12) **Answer to each question in Part B should begin on a new page.**
(13) Let $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{C}$ ($\mathbb{Z}_+$, $\mathbb{R}_+$, $\mathbb{Q}_+$ and $\mathbb{C}_+$) denote the set of (respectively positive) integers, real numbers, rational numbers and complex numbers respectively.
(14) For $n \geq 1$, the norm given by $\|(x_1, x_2, \ldots, x_n)\| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$ denote the standard norm on $\mathbb{R}^n$. The metric given by $d(x, y) = \|x - y\|$ is called the standard metric on $\mathbb{R}^n$.

MATHEMATICS

PART A

(1) The limit, $\lim_{R\to\infty} \dfrac{\int_R^\infty r^n e^{-\frac{r^2}{2}}\, dr}{R^{n-1}e^{-\frac{R^2}{2}}}$, $n \in \mathbb{Z}_+$, equals

(A) -1.
(B) 0.
(C) 1.
(D) ∞.

(2) The function $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(x) - f(y)| \leq c|x - y|$ for all $x, y \in \mathbb{R}$ and some constant $c \in \mathbb{R}_+$. Then,

(A) f must be bounded.
(B) f must be continuous but may not be uniformly continuous.
(C) f must be uniformly continuous but may not be differentiable.
(D) f must be differentiable.

(3) Consider the sequence $\{x_n\}_{n\geq 1}$ defined recursively as $x_1 = 1$,

$$x_n := \sup\left\{x \in [0, x_{n-1}) : \sin\left(\frac{1}{x}\right) = 0\right\}, \qquad n \geq 2.$$

Then, $\limsup_{n\to\infty} x_n$ equals

(A) $-\infty$.
(B) 0.
(C) 1.
(D) ∞.

(4) Let $f : [0, 1] \to \mathbb{R}$ be continuous. Then $\int_0^1 f(x)e^{-x}dx$ equals

(A) $f'(0) - f'(1)e^{-1}$.
(B) $\int_0^1 e^{-x}(\int_0^x f(y)\, dy)\, dx$.
(C) $f(c)(1 - e^{-1})$, for some $c \in [0, 1]$.
(D) $e^{-c}\int_0^1 f(x)\, dx$, for some $c \in [0, 1]$.

(5) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous with $f(0) = f(1) = 0$. Which of the following is **not** possible?

(A) $f([0,1]) = \{0\}$.

(B) $f([0,1]) = [0,1)$.

(C) $f([0,1]) = [0,1]$.

(D) $f([0,1]) = [-\frac{1}{2}, \frac{1}{2}]$.

(6) The series $\displaystyle\sum_{n=1}^{\infty} \frac{e^{i\sqrt{n}x}}{n^2}$ converges

(A) only at $x = 0$.

(B) only for $|x| \leq 1$.

(C) converges pointwise for all $x \in \mathbb{R}$, but not uniformly.

(D) converges uniformly on $\mathbb{R}$.

(7) Let $P(x)$ be a non-zero polynomial of degree N. The radius of convergence of the power series

$$\sum_{n=0}^{\infty} P(n)x^n$$

(A) depends on N.

(B) is 1 for all N.

(C) is 0 for all N.

(D) is ∞ for all N.

(8) The function $f(z) = \exp\left(\left(\frac{\cos z - 1}{z^2}\right)^2\right)$

(A) has a removable singularity at $z = 0$.

(B) has a pole of order 2 at $z = 0$.

(C) has a pole of order 4 at $z = 0$.

(D) has an essential singularity at $z = 0$.

(9) The region described by $\left|\frac{z-i}{z+i}\right| < 1$, where $z = x + iy \in \mathbb{C}$ is

(A) $\{z \in \mathbb{C} : x < 0\}$.

(B) $\{z \in \mathbb{C} : x > 0\}$.

(C) $\{z \in \mathbb{C} : y < 0\}$.

(D) $\{z \in \mathbb{C} : y > 0\}$.

(10) Let $A \neq \mathbb{R}$ be a dense subset of $\mathbb{R}$. If $U \subseteq \mathbb{R}$ is a non-empty open subset then

(A) $U \subseteq \overline{A \cap U}$.

(B) $\overline{A \cap U} = \emptyset$.

(C) $\overline{A \cap U} \subseteq U$.

(D) $\overline{A \cap U} = A \cap \overline{U}$.

(11) Let $X = \{1/n : n \in \mathbb{Z}, n \geq 1\}$ and let $\bar{X}$ be its closure. Then

(A) $\bar{X} \setminus X$ is a single point.

(B) $\bar{X} \setminus X$ is open in $\mathbb{R}$.

(C) $\bar{X} \setminus X$ is infinite but not open in $\mathbb{R}$.

(D) $\bar{X} \setminus X = \phi$.

(12) Let X be a subset of $\mathbb{R}$ homeomorpic to $(0, \pi)$. Then

(A) X must be bounded.

(B) X must be compact.

(C) The closure $\bar{X}$ of X must be unbounded.

(D) X must be open.

(13) Let $A \subseteq \mathbb{R}$ be an open set. If $(0,1) \cup A$ is connected, then

(A) A must be connected.

(B) A must have one or two componets.

(C) $A \setminus (0,1)$ has at most two components.

(D) A must be a Cantor set.

(14) Let A be an $n \times n$ matrix with entries 0 and 1 and $n > 1$. If there is exactly one non zero entry in each row and each column of A, then the determinant of A must be

(A) ± 1.

(B) 0.

(C) n.

(D) 1.

(15) Let A be an $n \times n$ matrix over real numbers such that $AB = BA$ for all $n \times n$ matrices B. Then

(A) A must be 0.

(B) A must be the identity.

(C) A must be a diagonal matrix.

(D) A must be either 0 or the identity.

(16) Let A be a matrix such that $A^3 = -I$. Then which of the following numbers can be an eigenvalue of A?

(A) i.

(B) 1.

(C) -1.

(D) $\frac{1}{2} + \frac{\sqrt{3}}{2} i$.

(17) Let V be the vector space of all polynomials whose degree is less than or equal to n. Let $D : V \to V$ be the differentiation operator on V, that is, $DP(x) = P'(x)$. Then the trace of D, $\mathrm{tr}(D)$ equals

(A) 0.

(B) 1.

(C) n.

(D) n^2.

(18) Let A be a 3×3 matrix over real numbers satisfying $A^{-1} = I - 2A$. Then the determinant of A, $\det(A)$ equals

(A) $-\frac{1}{2}$.

(B) $\frac{1}{2}$.

(C) 1 .

(D) 2 .

(19) For which of the following integers n is every group of order n abelian?

(A) $n = 6$.

(B) $n = 9$.

(C) $n = 12$.

(D) $n = 18$.

(20) Let G be the cyclic subgroup of order 18. The number of subgroups of G, including G and the trivial group, is

(A) 4.

(B) 6.

(C) 9.

(D) 18.

PART B

(1) Let $F_n : \mathbb{R} \to [0,1]$, $n \geq 0$, be continuous functions satisfying

 (i) $F_n(x) \leq F_n(y)$ for all $x \leq y$,

 (ii) $\lim_{x \to -\infty} F_n(x) = 0$, and

 (iii) $\lim_{x \to \infty} F_n(x) = 1$.

Suppose that F_n converges pointwise to F_0 on $\mathbb{R}$, that is $F_n(x) \to F_0(x)$ for all $x \in \mathbb{R}$, as $n \to \infty$. Show that F_n converges uniformly to F_0 on $\mathbb{R}$, as $n \to \infty$.

(2) Let $f : [0,1] \to \mathbb{R}$ be a continuous function of bounded variation. Then for any $p \in (1, \infty)$, show that

$$\lim_{n \to \infty} \sum_{k=1}^{n} \left| f\left(\frac{k}{n}\right) - f\left(\frac{k-1}{n}\right) \right|^p = 0.$$

(3) Given n points $z_1, z_2, \cdots, z_n$ on the unit circle $\{z \in \mathbb{C} : |z| = 1\}$, prove that there exists a point z on the unit circle such that $\prod_{i=1}^{n} |z - z_i| \geq 1$.

(4) Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Let $\Delta(0,R) = \{z = x + iy \in \mathbb{C} : |z| < R\}$, denote the open disc in the complex plane around the origin of radius $R > 0$. If $n - 1 \in \mathbb{Z}_+$, show that

$$\int_{\Delta(0,R)} \overline{z}^{\,n-1} f(z)\, dx\, dy = \frac{\pi R^{2n}}{n!} f^{(n-1)}(0),$$

where $f^{(k)}(0)$ denotes the k-th derivative of f at the origin.

(5) Let $f : \Delta(0,1) \to \mathbb{C}$ be analytic. Show that it is not possible for $f(z)$ to satisfy

$$f\left(\frac{1}{n}\right) = f\left(-\frac{1}{n}\right) = \frac{n}{n+1},$$

for all $n \geq 2$.

(6) For $A, B \subseteq \mathbb{R}^2$, define the distance $d(A,B) := \inf\{\|x - y\| \ : \ x \in A, \ x \in B\}$. Let $C, D \subseteq \mathbb{R}^2$ be two closed subsets. If $C \cap D = \emptyset$ and $d(C,D) = 0$ then show that both C and D are unbounded.

(7) Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear transformation and let $V \subset \mathbb{R}^n$ be a subspace such that $L(V) \subseteq V$. Show that $L(V) = V$. Here $L(V) = \{L(v) : v \in V\}$.

(8) Let A be a 2×2 invertible matrix over real numbers such that for some 2×2 invertible matrix P, $PAP^{-1} = A^2$. Show that either $A^3 = I$ or $QAQ^{-1} = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, for some invertible 2×2 matrix Q.

(9) Suppose V is an n-dimensional vector space over a field F. Let $W \subseteq V$ be a subspace of dimension $r < n$. Show that $W = \cap\{U : U$ is an $(n-1)$-dimensional subspace of V and $W \subseteq U\}$.

(10) Suppose G is a finite group. Show that every element x of G can be expressed as $x = y^2$ for some $y \in G$ if and only if the order of G is odd.

(11) Let G be a group with identity e. Let N_1, N_2, N_3 be three normal subgroups of G. If $N_i \cap N_j = \{e\}$ and $N_iN_j = G$ for $1 \leq i \neq j \leq 3$ then show the following:

(i) $xy = yx$ for $x \in N_i$, $y \in N_j$, $1 \leq i \neq j \leq 3$.

(ii) $yz = zy$ for $y, z \in N_i$, $1 \leq i \leq 3$.

(iii) G is commutative. **[4 + 4 + 2]**

(12) Let $y : \mathbb{R} \to \mathbb{R}$ be an infinitely many times differentiable function which satisfies

$$y'' + y' - y \geq 0, \quad y(0) = y(1) = 0.$$

If $y(x) \geq 0$ for all $x \in [0, 1]$, prove that y is identically zero in $[0, 1]$.