16

TRIGONOMETRIC FUNCTIONS - I

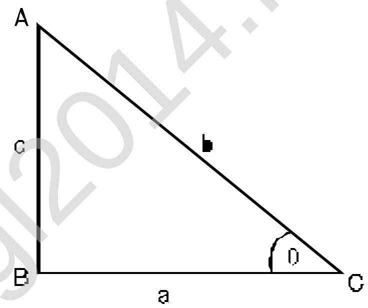

Fig.16.1

$\sin\theta = \frac{c}{b}$, $\cos\theta = \frac{a}{b}$, $\tan\theta = \frac{c}{a}$

and cosec $\theta = \frac{b}{c}$, $\sec\theta = \frac{b}{a}$, $\cot\theta = \frac{a}{c}$

We also developed relationships between these trigonometric ratios as

$\sin^2\theta + \cos^2\theta = 1$, $\sec^2\theta = 1 + \tan^2\theta$, $\text{cosec}^2\theta = 1 + \cot^2\theta$

We shall try to describe this knowledge gained so far in terms of functions, and try to develop this lesson using functional approach.

In this lesson, we shall develop the science of trigonometry using functional approach. We shall develop the concept of trigonometric functions using a unit circle. We shall discuss the radian measure of an angle and also define trigonometric functions of the type y = sin x, y = cos x, y = tan x, y = cot x, y = sec x, y = cosec x, y = a sin x, y = b cos x, etc., where *x, y* are real numbers.

We shall draw the graphs of functions of the type

y = sin x, y = cos x, y = tan x, y = cotx, y = secx, and y = cosecx y = a sin x, y = a cos x.

OBJECTIVES

After studying this lesson, you will be able to :

- define positive and negative angles;
- define degree and radian as a measure of an angle;
- convert measure of an angle from degrees to radians and vice-versa;

- state the formula $\ell = r\ \theta$ where r and θ have their usual meanings;
- solve problems using the relation $\ell = r\ \theta$;
- define trigonometric functions of a real number;
- draw the graphs of trigonometric functions; and
- interpret the graphs of trigonometric functions.

EXPECTED BACKGROUND KNOWLEDGE

- Definition of an angle.
- Concepts of a straight angle, right angle and complete angle.
- Circle and its allied concepts.
- Special products : $(a \pm b)^2 = a^2 + b^2 \pm 2ab$, $(a \pm b)^3 = a^3 \pm b^3 \pm 3ab(a \pm b)$
- Knowledge of Pythagoras Theorem and Py thagorean numbers.

16.1 CIRCULAR MEASURE OF ANGLE

An angle is a union of two rays with the common end point. An angle is formed by the rotation of a ray as well. Negative and positive angles are formed according as the rotation is clock-wise or anticlock-wise.

16.1.1 A Unit Circle

It can be seen easily that when a line segment makes one complete rotation, its end point describes a circle. In case the length of the rotating line be one unit then the circle described will be a circle of unit radius. Such a circle is termed as ***unit circle.***

16.1.2 A Radian

A radian is another unit of measurement of an angle other than degree.

A radian is the measure of an angle subtended at the centre of a circle by an arc equal in length to the radius (r) of the circle. In a unit circle one radian will be the angle subtended at the centre of the circle by an arc of unit length.

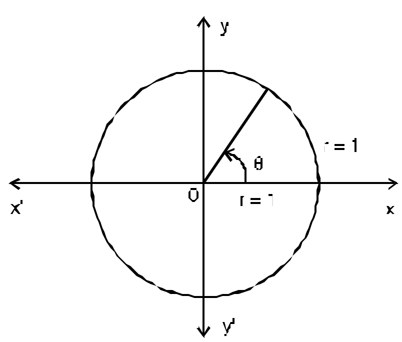

Fig. 16.2

Note : A radian is a constant angle; implying that the measure of the angle subtended by an are of a circle, with length equal to the radius is always the same irrespective of the radius of the circle.

16.1.3 Relation between Degree and Radian

An arc of unit length subtends an angle of 1 radian. The circumference 2π ($\because r = 1$) subtend an angle of 2π radians.

Hence 2π radians $= 360°$

π radians $= 180°$

$\frac{\pi}{2}$ radians $= 90°$

$\frac{\pi}{4}$ radians $= 45°$

$$1 \text{ radian} = \left(\frac{360}{2\pi}\right)^{\circ} = \left(\frac{180}{\pi}\right)^{\circ}$$

$$\text{or } 1° = \frac{2\pi}{360} \text{ radians} = \frac{\pi}{180} \text{ radians}$$

Example 16.1 Convert

(i) 90° into radians (ii) 15° into radians

(iii) $\frac{\pi}{6}$ radians into degrees. (iv) $\frac{\pi}{10}$ radians into degrees.

Solution :

(i) $1° = \frac{2\pi}{360}$ radians

$\Rightarrow$ $90° = \frac{2\pi}{360} \times 90$ radians or $90° = \frac{\pi}{2}$ radians

(ii) $15° = \frac{2\pi}{360} \times 15$ radians or $15° = \frac{\pi}{12}$ radians

(iii) $1 \text{ radian} = \left(\frac{360}{2\pi}\right)^{\circ}$

$\frac{\pi}{6}$ radians $= \left(\frac{360}{2\pi} \times \frac{\pi}{6}\right)^{\circ}$

$\frac{\pi}{6}$ radians $= 30°$

(iv) $\frac{\pi}{10}$ radians $= \left(\frac{360}{2\pi} \times \frac{\pi}{10}\right)^{\circ}$

$\frac{\pi}{10}$ radians $= 18°$

CHECK YOUR PROGRESS 16.1

1. Convert the following angles (in degrees) into radians :

   (i) 60° (ii) 15° (iii) 75° (iv) 105° (v) 270°

2. Convert the following angles into degrees:

   (i) $\frac{\pi}{4}$ (ii) $\frac{\pi}{12}$ (iii) $\frac{\pi}{20}$ (iv) $\frac{\pi}{60}$ (v) $\frac{2\pi}{3}$

3. The angles of a triangle are 45°, 65° and 70°. Express these angles in radians

4. The three angles of a quadrilateral are $\frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}$. Find the fourth angle in radians.

5. Find the angle complementary to $\frac{\pi}{6}$.

16.1.4 Relation Between Length of an Arc and Radius of the Circle

An angle of 1 radian is subtended by an arc whose length is equal to the radius of the circle. An angle of 2 radians will be substened if arc is double the radius.

An angle of 2½ radians willbe subtended if arc is 2½ times the radius.

All this can be read from the following table :

Length of the arc (*l*)	**Angle subtended at the centre of the circle** q **(in radians)**
r	1
2r	2
(2½)r	2½
4r	4

Therefore, $\theta = \frac{\ell}{r}$ or $\ell = r\theta$

where r = radius of the circle,

θ = angle substended at the centre in radians

and ℓ = length of the arc.

The angle subtended by an arc of a circle at the centre of the circle is given by the ratio of the length of the arc and the radius of the circle.

Note : In arriving at the above relation, we have used the radian measure of the angle and not the degree measure. Thus the relation $\theta = \frac{\ell}{r}$ is valid only when the angle is measured in radians.

Example 16.2 Find the angle in radians subtended by an arc of length 10 cm at the centre of a circle of radius 35 cm.

Solution : $\ell = 10\text{cm}$ and $r = 35\text{cm}$.

$$\theta = \frac{\ell}{r} \text{ radians} \qquad \text{or} \qquad \theta = \frac{10}{35} \text{ radians}$$

or $$\theta = \frac{2}{7} \text{ radians}$$

Example 16.3 If D and C represent the number of degrees and radians in an angle prove that

$$\frac{D}{180} = \frac{C}{\pi}$$

Solution : $$1 \text{ radian} = \left(\frac{360}{2\pi}\right)^\circ \text{ or } \left(\frac{180}{\pi}\right)^\circ$$

$$\therefore \quad C \text{ radians} = \left(C \times \frac{180}{\pi}\right)^\circ$$

Since D is the degree measure of the same angle, therefore,

$$D = C \times \frac{180}{\pi}$$

which implies $$\frac{D}{180} = \frac{C}{\pi}$$

Example 16.4 A railroad curve is to be laid out on a circle. What should be the radius of a circular track if the railroad is to turn through an angle of 45° in a distance of 500m?

Solution : Angle θ is given in degrees. To apply the formula $\ell = r\theta$, θ must be changed to radians.

$$\theta = 45° = 45 \times \frac{\pi}{180} \text{ radians} \qquad ....(1)$$

$$= \frac{\pi}{4} \text{ radians}$$

$$\ell = 500 \text{ m} \qquad ....(2)$$

$$\ell = r\,\theta \text{ gives } r = \frac{\ell}{\theta}$$

$$\therefore \quad r = \frac{500}{\frac{\pi}{4}} \text{ m} \qquad [\text{using (1) and (2)}]$$

$$= 500 \times \frac{4}{\pi} \text{ m}$$

$$= 2000 \times 0.32 \text{ m} \qquad \left(\frac{1}{\pi} = 0.32\right)$$

$$= 640 \text{ m}$$

Example 16.5 A train is travelling at the rate of 60 km per hour on a circular track. Through what angle will it turn in 15 seconds if the radius of the track is $\frac{5}{6}$ km.

Solution : The speed of the train is 60 km per hour. In 15 seconds, it will cover

$$\frac{60 \times 15}{60 \times 60} \text{ km}$$

$$= \frac{1}{4} \text{ km}$$

$\therefore$ We have, $\qquad \ell = \frac{1}{4}$ km $\quad$ and $\quad r = \frac{5}{6}$ km

$$\therefore \qquad \theta = \frac{\ell}{r} = \frac{\frac{1}{4}}{\frac{5}{6}} \text{ radians}$$

$$= \frac{3}{10} \text{ radians}$$

CHECK YOUR PROGRESS 16.2

1. Express the following angles in radians :

   (a) 30° (b) 60° (c) 150°

2. Express the following angles in degrees :

   (a) $\frac{\pi}{5}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{9}$

3. Find the angle in radians and in degrees subtended by an arc of length 2.5 cm at the centre of a circle of radius 15 cm.

4. A train is travelling at the rate of 20 km per hour on a circular track. Through what angle will it turn in 3 seconds if the radius of the track is $\frac{1}{12}$ of a km?.

5. A railroad curve is to be laid out on a circle. What should be the radius of the circular track if the railroad is to turn through an angle of 60° in a distance of 100 m?

6. Complete the following table for l, r, θ having their usual meanings.

l		*r*	*q*
(a)	1.25m	....	135°
(b)	30 cm	....	$\frac{\pi}{4}$
(c)	0.5 cm	2.5 m	....
(d)	....	6 m	120°
(e)	....	150 cm	$\frac{\pi}{15}$
(f)	150 cm	40 m	....
(g)	....	12 m	$\frac{\pi}{6}$
(h)	1.5 m	0.75 m	....
(i)	25 m	....	75°

16.2 TRIGONOMETRIC FUNCTIONS

While considering, a unit circle you must have noticed that for every real number between *0* and 2π, there exists a ordered pair of numbers x and y. This ordered pair (*x*, *y*) represents the coordinates of the point *P*.

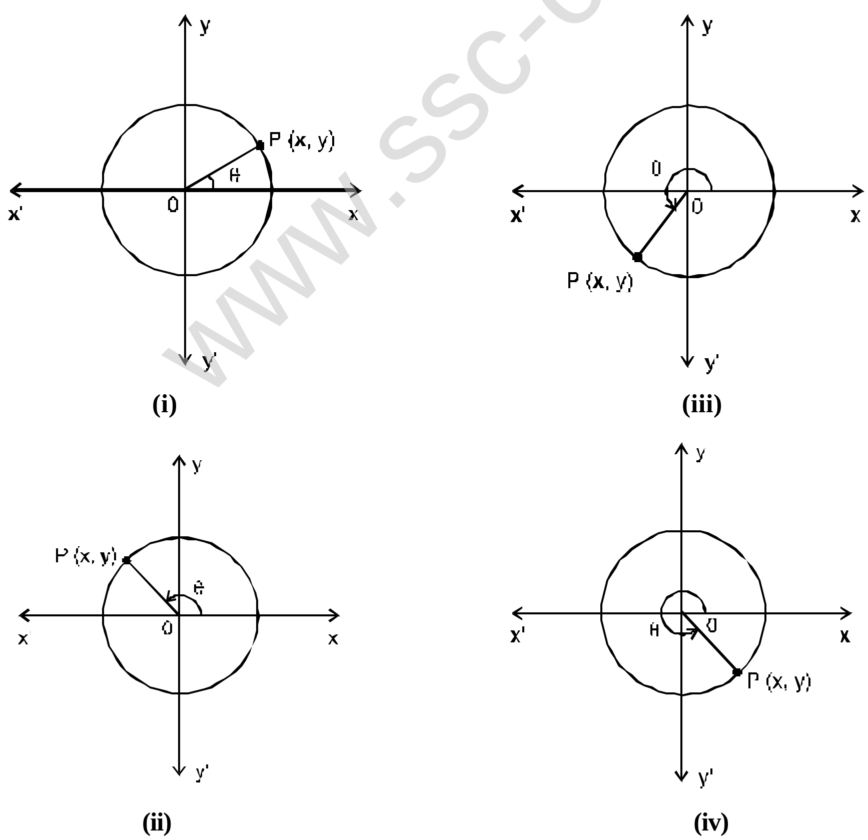

Fig. 16.3

If we consider $\theta = 0$ on the unit circle, we will have a point whose coordinates are (1,0).

If $\theta = \frac{\pi}{2}$, then the corresponding point on the unit circle will have its coordinates (0,1).

In the above figures you can easily observe that no matter what the position of the point, corresponding to every real number θ we have a unique set of coordinates (x, y). The values of x and y will be negative or positive depending on the quadrant in which we are considering the point.

Considering a point P (on the unit circle) and the corresponding coordinates (x, y), we define trigonometric functions as :

$$\sin\theta = y, \quad \cos\theta = x$$

$$\tan\theta = \frac{y}{x} \ (\text{for } x \neq 0), \quad \cot\theta = \frac{x}{y} \ (\text{for } y \neq 0)$$

$$\sec\theta = \frac{1}{x} \ (\text{for } x \neq 0), \quad \text{cosec}\,\theta = \frac{1}{y} \ (\text{for } y \neq 0)$$

Now let the point P move from its original position in anti-clockwise direction. For various positions of this point in the four quadrants, various real numbers θ will be generated. We summarise, the above discussion as follows. For values of θ in the :

I	quadrant, both x and y are positve.
II	quadrant, x will be negative and y will be positive.
III	quadrant, x as well as y will be negative.
IV	quadrant, x will be positive and y will be negative.

or	I quadrant	II quadrant	III quadrant	IV quadrant
	All positive	sin positive	tan positive	cos positive
		cosec positive	cot positive	sec positive

Where what is positive can be rememebred by :

	All	sin	tan	cos
Quardrant	I	II	III	IV

If (x, y) are the coordinates of a point P on a unit circle and θ, the real number generated by the position of the point, then $\sin\theta = y$ and $\cos\theta = x$. This means the coordinates of the point P can also be written as $(\cos\theta, \sin\theta)$

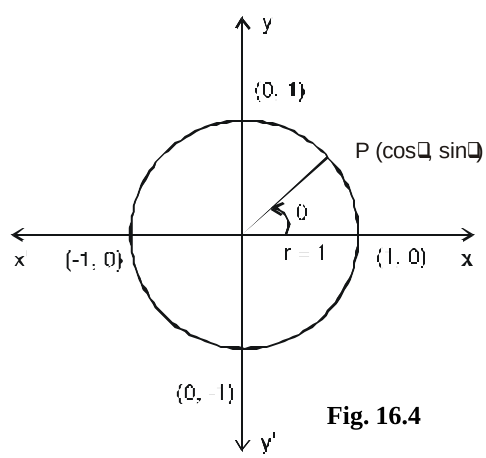

Fig. 16.4

From Fig. 16.5, you can easily see that the values of x will be between -1 and $+1$ as P moves on the unit circle. Same will be true for y also.

Thus, for all P on the unit circle

$-1 \le x \le 1$ and $-1 \le y \le 1$

Thereby, we conclude that for all real numbers θ

$$-1 \le \cos\theta \le 1 \text{ and } -1 \le \sin\theta \le 1$$

In other words, $\sin\theta$ and $\cos\theta$ can not be numerically greater than 1

Example 16.6 What will be sign of the following ?

(i) $\sin\frac{7\pi}{18}$ (ii) $\cos\frac{4\pi}{9}$ (iii) $\tan\frac{5\pi}{9}$

Solution :

(i) Since $\frac{7\pi}{18}$ lies in the first quadrant, the sign of $\sin\frac{7\pi}{18}$ will be posilive.

(ii) Since $\frac{4\pi}{9}$ lies in the first quadrant, the sign of $\cos\frac{4\pi}{9}$ will be positive.

(iii) Since $\frac{5\pi}{9}$ lies in the second quadrant, the sign of $\tan\frac{5\pi}{9}$ will be negative.

Example 16.7 Write the values of (i) $\sin\frac{\pi}{2}$ (ii) $\cos 0$ (iii) $\tan\frac{\pi}{2}$

Solution : (i) From Fig.16.5, we can see that the coordinates of the point *A* are (0,1)

$\therefore \sin\frac{\pi}{2} = 1$, as $\sin\theta = y$

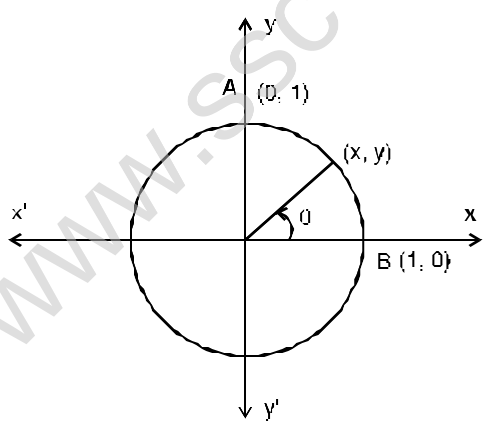

Fig.16.5

(ii) Coordinates of the point B are (1, 0)

$\therefore \cos 0 = 1$, as $\cos\theta = x$

(iii) $\tan\frac{\pi}{2} = \dfrac{\sin\frac{\pi}{2}}{\cos\frac{\pi}{2}} = \frac{1}{0}$ which is not defined

Thus $\tan\frac{\pi}{2}$ is not defined.

Example 16.8 Write the minimum and maximum values of $\cos\theta$.

Solution : We know that $-1 \le \cos\theta \le 1$

$\therefore$ The maximum value of $\cos\theta$ is 1 and the minimum value of $\cos\theta$ is -1.

CHECK YOUR PROGRESS 16.3

1. What will be the sign of the following ?

(i)	$\cos\dfrac{2\pi}{3}$	(ii)	$\tan\dfrac{5\pi}{6}$	(iii)	$\sec\dfrac{2\pi}{3}$
(iv)	$\sec\dfrac{35\pi}{18}$	(v)	$\tan\dfrac{25\pi}{18}$	(vi)	$\cot\dfrac{3\pi}{4}$
(vii)	$\operatorname{cosec}\dfrac{8\pi}{3}$	(viii)	$\cot\dfrac{7\pi}{8}$		

2. Write the value of each of the following :

(i)	$\cos\dfrac{\pi}{2}$	(ii)	$\sin 0$	(iii)	$\cos\dfrac{2\pi}{3}$
(iv)	$\tan\dfrac{3\pi}{4}$	(v)	$\sec 0$	(vi)	$\tan\dfrac{\pi}{2}$
(vii)	$\tan\dfrac{3\pi}{2}$	(viii)	$\cos 2\pi$		

16.2.1 Relation Between Trigonometric Functions

By definition $x = \cos\theta$

$y = \sin\theta$

As $\tan\theta = \dfrac{y}{x}$, $(x \ne 0)$

$= \dfrac{\sin\theta}{\cos\theta}$, $\theta \ne \dfrac{n\pi}{2}$

and $\cot\theta = \dfrac{x}{y}$, $(y \ne 0)$

i.e., $\cot\theta = \dfrac{\cos\theta}{\sin\theta} = \dfrac{1}{\tan\theta}$ $\quad (\theta \ne n\pi)$

Similarly, $\sec\theta = \dfrac{1}{\cos\theta}$ $\quad \left(\theta \ne \dfrac{n\pi}{2}\right)$

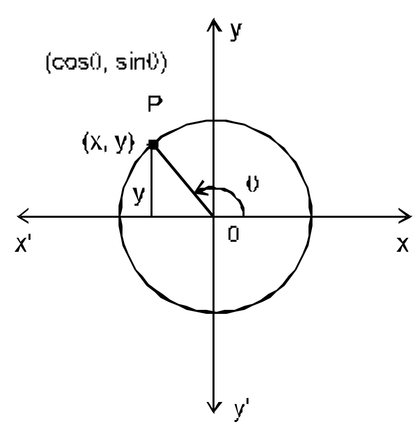

Fig. 16.6

and $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$ $\qquad (\theta \neq n\pi)$

Using Pythagoras theorem we have, $x^2 + y^2 = 1$

i.e., $(\cos\theta)^2 + (\sin\theta)^2 = 1$

or, $\mathbf{\cos^2\theta + \sin^2\theta = 1}$

Note : $(\cos\theta)^2$ is written as $\cos^2\theta$ and $(\sin\theta)^2$ as $\sin^2\theta$

Again $x^2 + y^2 = 1$

or $1 + \left(\dfrac{y}{x}\right)^2 = \left(\dfrac{1}{x}\right)^2$, for $x \neq 0$ $\qquad$ or, $1 + (\tan\theta)^2 = (\sec\theta)^2$

i.e. $\mathbf{\sec^2\theta = 1 + \tan^2\theta}$

Similarly, $\mathbf{\operatorname{cosec}^2\theta = 1 + \cot^2\theta}$

Example 16.9 Prove that $\sin^4\theta + \cos^4\theta = 1 - 2\sin^2\theta\cos^2\theta$

Solution : L.H.S. $= \sin^4\theta + \cos^4\theta$

$$= \sin^4\theta + \cos^4\theta + 2\sin^2\theta\cos^2\theta - 2\sin^2\theta\cos^2\theta$$

$$= (\sin^2\theta + \cos^2\theta)^2 - 2\sin^2\theta\cos^2\theta$$

$$= 1 - 2\sin^2\theta\cos^2\theta \quad (\because \sin^2\theta + \cos^2\theta = 1)$$

= R.H.S.

Example 16.10 Prove that $\sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}} = \sec\theta - \tan\theta$

Solution : L.H.S. $= \sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}}$

$$= \sqrt{\frac{(1-\sin\theta)(1-\sin\theta)}{(1+\sin\theta)(1-\sin\theta)}}$$

$$= \sqrt{\frac{(1-\sin\theta)^2}{1-\sin^2\theta}}$$

$$=\sqrt{\frac{(1-\sin\theta)^2}{\cos^2\theta}}$$

$$=\frac{1-\sin\theta}{\cos\theta}$$

$$=\frac{1}{\cos\theta}-\frac{\sin\theta}{\cos\theta}$$

$$=\sec\theta-\tan\theta=\text{R.H.S.}$$

Example 16.11 If $\sin\theta=\frac{21}{29}$, prove that $\sec\theta+\tan\theta=2\frac{1}{2}$, given that θ lies in the first quadrant.

Solution : $\sin\theta=\frac{21}{29}$

Also, $\sin^2\theta+\cos^2\theta=1$

$$\Rightarrow \cos^2\theta=1-\sin^2\theta=1-\frac{441}{841}=\frac{400}{841}=\left(\frac{20}{29}\right)^2$$

$$\Rightarrow \cos\theta=\frac{20}{29}$$ ($\cos\theta$ is positive as θ lies in the first quadrant)

$$\therefore \tan\theta=\frac{21}{20}$$

$$\therefore \sec\theta+\tan\theta=\frac{29}{20}+\frac{21}{20}=\frac{29+21}{20}$$

$$=\frac{5}{2}=2\frac{1}{2}=\text{R.H.S.}$$

CHECK YOUR PROGRESS 16.4

1. Prove that $\sin^4\theta-\cos^4\theta=\sin^2\theta-\cos^2\theta$

2. If $\tan\theta=\frac{1}{2}$, find the other five trigonometric functions.

3. If $\operatorname{cosec}\theta=\frac{b}{a}$, find the other five trigonometric functions, if θ lies in the first quadrant.

4. Prove that $\sqrt{\frac{1+\cos\theta}{1-\cos\theta}}=\operatorname{cosec}\theta+\cot\theta$

5. If $\cot\theta + \operatorname{cosec}\theta = 1.5$, show that $\cos\theta = \frac{5}{13}$

6. If $\tan\theta + \sec\theta = m$, find the value of $\cos\theta$

7. Prove that $(\tan A + 2)(2\tan A + 1) = 5\tan A + 2\sec^2 A$

8. Prove that $\sin^6\theta + \cos^6\theta = 1 - 3\sin^2\theta\cos^2\theta$

9. Prove that $\frac{\cos\theta}{1-\tan\theta} + \frac{\sin\theta}{1-\cot\theta} = \cos\theta + \sin\theta$

10. Prove that $\frac{\tan\theta}{1+\cos\theta} + \frac{\sin\theta}{1-\cos\theta} = \cot\theta + \operatorname{cosec}\theta\sec\theta$

16.3 TRIGONOMETRIC FUNCTIONS OF SOME SPECIFIC REAL NUMBERS

The values of the trigonometric functions of 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$ and $\frac{\pi}{2}$ are summarised below in the form of a table :

Real Numbers → / Function ↓	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not defined

As an aid to memory, we may think of the following pattern for above mentioned values of sin function :

$$\sqrt{\frac{0}{4}}, \sqrt{\frac{1}{4}}, \sqrt{\frac{2}{4}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{4}}$$

On simplification, we get the values as given in the table. The values for cosines occur in the reverse order.

Example 16.12 Find the value of the following :

(a) $\sin\frac{\pi}{4}\sin\frac{\pi}{3} - \cos\frac{\pi}{4}\cos\frac{\pi}{3}$ (b) $4\tan^2\frac{\pi}{4} - \operatorname{cosec}^2\frac{\pi}{6} - \cos^2\frac{\pi}{3}$

Solution :

(a) $\sin\frac{\pi}{4}\sin\frac{\pi}{3}-\cos\frac{\pi}{4}\cos\frac{\pi}{3}$

$$=\left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right)-\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right)$$

$$=\frac{\sqrt{3}-1}{2\sqrt{2}}$$

(b) $4\tan^2\frac{\pi}{4}-\operatorname{cosec}^2\frac{\pi}{6}-\cos^2\frac{\pi}{3}$

$$=4\,(1)^2-(2)^2-\left(\frac{1}{2}\right)^2$$

$$=4-4-\frac{1}{4}=-\frac{1}{4}$$

Example 16.13 If $A=\frac{\pi}{3}$ and $B=\frac{\pi}{6}$, verify that

$$\cos(A+B)=\cos A\cos B-\sin A\sin B$$

Solution : L.H.S. $=\cos(A+B)$

$$=\cos\left(\frac{\pi}{3}+\frac{\pi}{6}\right)=\cos\frac{\pi}{2}=0$$

R.H.S. $=\cos\frac{\pi}{3}\cos\frac{\pi}{6}-\sin\frac{\pi}{3}\sin\frac{\pi}{6}$

$$=\frac{1}{2}\cdot\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2}\cdot\frac{1}{2}$$

$$=\frac{\sqrt{3}}{4}-\frac{\sqrt{3}}{4}=0$$

$\therefore$ L.H.S. = 0 = R.H.S.

cos (A + B) = cos A cos B − sin A sin B

CHECK YOUR PROGRESS 16.5

1. Find the value of

(i) $\sin^2\frac{\pi}{6}+\tan^2\frac{\pi}{4}+\tan^2\frac{\pi}{3}$ (ii) $\sin^2\frac{\pi}{3}+\text{cosec}^2\frac{\pi}{6}+\sec^2\frac{\pi}{4}-\cos^2\frac{\pi}{3}$

(iii) $\cos\frac{2\pi}{3}\cos\frac{\pi}{3}-\sin\frac{2\pi}{3}\sin\frac{\pi}{3}$ (iv) $4\cot^2\frac{\pi}{3}+\text{cosec}^2\frac{\pi}{4}+\sec^2\frac{\pi}{3}\tan^2\frac{\pi}{4}$

(v) $\left(\sin\frac{\pi}{6}+\sin\frac{\pi}{4}\right)\left(\cos\frac{\pi}{3}-\cos\frac{\pi}{4}\right)+\frac{1}{4}$

2. Show that

$$\left(1+\tan\frac{\pi}{6}\tan\frac{\pi}{3}\right)+\left(\tan\frac{\pi}{6}-\tan\frac{\pi}{3}\right)=\sec^2\frac{\pi}{6}\sec^2\frac{\pi}{3}$$

3. Taking $A=\frac{\pi}{3}$, $B=\frac{\pi}{6}$, verify that

(i) $\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$ (ii) $\cos(A+B)=\cos A\cos B-\sin A\sin B$

4. If $\theta=\frac{\pi}{4}$, verify the following :

(i) $\sin 2\theta=2\sin\theta\cos\theta$ (ii) $\cos 2\theta=\cos^2\theta-\sin^2\theta$

$$=2\cos^2\theta-1$$

$$=1-2\sin^2\theta$$

5. If $A=\frac{\pi}{6}$, verify that

(i) $\cos 2A=2\cos^2 A-1$ (ii) $\tan 2A=\frac{2\tan A}{1-\tan^2 A}$

(iii) $\sin 2A=2\sin A\cos A$

16.4 GRAPHS OF TRIGONOMETRIC FUNCTIONS

Given any function, a pictorial or a graphical representation makes a lasting impression on the minds of learners and viewers. The importance of the graph of functions stems from the fact that this is a convenient way of presenting many properties of the functions. By observing the graph we can examine several characteristic properties of the functions such as (i) periodicity, (ii) intervals in which the function is increasing or decreasing (iii) symmetry about axes, (iv) maximum and minimum points of the graph in the given interval. It also helps to compute the areas enclosed by the curves of the graph.

16.4.1 Variations of sinq as q Varies Continuously From 0 to 2p.

Let $X'OX$ and $Y'OY$ be the axes of coordinates. With centre O and radius OP = unity, draw a circle. Let OP starting from OX and moving in anticlockwise direction make an angle θ with the x-axis, i.e. $\angle XOP = \theta$. Draw $PM \perp X'OX$, then $\sin\theta = MP$ as $OP = 1$.

$\therefore$ The variations of sin θ are the same as those of MP.

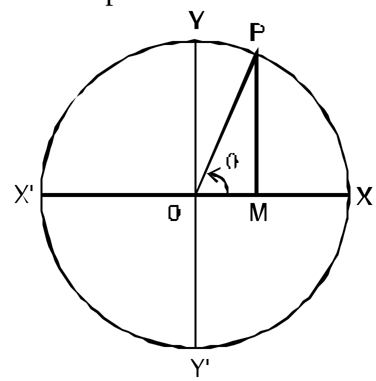

Fig. 16.7

I Quadrant :

As θ increases continuously from 0 to $\frac{\pi}{2}$

PM is positive and increases from 0 to 1.

$\therefore$ sin θ is positive.

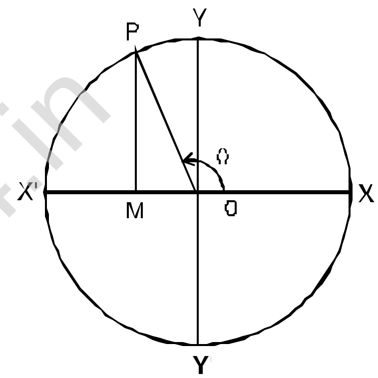

Fig. 16.8

II Quadrant $\left[\frac{\pi}{2}, \pi\right]$

In this interval, θ lies in the second quadrant.

Therefore, point P is in the second quadrant. Here PM = y is positive, but decreases from 1 to 0 as θ varies from $\frac{\pi}{2}$ to π. Thus sin θ is positive.

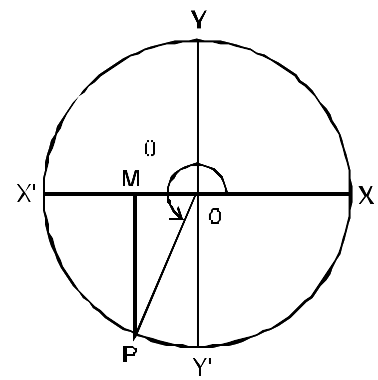

Fig. 16.9

III Quadrant $\left[\pi, \frac{3\pi}{2}\right]$

In this interval, θ lies in the third quandrant. Therefore, point P can move in the third quadrant only. Hence $PM = y$ is negative and decreases from 0 to -1 as θ varies from π to $\frac{3\pi}{2}$. In this interval $\sin\theta$ decreases from 0 to -1. In this interval sin θ is negative.

IV Quadrant $\left[\frac{3\pi}{2}, 2\pi\right]$

In this interval, θ lies in the fourth quadrant. Therefore, point P can move in the fourth quadrant only. Here again PM = y is negative but increases from -1 to 0 as θ varies from $\frac{3\pi}{2}$ to 2π. Thus sin θ is negative in this interval.

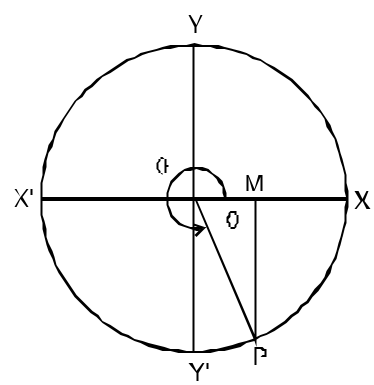

Fig. 16.10

16.4.2 Graph of sin θ as θ varies from 0 to 2π.

Let $X'OX$ and $Y'OY$ be the two coordinate axes of reference. The values of θ are to be measured along x-axis and the values of sine θ are to be measured along y-axis.

(Approximate value of $\sqrt{2} = 1.41, \frac{1}{\sqrt{2}} = .707, \frac{\sqrt{3}}{2} = .87$)

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$\sin\theta$	0	.5	.87	1	.87	.5	0	$-.5$	$-.87$	-1	$-.87$	$-.5$	0

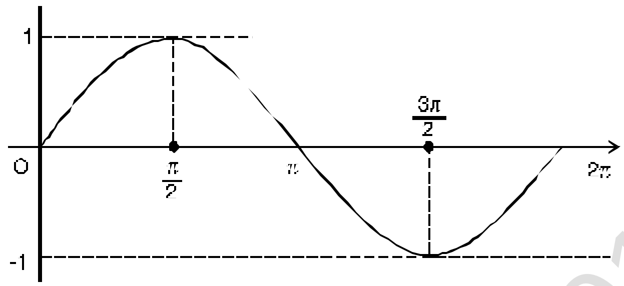

Fig. 16.11

Some Observations

(i) Maximum value of sin θ is 1.

(ii) Minimum value of sin θ is -1.

(iii) It is continuous everywhere.

(iv) It is increasing from 0 to $\frac{\pi}{2}$ and from $\frac{3\pi}{2}$ to 2π. It is decreasing from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$.

With the help of the graph drawn in Fig. 16.12 we can always draw another graph. $y = \sin\theta$ in the interval of $[2\pi, 4\pi]$ (see Fig. 16.11)

What do you observe ?

The graph of $y = \sin\theta$ in the interval $[2\pi, 4\pi]$ is the same as that in 0 to 2π. Therefore, this graph can be drawn by using the property $\sin(2\pi + \theta) = \sin\theta$. Thus, sin θ repeats itself when θ is increased by 2π. This is known as the periodicity of sin θ.

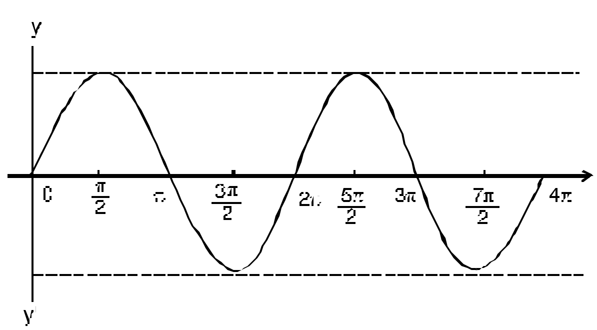

Fig. 16.12

LET US SUM UP

- An angle is generated by the rotation of a ray.
- The angle can be negative or positive according as rotation of the ray is clockwise or anticlockwise.
- A degree is one of the measures of an angle and one complete rotation generates an angle of 360°.
- An angle can be measured in radians, 360° being equivalent to 2π radians.
- If an arc of length *l* subtends an angle of θ radians at the centre of the circle with radius *r*, we have $l = r\theta$.
- If the coordinates of a point *P* of a unit circle are (*x*, *y*) then the six trigonometric functions are defined as $\sin\theta = y$, $\cos\theta = x$, $\tan\theta = \frac{y}{x}$, $\cot\theta = \frac{x}{y}$, $\sec\theta = \frac{1}{x}$ and $\operatorname{cosec}\theta = \frac{1}{y}$.

  The coordinates (*x*, *y*) of a point *P* can also be written as $(\cos\theta, \sin\theta)$.

  Here θ is the angle which the line joining centre to the point *P* makes with the positive direction of x-axis.
- The values of the trigonometric functions $\sin\theta$ and $\cos\theta$ when θ takes values 0, $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$ are given by

	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0

- Graphs of $\sin\theta$, $\cos\theta$ are continous every where
  - — Maximum value of both $\sin\theta$ and $\cos\theta$ is 1.
  - — Minimum value of both $\sin\theta$ and $\cos\theta$ is -1.
  - — Period of these functions is 2π.

- $\tan\theta$ and $\cot\theta$ can have any value between $-\infty$ and $+\infty$.
  - — The function $\tan\theta$ has discontinuities (breaks) at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ in $(0, 2\pi)$.
  - — Its period is π.
  - — The graph of $\cot\theta$ has discontinuities (breaks) at 0, $\pi, 2\pi$. Its period is π.
- $\sec\theta$ cannot have any value numerically less than 1.

(i) It has breaks at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$. It repeats itself after 2π.

(ii) $\operatorname{cosec}\theta$ cannot have any value between -1 and $+1$.

It has discontinuities (breaks) at 0, π, 2π.

It repeats itself after 2π.

SUPPORTIVE WEB SITES

- *http://www.wikipedia.org*
- *http://mathworld.wolfram.com*

TERMINAL EXERCISE

1. A train is moving at the rate of 75 km/hour along a circular path of radius 2500 m. Through how many radians does it turn in one minute ?

2. Find the number of degrees subtended at the centre of the circle by an arc whose length is 0.357 times the radius.

3. The minute hand of a clock is 30 cm long. Find the distance covered by the tip of the hand in 15 minutes.

4. Prove that

(a) $\sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}} = \sec\theta - \tan\theta$

(b) $\dfrac{1}{\sec\theta + \tan\theta} = \sec\theta - \tan\theta$

(c) $\dfrac{\tan\theta}{1+\tan^2\theta} - \dfrac{\cot\theta}{1+\cot^2\theta} = 2\sin\theta\cos\theta$

(d) $\dfrac{1+\sin\theta}{1-\sin\theta} = (\tan\theta + \sec\theta)^2$

(e) $\sin^8\theta - \cos^8\theta = \left(\sin^2\theta - \cos^2\theta\right)\left(1 - 2\sin^2\theta\cos^2\theta\right)$

(f) $\sqrt{\sec^2\theta + \operatorname{cosec}^2\theta} = \tan\theta + \cot\theta$

5. If $\theta = \frac{\pi}{4}$, verify that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

6. Evaluate :

(a) $\sin\frac{25\pi}{6}$ (b) $\sin\frac{21\pi}{4}$ (c) $\tan\left(\frac{3\pi}{4}\right)$

(d) $\sin\frac{17}{4}\pi$ (e) $\cos\frac{19}{3}\pi$

ANSWERS

TERMINAL EXERCISE

1. $\frac{1}{2}$ radian 2. 20.45° 3. 15π cm

6. (a) $\frac{1}{2}$ (b) $-\frac{1}{\sqrt{2}}$ (c) -1 (d) $\frac{1}{\sqrt{2}}$ (e) $\frac{1}{2}$

CHECK YOUR PROGRESS 16.5

1. (i) $4\frac{1}{4}$ (ii) $6\frac{1}{2}$ (iii) -1 (iv) $\frac{22}{3}$ (v) Zero

ANSWERS

CHECK YOUR PROGRESS 16.1

1. (i) $\frac{\pi}{3}$ (ii) $\frac{\pi}{12}$ (iii) $\frac{5\pi}{12}$ (iv) $\frac{7\pi}{12}$ (v) $\frac{3\pi}{2}$

2. (i) 45° (ii) 15° (iii) 9° (iv) 3° (v) 120°

3. $\frac{\pi}{4}, \frac{13\pi}{36}, \frac{14\pi}{36}$ 4. $\frac{5\pi}{6}$ 5. $\frac{\pi}{3}$

CHECK YOUR PROGRESS 16.2

1. (a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{5\pi}{6}$

2. (a) 36° (b) 30° (c) 20°

3. $\frac{1}{6}$ radian; 9.55° 4. $\frac{1}{5}$ radian 5. 95.54 m

6. (a) 0.53 m (b) 38.22 cm (c) 0.002 radian
 (d) 12.56 m (e) 31.4 cm (f) 3.75 radian
 (g) 6.28 m (h) 2 radian (i) 19.11 m.

CHECK YOUR PROGRESS 16.3

1. (i) – ive (ii) – ive (iii) – ive (iv) + ive
 (v) + ive (vi) – ive (vii) + ive (viii) – ive

2. (i) zero (ii) zero (iii) $-\frac{1}{2}$ (iv) – 1
 (v) 1 (vi) Not defined (vii) Not defined (viii) 1

CHECK YOUR PROGRESS 16.4

2. $\sin\theta = \frac{1}{\sqrt{5}}$, $\cos\theta = \frac{2}{\sqrt{5}}$, $\cot\theta = 2$, $\operatorname{cosec}\theta = \sqrt{5}$, $\sec\theta = \frac{\sqrt{5}}{2}$

3. $\sin\theta = \frac{a}{b}$, $\cos\theta = \frac{\sqrt{b^2 - a^2}}{b}$, $\sec\theta = \frac{b}{\sqrt{b^2 - a^2}}$,

 $\tan\theta = \frac{a}{\sqrt{b^2 - a^2}}$, $\cot\theta = \frac{\sqrt{b^2 - a^2}}{a}$ 6. $\frac{2m}{1 + m^2}$