

42. If \vec{a} and \vec{b} are vectors such that $|\vec{a} + \vec{b}| = \sqrt{29}$ and $\vec{a} \times (2\hat{i} + 3\hat{j} + 4\hat{k}) = (2\hat{i} + 3\hat{j} + 4\hat{k}) \times \vec{b}$, then a possible value of $(\vec{a} + \vec{b}) \cdot (-7\hat{i} + 2\hat{j} + 3\hat{k})$ is

- (A) 0 (B) 3 (C) 4 (D) 8

Sol. Ans. (C)

Let $\vec{c} = 2\hat{i} + 3\hat{j} + 4\hat{k}$

$$\vec{a} \times \vec{c} = \vec{c} \times \vec{b}$$

$$\Rightarrow (\vec{a} + \vec{b}) \times \vec{c} = \vec{0}$$

$$\Rightarrow (\vec{a} + \vec{b}) \parallel \vec{c}$$

Let $(\vec{a} + \vec{b}) = \lambda \vec{c}$

$$\Rightarrow |\vec{a} + \vec{b}| = |\lambda| |\vec{c}|$$

$$\Rightarrow \sqrt{29} = |\lambda| \cdot \sqrt{29}$$

$$\Rightarrow \lambda = \pm 1$$

$$\therefore \vec{a} + \vec{b} = \pm (2\hat{i} + 3\hat{j} + 4\hat{k})$$

Now $(\vec{a} + \vec{b}) \cdot (-7\hat{i} + 2\hat{j} + 3\hat{k}) = \pm (-14 + 6 + 12)$
 $= \pm 4$

43. Let PQR be a triangle of area Δ with $a = 2$, $b = \frac{7}{2}$ and $c = \frac{5}{2}$, where a, b and c are the lengths of the sides

of the triangle opposite to the angles at P, Q and R respectively. Then $\frac{2\sin P - \sin 2P}{2\sin P + \sin 2P}$ equals

- (A) $\frac{3}{4\Delta}$ (B) $\frac{45}{4\Delta}$ (C) $\left(\frac{3}{4\Delta}\right)^2$ (D) $\left(\frac{45}{4\Delta}\right)^2$

Sol. Ans. (C)

$$a = 2 = QR$$

$$b = \frac{7}{2} = PR$$

$$c = \frac{5}{2} = PQ$$

$$s = \frac{a+b+c}{2} = \frac{8}{4} = 4$$

$$\frac{2\sin P - 2\sin P \cos P}{2\sin P + 2\sin P \cos P} = \frac{2\sin P(1 - \cos P)}{2\sin P(1 + \cos P)} = \frac{1 - \cos P}{1 + \cos P} = \frac{2\sin^2 \frac{P}{2}}{2\cos^2 \frac{P}{2}} = \tan^2 \frac{P}{2}$$

$$= \frac{(s-b)(s-c)}{s(s-a)} = \frac{(s-b)^2(s-c)^2}{\Delta^2} = \frac{\left(4-\frac{7}{2}\right)^2\left(4-\frac{5}{2}\right)^2}{\Delta^2} = \left(\frac{3}{4\Delta}\right)^2$$

44. Four fair dice D_1, D_2, D_3 and D_4 each having six faces numbered 1,2,3,4,5 and 6 are rolled simultaneously. The probability that D_4 shows a number appearing on one of D_1, D_2 and D_3 is

- (A) $\frac{91}{216}$ (B) $\frac{108}{216}$ (C) $\frac{125}{216}$ (D) $\frac{127}{216}$

Sol. Ans. (A)

Favourable : D_4 shows a number and
 only 1 of D_1, D_2, D_3 shows same number
 or only 2 of D_1, D_2, D_3 shows same number
 or all 3 of D_1, D_2, D_3 shows same number

$$\begin{aligned} \text{Required Probability} &= \frac{{}^6C_1({}^3C_1 \times 5 \times 5 + {}^3C_2 \times 5 + {}^3C_3)}{216 \times 6} \\ &= \frac{6 \times (75 + 15 + 1)}{216 \times 6} \\ &= \frac{6 \times 91}{216 \times 6} \\ &= \frac{91}{216} \end{aligned}$$

45. The value of the integral $\int_{-\pi/2}^{\pi/2} \left(x^2 + \ln \frac{\pi+x}{\pi-x} \right) \cos x \, dx$ is

- (A) 0 (B) $\frac{\pi^2}{2} - 4$ (C) $\frac{\pi^2}{2} + 4$ (D) $\frac{\pi^2}{2}$

Sol. Ans. (B)

$$\begin{aligned} \int_{-\pi/2}^{\pi/2} \left(x^2 + \ln \left(\frac{\pi+x}{\pi-x} \right) \right) \cos x \, dx &= 2 \int_0^{\pi/2} x^2 \cos x \, dx + 0 \quad \left(\because \ln \left(\frac{\pi+x}{\pi-x} \right) \text{ is an odd function} \right) \\ &= 2 \left[(x^2 \sin x) \Big|_0^{\pi/2} - \int_0^{\pi/2} 2x \sin x \, dx \right] = 2 \left(\frac{\pi^2}{4} - 0 \right) - 4 \int_0^{\pi/2} x \sin x \, dx \\ &= \frac{\pi^2}{2} - 4 \left[(-x \cos x) \Big|_0^{\pi/2} + \int_0^{\pi/2} \cos x \, dx \right] \\ &= \frac{\pi^2}{2} - 4 \end{aligned}$$

46. If P is a 3×3 matrix such that $P^T = 2P + I$, where P^T is the transpose of P and I is the 3×3 identity matrix

then there exists a column matrix $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ such that

- (A) $PX = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (B) $PX = X$ (C) $PX = 2X$ (D) $PX = -X$

Sol. Ans. (D)

$$\begin{aligned} P^T &= 2P + I \\ \Rightarrow (P^T)^T &= (2P + I)^T \\ \Rightarrow P &= 2P^T + I \\ \Rightarrow P &= 2(2P + I) + I \\ \Rightarrow 3P &= -3I \quad \Rightarrow \quad P = -I \\ \Rightarrow PX &= -IX = -X \end{aligned}$$

47. Let a_1, a_2, a_3, \dots be in harmonic progression with $a_1 = 5$ and $a_{20} = 25$. The least positive integer n for which $a_n < 0$ is

- (A) 22 (B) 23 (C) 24 (D) 25

Sol. Ans. (D)

Corresponding A.P.

$$\frac{1}{5}, \dots, \frac{1}{25} \text{ (20th term)}$$

$$\frac{1}{25} = \frac{1}{5} + 19d \quad \Rightarrow \quad d = \frac{1}{19} \left(\frac{-4}{25} \right) = -\frac{4}{19 \times 25}$$

$$a_n < 0$$

$$\frac{1}{5} - \frac{4}{19 \times 25} \times (n-1) < 0$$

$$\frac{19 \times 5}{4} < n - 1$$

$$n > 24.75$$

48. Let $\alpha(a)$ and $\beta(a)$ be the roots of the equation $(\sqrt[3]{1+a} - 1)x^2 + (\sqrt{1+a} - 1)x + (\sqrt[6]{1+a} - 1) = 0$ where $a > -1$.

Then $\lim_{a \rightarrow 0^+} \alpha(a)$ and $\lim_{a \rightarrow 0^+} \beta(a)$ are

- (A) $-\frac{5}{2}$ and 1 (B) $-\frac{1}{2}$ and -1 (C) $-\frac{7}{2}$ and 2 (D) $-\frac{9}{2}$ and 3

Sol. Ans. (B)

$$((1+a)^{1/3} - 1)^{-2} + ((1+a)^{1/2} - 1)^{-1} + ((1+a)^{1/6} - 1) = 0$$

let $a + 1 = t^6$

$$\begin{aligned} \therefore (t^2 - 1)^r + (t^3 - 1)^r + (t - 1) &= 0 \\ (t + 1)x^2 + (t^2 + t + 1)x + 1 &= 0 \end{aligned}$$

As $a \rightarrow 0$, $t \rightarrow 1$

$$2x^2 + 3x + 1 = 0 \Rightarrow x = -1 \text{ and } r = -\frac{1}{2}$$

Section II : Paragraph Type

This section contains 6 multiple choice questions relating to three paragraphs with two questions on each paragraph. Each question has four choices (A), (B), (C) and (D) out of which **ONLY ONE** is correct.

Paragraph for Question Nos. 49 to 50

Let $f(x) = (1 - x)^2 \sin^2 x + x^2$ for all $x \in \mathbb{R}$ and let $g(x) = \int_1^x \left(\frac{2(t-1)}{t+1} - \ln t \right) f(t) dt$ for all $x \in (1, \infty)$.

49. Which of the following is true ?
- (A) g is increasing on $(1, \infty)$
 - (B) g is decreasing on $(1, \infty)$
 - (C) g is increasing on $(1, 2)$ and decreasing on $(2, \infty)$
 - (D) g is decreasing on $(1, 2)$ and increasing on $(2, \infty)$

Sol. **Ans. (B)**

$$f(x) = (1 - x)^2 \sin^2 x + x^2 : x \in \mathbb{R}$$

$$g(x) = \int_1^x \left(\frac{2(t-1)}{t+1} - \ln t \right) f(t) dt$$

$$\therefore g'(x) = \left(\frac{2(x-1)}{x+1} - \ln x \right) f(x) \cdot 1$$

$$\text{let } \phi(x) = \frac{2(x-1)}{x+1} - \ln x$$

$$\phi'(x) = \frac{2[(x+1) - (x-1) \cdot 1]}{(x+1)^2} - \frac{1}{x} = \frac{4}{(x+1)^2} - \frac{1}{x} = \frac{-x^2 + 2x - 1}{x(x+1)^2} = \frac{-(x-1)^2}{x(x+1)^2}$$

$$\therefore \phi'(x) \leq 0$$

$$\therefore \text{for } x \in (1, \infty), \phi(x) < 0$$

$$\therefore g'(x) < 0 \quad \text{for } x \in (1, \infty)$$

50. Consider the statements :

P : There exists some $x \in \mathbb{R}$ such that $f(x) + 2x = 2(1 + x^2)$

Q : There exists some $x \in \mathbb{R}$ such that $2f(x) + 1 = 2x(1 + x)$

Then

(A) both P and Q are true

(B) P is true and Q is false

(C) P is false and Q is true

(D) both P and Q are false

Sol. **Ans. (C)**

$$f(x) + 2x = (1 - x)^2 \sin^2 x + x^2 + 2x$$

$$\therefore f(x) + 2x = 2(1 + x^2)$$

$$\Rightarrow (1 - x)^2 \sin^2 x + x^2 + 2x = 2 + 2x^2$$

$$(1 - x)^2 \sin^2 x = x^2 - 2x + 1 + 1 \\ = (1 - x)^2 + 1$$

$$\Rightarrow (1 - x)^2 \cos^2 x = -1$$

which can never be possible

P is not true

$$\Rightarrow \text{Let } H(x) = 2f(x) + 1 - 2x(1 + x)$$

$$H(0) = 2f(0) + 1 - 0 = 1$$

$$H(1) = 2f(1) + 1 - 2 = -3$$

\Rightarrow so $H(x)$ has a solution

so Q is true

Paragraph for Question Nos. 51 to 52

Let a_n denote the number of all n -digit positive integers formed by the digits 0, 1 or both such that no consecutive digits in them are 0. Let b_n = the number of such n -digit integers ending with digit 1 and c_n = the number of such n -digit integers ending with digit 0.

51. Which of the following is correct ?

(A) $a_{17} = a_{16} + a_{15}$

(B) $c_{17} \neq c_{16} + c_{15}$

(C) $b_{17} \neq b_{16} + c_{15}$

(D) $a_{17} = c_{17} + b_{16}$

Sol. **Ans. (A)**

$$\underline{1} \dots \dots \dots \underline{1} \# a_{n-1}$$

$$\dots \dots \dots \underline{10} \# a_{n-2}$$

So A choice is correct

consider B choice $c_{17} \neq c_{16} + c_{15}$

$$c_{15} \neq c_{14} + c_{13} \text{ is not true}$$

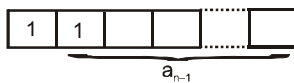
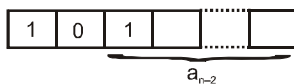
consider C choice $b_{17} \neq b_{16} + c_{15}$

$$a_{16} \neq a_{15} + a_{14} \text{ is not true}$$

consider D choice $a_{17} = c_{17} + b_{16}$

$$a_{17} = a_{15} + a_{15} \text{ which is not true}$$

Aliter



using the Recursion formula

$$a_n = a_{n-1} + a_{n-2}$$

$$\text{Similarly } b_n = b_{n-1} + b_{n-2} \text{ and } c_n = c_{n-1} + c_{n-2} \quad \forall n \geq 3$$

$$\text{and } a_n = b_n + c_n \quad \forall n \geq 1$$

$$\text{so } a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 5, a_5 = 8, \dots$$

$$b_1 = 1, b_2 = 1, b_3 = 2, b_4 = 3, b_5 = 5, b_6 = 8, \dots$$

$$c_1 = 0, c_2 = 1, c_3 = 1, c_4 = 2, c_5 = 3, c_6 = 5, \dots$$

$$\text{using this } b_{n-1} = c_n \quad \forall n \geq 2$$

52. The value of b_6 is
 (A) 7 (B) 8 (C) 9 (D) 11

Sol. Ans. (B)

$$b_6 = a_5$$

$$a_5 = \binom{1}{0} + \binom{1}{1} + \binom{1}{2} + \binom{1}{3} + \binom{1}{4} + \binom{1}{5}$$

$${}^3C_0 + {}^3C_1 + 1 + {}^2C_1 + 1$$

$$1 + 3 + 1 + 2 + 1$$

$$4 + 4 = 8$$

Paragraph for Question Nos. 53 to 54

A tangent PT is drawn to the circle $x^2 + y^2 = 4$ at the point $P(\sqrt{3}, 1)$. A straight line L, perpendicular to PT is a tangent to the circle $(x - 3)^2 + y^2 = 1$.

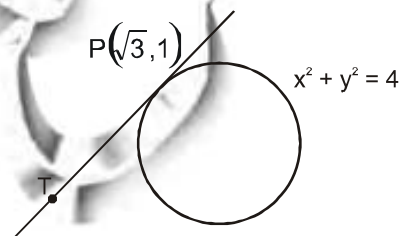
53. A common tangent of the two circles is
 (A) $x = 4$ (B) $y = 2$ (C) $x + \sqrt{3}y = 4$ (D) $x + 2\sqrt{2}y = 6$

Ans. (D)

54. A possible equation of L is
 (A) $x - \sqrt{3}y = 1$ (B) $x + \sqrt{3}y = 1$ (C) $x - \sqrt{3}y = -1$ (D) $x + \sqrt{3}y = 5$

Ans. (A)

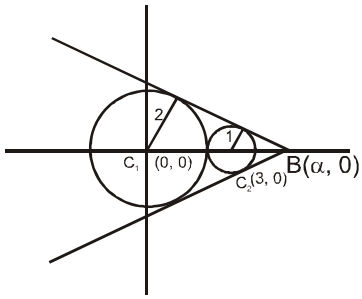
Sol. Q.No. 53 to 54



Equation of tangent at $(\sqrt{3}, 1)$

$$\sqrt{3}x + y = 4$$

53.



B divides $C_1 C_2$ in 2 : 1 externally

$$\therefore B(6, 0)$$

Hence let equation of common tangent is

$$y - 0 = m(x - 6)$$

$$mx - y - 6m = 0$$

length of \perp dropped from center $(0, 0)$ = radius

$$\left| \frac{6m}{\sqrt{1+m^2}} \right| = 2 \Rightarrow m = \pm \frac{1}{2\sqrt{2}}$$

$$\therefore \text{equation is } x + 2\sqrt{2}y = 6 \text{ or } x - 2\sqrt{2}y = 6$$

54.

Equation of L is

$$x - y\sqrt{3} + c = 0$$

length of perpendicular dropped from centre = radius of circle

$$\therefore \left| \frac{3+c}{2} \right| = 1 \Rightarrow c = -1, -5$$

$$\therefore x - \sqrt{3}y = 1 \text{ or } x - \sqrt{3}y = 5$$

Section III : Multiple Correct Answer(s) Type

This section contains **6 multiple choice questions**. Each question has four choices (A), (B), (C) and (D) out of which **ONE or MORE** are correct.

55. Let X and Y be two events such that $P(X|Y) = \frac{1}{2}$, $P(Y|X) = \frac{1}{3}$ and $P(X \cap Y) = \frac{1}{6}$. Which of the following is (are) correct ?

(A) $P(X \cup Y) = \frac{2}{3}$

(B) X and Y are independent

(C) X and Y are not independent

(D) $P(X^c \cap Y) = \frac{1}{3}$

Sol. Ans. (AB)

$$P(X|Y) = \frac{1}{2}$$

$$\frac{P(X \cap Y)}{P(Y)} = \frac{1}{2} \Rightarrow P(Y) = \frac{1}{3}$$

$$P(Y|X) = \frac{1}{3}$$

$$\frac{P(X \cap Y)}{P(X)} = \frac{1}{3} \Rightarrow P(X) = \frac{1}{2}$$

$$P(X \cup Y) = P(X) + P(Y) - P(X \cap Y) = \frac{2}{3}$$

$$P(X \cap Y) = P(X) \cdot P(Y) \Rightarrow X \text{ and } Y \text{ are independent}$$

$$P(X^c \cap Y) = P(Y) - P(X \cap Y)$$

$$= \frac{1}{3} - \frac{1}{6} = \frac{1}{6}$$

A is correct

B is correct

D is not correct

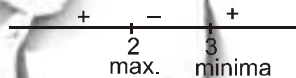
56. If $f(x) = \int_0^x e^{t^2} (t-2)(t-3) dt$ for all $x \in (0, \infty)$, then

- (A) f has a local maximum at $x = 2$
- (B) f is decreasing on $(2, 3)$
- (C) there exists some $c \in (0, \infty)$ such that $f''(c) = 0$
- (D) f has a local minimum at $x = 3$

Sol. Ans. (ABCD)

$$f(x) = \int_0^x e^{t^2} (t-2)(t-3) dt$$

$$f'(x) = e^{x^2} (x-2)(x-3)$$



- (i) $x = 2$ is local maxima
- (ii) $x = 3$ is local minima
- (iii) It is decreasing in $x \in (2, 3)$

$$(iv) f''(x) = e^{x^2} (x-2) + e^{x^2} (x-3) + 2x e^{x^2} (x-2)(x-3)$$

$$= e^{x^2} (x-2 + x-3 + 2x(x-2)(x-3))$$

$$f''(x) = 0$$

$$f''(x) = e^{x^2} (2x^3 - 10x^2 + 14x - 5)$$

$$f''(0) < 0 \text{ and } f''(1) > 0$$

$$\text{so } f''(c) = 0 \text{ where } c \in (0, 1)$$

57. For every integer n , let a_n and b_n be real numbers. Let function $f : \mathbb{R} \rightarrow \mathbb{R}$ be given by

$$f(x) = \begin{cases} a_n + \sin \pi x, & \text{for } x \in [2n, 2n + 1] \\ b_n + \cos \pi x, & \text{for } x \in (2n - 1, 2n), \end{cases} \text{ for all integers } n.$$

If f is continuous, then which of the following hold(s) for all n ?

- (A) $a_{n-1} - b_{n-1} = 0$ (B) $a_n - b_n = 1$ (C) $a_n - b_{n+1} = 1$ (D) $a_{n-1} - b_n = -1$

Sol. **Ans. (BD)**

$$\left. \begin{aligned} f(2n) &= a_n \\ f(2n^+) &= a_n \\ f(2n^-) &= b_n + 1 \end{aligned} \right\} \begin{aligned} a_n &= b_n + 1 \\ a_n - b_n &= 1 \end{aligned} \text{ So B is correct}$$

$$\left. \begin{aligned} f(2n+1) &= a_n \\ f((2n+1)^-) &= a_n \\ f((2n+1)^+) &= b_{n+1} - 1 \end{aligned} \right\} \begin{aligned} a_n &= b_{n+1} - 1 \\ a_n - b_{n+1} &= -1 \\ a_{n+1} - b_n &= -1 \end{aligned}$$

So D is correct

58. If the straight lines $\frac{x-1}{2} = \frac{y+1}{k} = \frac{z}{2}$ and $\frac{x+1}{5} = \frac{y+1}{2} = \frac{z}{k}$ are coplanar, then the plane(s) containing these two lines is(are)

- (A) $y + 2z = -1$ (B) $y + z = -1$ (C) $y - z = -1$ (D) $y - 2z = -1$

Sol. **Ans. (BC)**

For co-planer lines $[\vec{a} - \vec{c} \vec{b} \vec{d}] = 0$

$$\vec{a} \equiv (1, -1, 0) \quad \vec{c} \equiv (-1, -1, 0)$$

$$\vec{b} = 2\hat{i} + k\hat{j} + 2\hat{k} \quad \vec{d} = 5\hat{i} + 2\hat{j} + k\hat{k}$$

$$\text{Now } \begin{vmatrix} 2 & 0 & 0 \\ 2 & k & 2 \\ 5 & 2 & k \end{vmatrix} = 0 \Rightarrow k = \pm 2$$

$$\vec{n}_1 = \vec{b}_1 \times \vec{d}_1 = 6\hat{j} - 6\hat{k} \quad \text{for } k = 2$$

$$\vec{n}_2 = \vec{b}_2 \times \vec{d}_2 = 14\hat{j} + 14\hat{k} \quad \text{for } k = -2$$

$$\text{so the equation of planes are } (\vec{r} - \vec{a})\vec{n}_1 = 0 \Rightarrow y - z = -1 \quad \dots (1)$$

$$(\vec{r} - \vec{a})\vec{n}_2 = 0 \Rightarrow y + z = -1 \quad \dots (2)$$

so answer is (B,C)

