TEST CODE: PMB

SYLLABUS

Convergence and divergence of sequence and series;
Cauchy sequence and completeness;
Bolzano-Weierstrass theorem;
continuity, uniform continuity, differentiability;
directional derivatives, Jacobians, Taylor Expansion;
integral calculus of one variable – existence of Riemann integral;
Fundamental theorem of calculus, change of variable;
elementary topological notions for metric space – open, closed and compact sets, connectedness;
elements of ordinary differential equations.

Equivalence relations and partitions;
vector spaces, subspaces, basis, dimension, direct sum;
matrices, systems of linear equations, determinants;
diagonalization, triangular forms;
linear transformations and their representation as matrices;
groups, subgroups, quotients, homomorphisms, products;
Lagrange's theorem, Sylow's theorems;
rings, ideals, maximal ideals, prime ideals, quotients,
integral domains, unique factorization domains, polynomial rings;
fields, algebraic extensions, separable and normal extensions, finite fields.

SAMPLE QUESTIONS

1. Let k be a field and $k[x, y]$ denote the polynomial ring in the two variables x and y with coefficients from k. Prove that for any $a, b \in k$ the ideal generated by the linear polynomials $x - a$ and $y - b$ is a maximal ideal of $k[x, y]$.

2. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation. Show that there is a line L such that $T(L) = L$.

3. Let $A \subseteq \mathbb{R}^n$ and $f : A \to \mathbb{R}^m$ be a uniformly continuous function. If $\{x_n\}_{n\geq 1} \subseteq A$ is a Cauchy sequence then show that $\lim_{n\to\infty} f(x_n)$ exists.

4. Let $N > 0$ and let $f : [0, 1] \to [0, 1]$ be denoted by $f(x) = 1$ if $x = 1/i$ for some integer $i \leq N$ and $f(x) = 0$ for all other values of x. Show that f is Riemann integrable.

5. Let $F : \mathbf{R}^n \to \mathbf{R}$ be defined by

$$F(x_1, x_2, \ldots, x_n) = \max\{|x_1|, |x_2|, \ldots, |x_n|\}.$$

   Show that F is a uniformly continuous function.

6. Show that every isometry of a compact metric space into itself is onto.

7. Let $\mathbf{T} = \{z \in \mathbf{C} : |z| = 1\}$ and $f : [0,1] \to \mathbf{C}$ be continuous with $f(0) = 0, f(1) = 2$. Show that there exists at least one t_0 in $[0,1]$ such that $f(t_0)$ is in $\mathbf{T}$.

8. Let f be a continuous function on $[0,1]$. Evaluate

$$\lim_{n\to\infty} \int_0^1 x^n f(x)dx.$$

9. Find the most general curve whose normal at each point passes though $(0,0)$. Find the particular curve through $(2,3)$.

10. Suppose f is a continuous function on $\mathbf{R}$ which is periodic with period 1, that is, $f(x+1) = f(x)$ for all x. Show that

    (i) the function f is bounded above and below,

    (ii) it achieves both its maximum and minimum and

    (iii) it is uniformly continuous.

11. Let $A = (a_{ij})$ be an $n \times n$ matrix such that $a_{ij} = 0$ whenever $i \geq j$. Prove that A^n is the zero matrix.

12. Determine the integers n for which $\mathbf{Z}_n$, the set of integers modulo n, contains elements x, y so that $x + y = 2,\ 2x - 3y = 3$.

13. Let a_1, b_1 be arbitrary positive real numbers. Define

$$a_{n+1} = \frac{a_n + b_n}{2}, b_{n+1} = \sqrt{a_n b_n}$$

    for all $n \geq 1$. Show that a_n and b_n converge to a common limit.

14. Show that the only field automorphism of $\mathbf{Q}$ is the identity. Using this prove that the only field automorphism of $\mathbf{R}$ is the identity.

15. Consider a circle which is tangent to the y-axis at 0. Show that the slope at any point (x,y) satisfies $\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$.

16. Consider an $n \times n$ matrix $A = (a_{ij})$ with $a_{12} = 1, a_{ij} = 0\ \forall\ (i,j) \neq (1,2)$. Prove that there is no invertible matrix P such that PAP^{-1} is a diagonal matrix.

17. Let G be a nonabelian group of order 39. How many subgroups of order 3 does it have?

18. Let $n \in \mathbf{N}$, let p be a prime number and let $\mathbf{Z}_{p^n}$ denote the ring of integers modulo p^n under addition and multiplication modulo p^n. Let $f(x)$ and $g(x)$ be polynomials with coefficients from the ring $\mathbf{Z}_{p^n}$ such that $f(x) \cdot g(x) = 0$. Prove that $a_i b_j = 0\ \forall\ i,j$ where a_i and b_j are the coefficients of f and g respectively.

19. Show that the fields $\mathbf{Q}(\sqrt{2})$ and $\mathbf{Q}(\sqrt{3})$ are isomorphic as $\mathbf{Q}$-vector spaces but not as fields.

20. Suppose $a_n \geq 0$ and $\sum a_n$ is convergent. Show that $\sum 1/(n^2 a_n)$ is divergent.

Model question paper

Group A

1. Let f be a bounded twice differentiable real valued function on $\mathbb{R}$ such that $f''(x) \geq 0$ for all x. Show that f is a constant.

2. Let $u, v : [a, b] \to \mathbb{R}$ be continuous. Define $f : [a, b] \to \mathbb{R}$ by

$$f(x) = \begin{cases} u(x) & \text{if } x \text{ is rational} \\ v(x) & \text{if } x \text{ is irrational.} \end{cases}$$

   Show that f is Riemann integrable on $[a, b]$ if and only if $u(x) = v(x)$ for all $x \in [a, b]$

3. In each case, give an example of a function f, continuous on S and such that $f(S) = T$, or else explain why there can be no such f :

   (a) $S = (0, 1)$, $T = (0, 1]$.
   (b) $S = \mathbb{R}$, $T = \{$the set of rational numbers$\}$.
   (c) $S = [0, 1] \cup [2, 3]$, $T = \{0, 1\}$.
   (d) $S = (0, 1)$, $T = \mathbb{R}$.
   (e) $S = [0, 1]$, $T = (0, 1)$.

4. If $f(x, y) = \sqrt[3]{x^3 + y^3}$, then is f differentiable at $(0, 0)$? Justify your answer.

5. Suppose that $g : \mathbb{R}^n \to \mathbb{R}$ has continuous partial derivatives and satisfy

$$\left| \frac{\partial g}{\partial x_j}(\mathbf{x}) \right| \leq M$$

   for all $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and for all $j = 1, 2, \ldots, n$.

   (a) Show that
$$|g(\mathbf{x}) - g(\mathbf{y})| \leq \sqrt{n} M \|\mathbf{x} - \mathbf{y}\|$$
   for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Here $\| \cdot \|$ denotes the Euclidian norm on $\mathbb{R}^n$.

   (b) Using (a) or otherwise, prove that g is uniformly continuous.

6. Let C be a subset of a compact metric space (X, d). Assume that, for every continuous function $h : X \to \mathbb{R}$, the restriction of h to C attains a maximum on C. Prove that C is compact.

Group B

1. For the following statements indicate True/False. If the statement is true then give a short proof. Otherwise give a counter-example.

(a) There is no non-trivial group homomorphism from the symmetric group S_3 to $\mathbb{Z}_3$.

(b) If a group G is such that each proper subgroup of G is cyclic, then G is cyclic.

(c) Let R be a ring with unity. Consider $X^2 - 1 \in R[X]$. Then $X^2 - 1$ has at most two roots in R.

(d) Let R be a ring such that $R[X]$ is a principal ideal domain (PID). Then R is a field.

2. Let R be a commutative ring with 1 such that every ascending chain of ideals terminates. Let $f : R \longrightarrow R$ be a surjective homomorphism. Prove that f is an isomorphism.

3. Find the gcd of $4X^4 - 2X^2 + 1$ and $-3X^3 + 4X^2 + X + 1$ in $\mathbb{Z}_7[X]$.

4. Find conditions on $a, b, c \in \mathbb{R}$ to ensure that the following system is consistent, and in that case, find the general solutions :

$$\begin{array}{rcrcrcl} x & + & 3y & - & 2z & = & a \\ -x & - & 5y & + & 3z & = & b \\ 2x & - & 8y & + & 3z & = & c \end{array}$$

5. Prove or disprove :

(a) There exists a linear map $T : \mathbb{R}^2 \to \mathbb{R}^4$ such that

$$\text{Range}(T) = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 + x_2 + x_3 + x_4 = 0\}.$$

(b) There exists a linear map $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that

$$\text{Range}(T) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}.$$

6. Let V be a finite dimensional vector space over $\mathbb{R}$ and $T : V \to V$ be a linear transformation with dimension of Range of $T = k$. Show that T can have at most $k + 1$ distinct eigenvalues.