UNIT 1 ALGEBRA

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1.1 INTRODUCTION

The concept of imaginary numbers has its historical origin in the fact that the solution of the quadratic equation $ax^2 + bx + c = 0$ leads to an expression $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ which is not found meaningfull when $b^2 - 4ac < 0$. This is because of the fact that the square of a real number is never negative. So it created the need of the extension of the system of real numbers. Euler was the first mathematician who introduced the symbol i far $\sqrt{-1}$ with the properties $i^2 = -1$ and accordingly a root of the equation $x^2 + 1 = 0$. Also symbol of the form $a + ib$ where a and b are real numbers was called a complex number.

In this unit, we will study the system of complex numbers, the algebraic operations on complex numbers and the fundamental laws of these operations.

Objectives

After studying this unit, you should be able to

- identify a complex number,
- determine its complex conjugate, modulus and argument,
- describe the basic properties of complex numbers, its modulus and argument, and
- explain the De Moivres theorem and give some useful properties of the theorem. understand the matrix and its applications,
- compute the inverse of a matrix,
- understand the relationship of linear transformations with matrices,
- calculate the rank of a matrix,
- understand certain aspects of matrices in terms of determinants,
- solve simultaneous equations by Cramer's Rule, and
- explain infinite series and its convergence.

1.2 COMPLEX NUMBERS

A number of the form $a + ib$, where a and b are real numbers and $i = \sqrt{-1}$ is called a complex number. The real number a is called the real part and the real number b is called the imaginary part of the complex number $a + ib$.

Let $a + ib$ be denoted by z, i.e. $z = a + ib$.

If $a = 0$, then $z = ib$ and z is said to be purely imaginary. If $b = 0$, then $z = a$ and z is said to be real.

$a - ib$ is said to be the conjugate of $z = a + ib$ and is denoted by $\bar{z}$.

1.2.1 Algebra of Complex Numbers

Let $z_1 = a + ib$ and $z_2 = c + id$ be two complex numbers. Then z_1 is said to be equal to z_2 if and only if $a = c$ and $b = d$.

Addition

$z_1 + z_2 = (a + ib) + (c + id)$ is defined as $(a + c) + i\,(b + d)$.

Subtraction

$z_1 - z_2 = (a + ib) - (c + id)$ is defined as $(a - c) + i\,(b - d)$.

Multiplication

$z_1 z_2 = (a + ib)\,(c + id)$ is defined as $(ac - bd) + i\,(bc + ad)$.

Division

Let $z_2 \neq 0$, i.e. $c \neq 0$ and $d \neq 0$.

We will prove that there exists a complex number $z = x + iy$ such that

$$z_1 = z \,.\, z_2$$

This z is called the quotient of z_1 and z_2 and is denoted by $\frac{z_1}{z_2}$.

$$z_1 = z \cdot z_2$$

$$\Rightarrow \quad (a + ib) = (x + iy)(c + id)$$

$$= (cx - dy) + i(dx + cy)$$

$$\therefore \quad a = cx - dy, \; b = dx + cy$$

Solving for x and y, we have

$$x = \frac{ac + bd}{c^2 + d^2}, \; y = \frac{bc - ad}{c^2 + d^2}$$

$$\therefore \quad \frac{z_1}{z_2} = \frac{ac + bd}{c^2 + d^2} + i\,\frac{bc - ad}{c^2 + d^2}$$

1.2.2 Complex Numbers Defined as Ordered Pair of Real Numbers

A complex number $z = a + ib$ is represented by an ordered pair (a, b) of real numbers and the set of all complex numbers is represented by C.

i.e. $\quad C = \{z : z = (a, b) \; \forall \; a, b \in R\}$

$\therefore$ Two complex numbers (a, b) and (c, d) are said to be equal iff $a = c$ and $b = d$.

The complex number (a, b) is said to be zero complex number iff $a = 0$ and $b = 0$.

The complex number $(-a, -b)$ is called the negative of the complex number (a, b) and vice-versa. We denote $(-a, -b)$ by $-(a, b)$ and $(a, -b)$ is the complex conjugate of (a, b).

Hence $\quad (a, b) + (c, d) = (a + c, b + d)$

$$(a, b) \quad (c, d) = (ac - bd, ad + bc)$$

$$(a, b) - (c, d) = (a, b) + (-c, -d)$$

$$= (a - c, b - d)$$

and $\quad \dfrac{(a, b)}{(c, d)} = \left(\dfrac{ac + bd}{c^2 + d^2}, \dfrac{bc - ad}{c^2 + d^2}\right)$

1.2.3 Basic Properties of Complex Numbers

If z_1, z_2, z_3 are three complex numbers, then it can be proved that

(i) $z_1 + z_2 = z_2 + z_1$

(ii) $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$

(iii) $z_1 z_2 = z_2 z_1$

(iv) $(z_1 z_2) z_3 = z_1 (z_2 z_3)$

(v) $z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$

(vi) $(z_1 + z_2) z_3 = z_1 z_3 + z_2 z_3$

(vii) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

(viii) $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$

(ix) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}, z_2 \neq 0$

We will prove (viii).

$$z_1 z_2 = (a + ib)(c + id)$$

$$= (ac - bd) + i(ad + bc)$$

$$\therefore \quad \overline{z_1 z_2} = (ac - bd) - i(ad + bc)$$

$$= (a - ib)(c - id)$$

$$= \overline{z_1}\,\overline{z_2}$$

The other properties can be proved.

Example 1.1

Express $\frac{(1 + i)(2 + i)}{3 + i}$ in the form $a + ib$.

Solution

$$\frac{(1 + i)(2 + i)}{3 + i} = \frac{2 + i + 2i - 1}{3 + i}$$

$$= \frac{1 + 3i}{3 + i}$$

$$= \frac{(1 + 3i)(3 - i)}{(3 + i)(3 - i)}$$

$$= \frac{3 - i + 9i + 3}{9 + 1}$$

$$= \frac{6 + 8i}{10}$$

$$= \frac{3}{5} + \frac{4}{5} i$$

Example 1.2

Express $\frac{(6 + i)(2 - i)}{(4 + 3i)(1 - 2i)}$ in the form $a + ib$.

Solution

$$\frac{(6 + i)(2 - i)}{(4 + 3i)(1 - 2i)} = \frac{12 + 1 + i(2 - 6)}{4 + 6 + i(3 - 8)} = \frac{13 - 4i}{10 - 5i}$$

$$= \frac{(13 - 4i)(10 + 5i)}{(10 - 5i)(10 + 5i)} = \frac{150 + 25i}{100 + 25}$$

$$= \frac{6 + i}{5} = \frac{6}{5} + \frac{1}{5} i.$$

1.3 GEOMETRICAL REPRESENTATION OF COMPLEX NUMBERS

1.3.1 Argand Diagram

Mathematician Argand represented a complex number in a diagram known as Argand diagram. A complex number $x + iy$ can be represented by a point P whose coordinate are (x, y). The axis of x is called the real axis and the axis of y the imaginary axis. The distance OP is the modulus and the angle, OP makes with the x-axis, is the argument of $x + iy$.

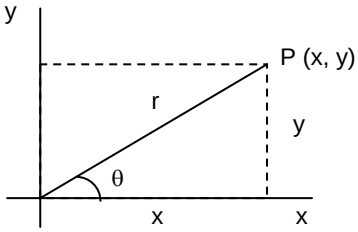

Figure 1.1

1.3.2 Modulus and Argument

Let $x + iy$ be a complex number.

Putting $x = r \cos \theta$ and $y = r \sin \theta$ so that $r = \sqrt{x^2 + y^2}$.

$$\cos \theta = \frac{x}{\sqrt{x^2 + y^2}} \qquad \text{and} \qquad \sin \theta = \frac{y}{\sqrt{x^2 + y^2}}$$

the positive value of the root being taken.

Then r is called the *modulus* or absolute value of the complex number $x + iy$ and is denoted by $| x + iy |$.

The angle θ is called the argument or amplitude of the complex number $x + iy$ and is denoted by argument $(x + iy)$ and $\theta = \tan^{-1} \frac{y}{x}$.

It is clear that θ will have infinite number of values differing by multiples of 2π. The values of θ lying in the range $-\pi < \theta \le \pi$ is called the *principal value* of the argument.

Note : $r^2 = x^2 + y^2 = (x + iy)(x - iy) = z\,\bar{z}$.

The complex number in polar form is $z = r(\cos \theta + i \sin \theta)$.

Example 1.3

Find the modulus and principal argument of the complex number $\frac{1 + 2i}{1 - (1 - i)^2}$.

Solution

$$\frac{1 + 2i}{1 - (1 - i)^2} = \frac{1 + 2i}{1 - (1 - 1 - 2i)}$$

$$= \frac{1 + 2i}{1 + 2i} = 1$$

$$= 1 + 0i$$

$$\left|\frac{1+2i}{1-(1-i)^2}\right| = 1$$

Principal argument of $\dfrac{1+2i}{1-(1-i)^2}$

$$= \tan^{-1}\frac{0}{1} = \tan^{-1} 0 = 0^{\circ}$$

Example 1.4

Express $\dfrac{1+2i}{1-3i}$ in the form $r(\cos\theta + i\sin\theta)$.

Solution

$$\frac{1+2i}{1-3i} = \frac{(1+2i)(1+3i)}{(1-3i)(1+3i)}$$

$$= \frac{1-6+5i}{1+9}$$

$$= \frac{-5+5i}{10}$$

$$= -\frac{1}{2} + \frac{i}{2}$$

$$-\frac{1}{2} + \frac{i}{2} = r(\cos\theta + i\sin\theta) \qquad \ldots (1.1)$$

$$\therefore \qquad r\cos\theta = -\frac{1}{2} \qquad \ldots (1.2)$$

$$r\sin\theta = \frac{1}{2} \qquad \ldots (1.3)$$

Squaring Eqs. (1.2) and (1.3) and then adding, we get

$$r^2(\cos^2\theta + \sin^2\theta) = \frac{1}{4} + \frac{1}{4}$$

$$r^2 = \frac{1}{2} \text{ or } r = \frac{1}{\sqrt{2}}$$

Putting the value of r in Eqs. (1.2) and (1.3), we have

$$\frac{1}{\sqrt{2}}\cos\theta = -\frac{1}{2} \text{ or } \cos\theta = -\frac{1}{\sqrt{2}} \qquad \ldots (1.4)$$

$$\frac{1}{\sqrt{2}}\sin\theta = \frac{1}{2} \text{ or } \sin\theta = \frac{1}{\sqrt{2}} \qquad \ldots (1.5)$$

From Eqs. (1.4) and (1.5), we have

$$\theta = \frac{3\pi}{4}$$

Putting the values of r and θ in Eq. (1.1), we get

$$\frac{1+2i}{1-3i} = \frac{1}{\sqrt{2}}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right).$$

1.3.3 Properties of Modulus and Argument

Theorem 1

If z_1 and z_2 are two complex numbers, then

(i) $|z_1 z_2| = |z_1||z_2|$

(ii) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$, **provided $z_2 \neq 0$**

(iii) $|z_1 + z_2| \leq |z_1| + |z_2|$

(iv) $|z_1 + z_2| \geq |z_1| - |z_2|$

(v) $|z_1 - z_2| \geq |z_1| - |z_2|$

Proof

(i)
$$\begin{aligned}|z_1 z_2|^2 &= (z_1 z_2)\,\overline{(z_1 z_2)}\\ &= (z_1 z_2)(\bar{z}_1 \bar{z}_2)\\ &= (z_1 \bar{z}_1)(z_2 \bar{z}_2)\\ &= |z_1|^2 |z_2|^2\end{aligned}$$

Since the modulus of a complex number is always non-negative,

$\therefore \quad |z_1 z_2| = |z_1||z_2|$

(ii) Can be similarly proved.

(iii)
$$\begin{aligned}|z_1 + z_2|^2 &= (z_1 + z_2)\,\overline{(z_1 + z_2)}\\ &= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2)\\ &= z_1\bar{z}_1 + z_2\bar{z}_2 + z_1\bar{z}_2 + z_2\bar{z}_1\\ &= |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + z_2\bar{z}_1\\ &= |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \overline{(z_1\bar{z}_2)}\\ &= |z_1|^2 + |z_2|^2 + 2 \text{ real part of } (z_1\bar{z}_2)\\ &\leq |z_1|^2 + |z_2|^2 + 2|z_1\bar{z}_2|\end{aligned}$$

(Θ Real part of $z_1 z_2 \leq |z_1 z_2|$)

$$|z_1|^2 + |z_2|^2 + 2|z_1||\bar{z}_2| = |z_1|^2 + |z_2|^2 + 2|z_1||\bar{z}_2| \quad (\Theta\ |\bar{z}_2| = |z_2|)$$
$$= [|z_1| + |z_2|]^2$$

$\therefore \quad |z_1 + z_2| \leq |z_1| + |z_2|$

(iv)
$$\begin{aligned}|z_1 + z_2|^2 &= (z_1 + z_2)\,\overline{(z_1 + z_2)}\\ &= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2)\\ &= z_1\bar{z}_1 + z_2\bar{z}_2 + z_1\bar{z}_2 + z_2\bar{z}_1\\ &= |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}\,|z_1\bar{z}_2|\end{aligned}$$

$$\geq |z_1|^2 + |z_2|^2 - 2|z_1 z_2|$$

$$\geq [|z_1| - |z_2|]^2$$

$\therefore \quad |z_1 + z_2| \geq |z_1| - |z_2|$

(v) $|z_1| = |z_1 - z_2 + z_2|$

$$\leq |z_1 - z_2| + |z_2| \text{ by (iii)}$$

$\therefore \quad |z_1| - |z_2| \leq |z_1 - z_2|$

i.e. $\quad |z_1 - z_2| \geq |z_1| - |z_2|$

Theorem 2

Prove that

(i) The argument of the product of two complex numbers is the sum of their arguments.

(ii) The argument of the quotient of two complex numbers is the difference of their arguments.

Proof

(i) Let $\quad z_1 = x_1 + i y_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$

and $z_2 = x_2 + i y_2 = r_2 (\cos \theta_2 + i \sin \theta_2)$ be two complex numbers.

Then $\quad |z_1| = r_1,\ |z_2| = r_2$,

and argument $z_1 = \theta_1$, argument $z_2 = \theta_2$.

Now $\quad z_1 z_2 = [r_1 (\cos \theta_1 + i \sin \theta_1)]\,[r_2 (\cos \theta_2 + i \sin \theta_2)]$

$$= r_1 r_2 (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2)$$

$$= r_1 r_2 [\cos (\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2)]$$

$\therefore$ Argument $z_1 z_2 = \theta_1 + \theta_2 =$ argument $z_1 +$ argument z_2.

Cor.

$|z_1 z_2| = r_1 r_2 = |z_1|\,|z_2|$.

(ii) $$\frac{z_1}{z_2} = \frac{r_1 (\cos \theta_1 + i \sin \theta_1)}{r_2 (\cos \theta_2 + i \sin \theta_2)} = \frac{r_1}{r_2} \frac{(\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 - i \sin \theta_2)}{(\cos \theta_2 + i \sin \theta_2)(\cos \theta_2 - i \sin \theta_2)}$$

$$= \frac{r_1}{r_2} [\cos (\theta_1 - \theta_2) + i \sin (\theta_1 - \theta_2)]$$

Hence the result.

Cor.

$$\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|}.$$

SAQ 1

(a) Express the following in the form $a + i\,b$, where a and b are real numbers

(i) $\dfrac{(3+4i)\,(2+i)}{1+i}$

(ii) $\dfrac{(1+2i)^3}{(1+i)\,(2-i)}$

(b) Find the modulus and principal argument of

(i) $\dfrac{(1+i)^2}{1-i}$

(ii) $-\sqrt{3} - i$

(c) Put the following complex numbers into polar form $r\,(\cos\theta + i\sin\theta)$

(i) $\dfrac{2+6\sqrt{3}\,i}{5+\sqrt{3}\,i}$

(ii) $\dfrac{(2+5i)\,(-3+i)}{(1-2i)^2}$

1.4 EXPONENTIAL AND CIRCULAR FUNCTIONS OF COMPLEX NUMBERS

Definition 1

If $z = x + i\,y$, we define

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots \qquad \ldots (1)$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots \qquad \ldots (2)$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \ldots \qquad \ldots (3)$$

From Eqs. (2) and (3), we have

$$\cos z + i\sin z = \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \ldots\right) + i\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots\right)$$

$$= 1 + \frac{iz}{1!} + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \ldots$$

$$= e^{iz}$$

$$\therefore \qquad \cos z + i\sin z = e^{iz} \qquad \ldots (4)$$

Similarly $\qquad \cos z - i\sin z = e^{-iz} \qquad \ldots (5)$

Hence from Eqs. (4) and (5), we have

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad \ldots (6)$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2} \qquad \ldots (7)$$

1.4.1 De Moivres Theorem

If n is an integer, then $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

Proof

We know that $\cos\theta + i\sin\theta = e^{i\theta}$

$$\therefore \qquad (\cos\theta + i\sin\theta)^n = (e^{i\theta})^n = e^{in\theta}$$

$$= \cos n\theta + i\sin n\theta$$

Cor.

If n is a fraction, then $\cos n\theta + i\sin n\theta$ *is one of the values of* $(\cos\theta + i\sin\theta)^n$.

Example 1.5

Express $\dfrac{(\cos\theta + i\sin\theta)^8}{(\sin\theta + i\cos\theta)^4}$ in the form $x + iy$.

Solution

$$\frac{(\cos\theta + i\sin\theta)^8}{(\sin\theta + i\cos\theta)^4} = \frac{(\cos\theta + i\sin\theta)^8}{(i)^4\left[\cos\theta + \frac{1}{i}\sin\theta\right]^4}$$

$$= \frac{(\cos\theta + i\sin\theta)^8}{(\cos\theta - i\sin\theta)^4} = \frac{(\cos\theta + i\sin\theta)^8}{[(\cos\theta + i\sin\theta)^{-1}]^4}$$

$$= \frac{(\cos\theta + i\sin\theta)^8}{(\cos\theta + i\sin\theta)^{-4}} = (\cos\theta + i\sin\theta)^8\,(\cos\theta + i\sin\theta)^4$$

$$= (\cos\theta + i\sin\theta)^{12}$$

Example 1.6

Prove that $(1 + \cos\theta + i\sin\theta)^n + (1 + \cos\theta - i\sin\theta)^n = 2^{n+1}\cos^n\frac{\theta}{2}\cos\frac{n\theta}{2}$ where n is an integer.

Solution

$$1 + \cos\theta + i\sin\theta = 1 + 2\cos^2\frac{\theta}{2} - 1 + i\,.\,2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$= 2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$$

and $\qquad 1 + \cos\theta - i\sin\theta = 1 + 2\cos^2\frac{\theta}{2} - 1 - i\,.\,2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$$= 2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right)$$

$$= 2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)^{-1}$$

$\therefore$ $(1 + \cos\theta + i\sin\theta)^n + (1 + \cos\theta - i\sin\theta)^n$

$$= \left[2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)\right]^n + \left[2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)^{-1}\right]^n$$

$$= 2^n\cos^n\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)^n + 2^n\cos^n\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)^{-n}$$

$$= 2^n\cos^n\frac{\theta}{2}\left(\cos\frac{n\theta}{2} + i\sin\frac{n\theta}{2}\right) + 2^n\cos^n\frac{\theta}{2}\left(\cos\frac{n\theta}{2} - i\sin\frac{n\theta}{2}\right)$$

$$= 2^n\cos^n\frac{\theta}{2}\left(\cos\frac{n\theta}{2} + i\sin\frac{n\theta}{2} + \cos\frac{n\theta}{2} - i\sin\frac{n\theta}{2}\right)$$

$$= 2^n\cos^n\frac{\theta}{2}\,.\,2\cos\frac{n\theta}{2}$$

$$= 2^{n+1}\cos^n\frac{\theta}{2}\cos\frac{n\theta}{2}$$

Example 1.7

If n is a positive integer, prove that $(\sqrt{3} + i)^n + (\sqrt{3} - i)^n = 2^{n+1}\cos\frac{n\pi}{6}$.

Solution

Let $\sqrt{3} + i = r(\cos\theta + i\sin\theta)$

Then $r = \sqrt{3+1} = 2$, $\tan\theta = \frac{1}{\sqrt{3}} = \tan\frac{\pi}{6}$

$\therefore$ $\theta = \frac{\pi}{6}$

Hence $\sqrt{3} + i = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

and $\sqrt{3} - i = 2\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)$

Hence $(\sqrt{3} + i)^n + (\sqrt{3} - i)^n = \left[2\cos\left(\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right]^n + \left[2\cos\left(\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)\right]^n$

$$= 2^n\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^n + 2^n\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)^n$$

$$= 2^n\left(\cos\frac{n\pi}{6} + i\sin\frac{n\pi}{6}\right) + 2^n\left(\cos\frac{n\pi}{6} - i\sin\frac{n\pi}{6}\right)$$

$$= 2^n\left(\cos\frac{n\pi}{6} + i\sin\frac{n\pi}{6} + \cos\frac{n\pi}{6} - i\sin\frac{n\pi}{6}\right)$$

$$= 2^n\,.\,2\cos\frac{n\pi}{6} = 2^{n+1}\cos\frac{n\pi}{6}$$

Example 1.8

Find the different values of $(1+i)^{\frac{1}{3}}$.

Solution

$$1 + i = r\,(\cos\theta + i\sin\theta)$$

Then $\quad r = \sqrt{1+1} = \sqrt{2}$

and $\quad \tan\theta = \frac{1}{1} = 1,$ i.e. $\theta = \frac{\pi}{4}$

$$\therefore \quad 1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$(1+i)^{\frac{1}{3}} = \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^{\frac{1}{3}}$$

$$= \left[\sqrt{2}\cos\left(2n\pi + \frac{\pi}{4}\right) + i\sin\left(2n\pi + \frac{\pi}{4}\right)\right]^{\frac{1}{3}}$$

$$= (2)^{\frac{1}{6}}\left[\cos\frac{1}{3}\left(2n\pi + \frac{\pi}{4}\right) + i\sin\frac{1}{3}\left(2n\pi + \frac{\pi}{4}\right)\right]$$

Putting $n = 0, 1, 2$, the three different values of $(1+i)^{\frac{1}{3}}$ are

$$(2)^{\frac{1}{6}}\left[\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right], \quad (2)^{\frac{1}{6}}\left[\cos\frac{9\pi}{12} + i\sin\frac{9\pi}{12}\right]$$

$$(2)^{\frac{1}{6}}\left[\cos\frac{17\pi}{12} + i\sin\frac{17\pi}{12}\right]$$

Example 1.9

Use De Moivers theorem to solve the equation $x^3 + 1 = 0$.

Solution

$$x^3 + 1 = 0 \Rightarrow x^3 = -1 = \cos\pi + i\sin\pi$$

$$\therefore \quad x^3 = \cos(2n\pi + \pi) + i\sin(2n\pi + \pi)$$

i.e. $\quad x = [\cos(2n\pi + \pi) + i\sin(2n\pi + \pi)]^{\frac{1}{3}}$

$$= \cos(2n+1)\frac{\pi}{3} + i\sin(2n+1)\frac{\pi}{3}$$

$n = 0, 1, 2$

i.e. the roots are

$$\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right), (\cos\pi + i\sin\pi), \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}$$

i.e. $\quad \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}, -1, \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}$

SAQ 2

(a) Find the value of

(i) $(1+i)^{\frac{1}{5}}$

(ii) $\left(\frac{1}{2}+\frac{\sqrt{-3}}{2}\right)^{\frac{3}{4}}$

(b) Use De Moivres theorem to solve following equations

(i) $x^4+1=0$

(ii) $x^{10}-x^5+1=0$ (**Hint :** Put $x^5=y$)

(iii) $x^6-x^5+x^4-x^3+x^2-x+1=0$

(**Hint :** Multiply the equation by $x+1$)

1.5 MATRICES

We introduce matrices and various operations involving matrices in this section. The notions of rank and inverse of a matrix are also introduced. The concepts of determinants have also been highlighted.

A matrix is a set of real or complex numbers (or elements) arranged in rows and columns to form a rectangular array. Let us consider the system of simultaneous equations,

$$x+3y+4z+5t=0$$

$$4x+9y+7z+6t=0$$

$$3x+6y+11z+2t=0$$

Now we arrange the coefficients of x, y, z and t of the above equations and enclose them within brackets and we get

$$A=\begin{bmatrix}1 & 3 & 4 & 5\\ 4 & 9 & 7 & 6\\ 3 & 6 & 11 & 2\end{bmatrix}$$

The above system of numbers arranged in a rectangular array in rows and columns and bounded by the brackets is called a matrix.

A matrix having m rows and n columns is called as $m \times n$ (i.e. 'm by n') matrix and is referred to as having order $m \times n$.

For example,

$$\begin{bmatrix}1 & 3 & 4 & 5\\ 4 & 9 & 7 & 6\\ 3 & 6 & 11 & 2\end{bmatrix}$$

is a 3×4 matrix, i.e. by '3 by 4' matrix.

It has got 3 rows and 4 columns and in all $3 \times 4 = 12$ elements.

Now, we have come to know that a matrix is simply an array (rectangular) of numbers :

A set of *mn* elements arranged in the form of an ordered set of *m* rows, each row consisting of an ordered set of *n* elements is called an $m \times n$ matrix.

We write it A_{ij}.

Thus,

$$A_{ij} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & .... & .... & a_{1j} \\ a_{21} & a_{22} & a_{23} & .... & .... & a_{2j} \\ a_{31} & a_{32} & a_{33} & .... & .... & a_{3j} \\ .... & .... & .... & .... & .... & .... \\ .... & .... & .... & .... & .... & .... \\ a_{i1} & a_{i2} & a_{i3} & .... & .... & a_{ij} \end{bmatrix}$$

The numbers a_{ij} are called the elements or entries of the matrix.

We also express the matrix A by its general element (a_{ij}). Thus $A = [a_{ij}]$. For the element a_{ij}, first suffix, *i* stands for the ith row and the second suffix *j* denotes the jth column.

SAQ 3

Fill in the blanks

(i) The matrix $\begin{bmatrix} 6 & 2 \\ 0 & 12 \\ 7 & 6 \end{bmatrix}$ is of order . . . .

(ii) The matrix $\begin{bmatrix} 2 & 6 & 9 & 12 \\ 6 & 12 & 5 & 2 \end{bmatrix}$ is of order . . . .

1.5.1 Different Types of Matrices

The following are the different types of matrix.

Row Matrix

A matrix having one row and any number of column is called row matrix. This is the type $1 \times n$ matrix. For example, [6 9], [1 2 3], [*a b c d x*] are all row matrices.

Column Matrix

A column matrix consists of 1 column only. For example

$$\begin{bmatrix} 6 \\ 2 \\ 9 \end{bmatrix},$$

is a column matrix of order 3×1.

Null Matrix

Any matrix in which all the elements are zero is called a zero matrix or null matrix.

For example,

$$[0],\ \begin{bmatrix}0 & 0\end{bmatrix},\ \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

are all null matrices.

Square Matrix

A matrix in which number of column equals number of rows is called a square matrix.

For example,

$$\begin{bmatrix}1 & 3 & 5\\ 6 & 7 & 9\\ 2 & 5 & 2\end{bmatrix},\ \begin{bmatrix}2 & 5\\ 9 & 2\end{bmatrix}$$

are all square matrix of order (3 × 3) and (2 × 2), respectively.

Diagonal Matrix

A square matrix is called a diagonal matrix if all its non-diagonal elements are zero.

For example,

$$\begin{bmatrix}2 & 0 & 0\\ 0 & 9 & 0\\ 0 & 0 & 7\end{bmatrix}$$

is a diagonal matrix.

Unit or Identity Matrix

A square matrix with 1's on the main diagonal and 0's elsewhere is called a unit or identity matrix.

For example,

$$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\ \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$ (denoted by $I_{2\times 2}$ and $I_{3\times 3}$ respectively).

are unit matrices of order (2 × 2) and (3 × 3), respectively.

Symmetric Matrix

A square matrix $[a_{ij}]$ is symmetric if $a_{ij} = a_{ji}$. (for all $i, j = 1, 2, \ldots, n$).

For example,

$$\begin{bmatrix}1 & 2 & 9\\ 2 & 7 & 6\\ 9 & 6 & 5\end{bmatrix}$$

Note that it is symmetrical about the leading diagonal. (A^T is defined later in Section 1.5.2).

Skew Symmetric Matrix

A square matrix $[a_{ij}]$ is skew-symmetric matrix, if $a_{ij} = -a_{ji}$.

(for all $i, j = 1, 2, \ldots, n$).

For example,

$$\begin{bmatrix} 0 & 2 & 9 \\ -2 & 0 & 6 \\ -9 & -6 & 0 \end{bmatrix}$$

is an example of a skew-symmetric matrix.

Triangular Matrix

A square matrix all of whose elements below the leading diagonal are zero, is called an upper triangular matrix. A square matrix, all of whose elements above the leading diagonal are zero, is called a lower triangular matrix.

$$\begin{bmatrix} 1 & 6 & 2 \\ 0 & 9 & 5 \\ 0 & 0 & 2 \end{bmatrix}$$

is an example of upper triangular matrix, whereas

$$\begin{bmatrix} 1 & 0 & 0 \\ 6 & 2 & 0 \\ 2 & 9 & 6 \end{bmatrix}$$

is an example of lower triangular matrix.

Equal Matrices

If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$, then $A = B$ if $a_{ij} = b_{ij}$, $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

Two matrices are said to be equal if corresponding elements throughout are equal. Therefore, the matrices must also be of the same order.

So if, $$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \\ 1 & 0 & 2 \end{bmatrix}$$

then, $a_{11} = 4,\ a_{12} = 5,\ a_{13} = 6,\ a_{21} = 7,$

$a_{22} = 8,\ a_{23} = 9,\ a_{31} = 1,\ a_{32} = 0,$ and $a_{33} = 2$

Therefore, if $[a_{ij}] = [x_{ij}]$,

Then $a_{ij} = x_{ij}$

for all values of i and j.

1.5.2 Transpose of a Matrix

If the rows and columns of a matrix are interchanged, i.e. the first row becomes the first column, the second row becomes the second column, the third row becomes the third column etc., then the new matrix so formed is known as the transpose of the original matrix. If A is the original matrix, its transpose is denoted by A' or A^T.

If $$A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \\ 9 & 2 \end{bmatrix},$$

Then $$A^T = \begin{bmatrix} 2 & 4 & 9 \\ 3 & 6 & 2 \end{bmatrix}$$

Note that for a symmetric matrix $A = A^T$, whereas for a skew-symmetric matrix $A = -A^T$, A being a square matrix.

1.5.3 Orthogonal Matrix

A square matrix A is called an orthogonal matrix if the product of the matrix A and its transpose A^T is an identity matrix. So, for orthogonal matrix

$$AA^T = I$$

Determinant of a Square Matrix

The determinant of a square matrix is the determinant having the same elements as those of the matrix. For example, the determinants of

$$\begin{bmatrix} 2 & 3 & 4 \\ 1 & 5 & 7 \\ 9 & 2 & 5 \end{bmatrix}$$

is $$\begin{vmatrix} 2 & 3 & 4 \\ 1 & 5 & 7 \\ 9 & 2 & 5 \end{vmatrix}$$

and the value of the determinant is

$$2\,(5 \times 5 - 2 \times 7) - 3\,(1 \times 5 - 7 \times 9) + 4\,(1 \times 2 - 5 \times 9)$$

$$= 2\,(11) - 3\,(-58) + 4\,(-43)$$

$$= 22 + 174 - 172 = 24$$

Note that the transpose of the matrix is

$$\begin{bmatrix} 2 & 1 & 9 \\ 3 & 5 & 2 \\ 4 & 7 & 5 \end{bmatrix}$$

and the determinants of the transpose is

$$\begin{vmatrix} 2 & 1 & 9 \\ 3 & 5 & 2 \\ 4 & 7 & 5 \end{vmatrix},$$

the value of which is

$$2 \times (5 \times 5 - 2 \times 7) - 1\,(3 \times 5 - 2 \times 4) + 9\,(3 \times 7 - 5 \times 4)$$

$$= 2 \times (11) - (7) + 9\,(1) = 24$$

1.5.4 Addition and Subtraction of Two Matrices

If A and B be two matrices of the same order, then their sum $A + B$ and A – B is defined as the matrix each element of which is the sum and the difference of the corresponding elements in A and B respectively.

To be added or subtracted, two matrices must be of the same order. Also order of the matrix obtained on addition of two matrices is the same as the order of matrices being added.

For example,

$$\begin{bmatrix} 4 & 9 & 6 \\ 6 & 2 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 5 & 6 \\ 4 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 7 & 14 & 12 \\ 10 & 6 & 8 \end{bmatrix}$$

and

$$\begin{bmatrix} 4 & 9 & 6 \\ 6 & 2 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 5 & 6 \\ 4 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 0 \\ 2 & -2 & 2 \end{bmatrix}$$

1.5.5 Multiplication of a Matrix by a Scalar

To multiply a matrix by a single number (i.e. scalar), each individual element of the matrix is multiplied by that scalar,

For example, if

$$A = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 2 & 5 \\ 9 & 3 & 0 \end{bmatrix}$$

then

$$5A = \begin{bmatrix} 15 & 10 & 30 \\ 5 & 10 & 25 \\ 45 & 15 & 0 \end{bmatrix}$$

Example 1.10

If $A = \begin{bmatrix} 0 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 6 & 3 \\ 1 & 4 & 5 \end{bmatrix}$, find $3A - B$.

Solution

We have

$$3A = 3\begin{bmatrix} 0 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 6 & 9 \\ 6 & 3 & 12 \end{bmatrix}$$

Now,

$$3A - B = \begin{bmatrix} 0 & 6 & 9 \\ 6 & 3 & 12 \end{bmatrix} - \begin{bmatrix} 7 & 6 & 3 \\ 1 & 4 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & 0 & 6 \\ 5 & -1 & 7 \end{bmatrix}$$

Example 1.11

Find x, y, z and w given that

$$3\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 6 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 4 & x+y \\ z+w & 3 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 3x & 3y \\ 3z & 3w \end{bmatrix} = \begin{bmatrix} x+4 & x+y+6 \\ z+w-1 & 2w+3 \end{bmatrix}$$

Equating corresponding elements

(i)

$$3x = x + 4 \qquad \ldots$$

$$3y = x + y + 6 \qquad \ldots \text{(ii)}$$

$$3z = z + w - 1 \qquad \ldots \text{(iii)}$$

$$3w = 2w + 3 \qquad \ldots$$

(iv)

From Eq. (i)

$$3x = x + 4$$

$$\Rightarrow x = 2$$

From Eq. (ii)

$$3y = x + y + 6$$

$$\Rightarrow 2y = x + 6$$

$$\Rightarrow y = 4$$

From Eq. (iv)

$$3w = 2w + 3$$

$$\Rightarrow w = 3$$

From Eq. (iii)

$$3z = z + w - 1$$

$$\Rightarrow 2z = w - 1$$

$$\Rightarrow z = 1$$

So,

$$x = 2, \quad y = 4, \quad z = 1, \quad w = 3.$$

SAQ 4

(a) Find x and y such that

$$\begin{bmatrix} -2 & -1 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 8 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(b) Prove that

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

is an orthogonal matrix.

(c) If the given matrix A is orthogonal, find the values of x, y and z

$$A = \frac{1}{3}\begin{bmatrix} x & 2 & 2 \\ 2 & 1 & y \\ 2 & z & 1 \end{bmatrix}$$

(d) Verify that the following matrix is orthogonal

$$A = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

(e) If

$$A = \begin{bmatrix} 2 & -3 \\ 0 & 4 \end{bmatrix},$$

represent it as $A = B + C$, where B is a symmetric matrix and C is a skew-symmetric matrix.

1.5.6 Multiplication of Two Matrices

Two matrices A and B can be multiplied only if the number of columns of matrix A is equal to the number of rows in matrix B.

The $(i, j)^{th}$ element of product matrix AB is the sum of products of the elements of ith row of A multiplied by the corresponding elements in the jth column of B.

If order of matrix A is $m \times n$ and the order of matrix B is $n \times p$, then the order of product matrix AB is $m \times p$.

Let us consider two matrices A and B, where

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

Then the product of A and B is denoted by AB and is given by

$$AB = \begin{bmatrix} a_{11}\,b_{11} + a_{12}\,b_{21} & a_{11}\,b_{12} + a_{12}\,b_{22} & a_{11}\,b_{13} + a_{12}\,b_{23} \\ a_{21}\,b_{11} + a_{22}\,b_{21} & a_{21}\,b_{12} + a_{22}\,b_{22} & a_{21}\,b_{13} + a_{22}\,b_{23} \end{bmatrix}$$

The entries in the product AB are determined in the following way :

(i) Each element of the top row of A is multiplied by the corresponding element in the first column of B. The sum of these products is the first entry of the product matrix.

(ii) We multiply the elements of the second row of A with the corresponding elements of second column of B and the sum of these products is the required entry for $(2, 2)^{th}$ element of product matrix, and so on.

Two matrices can be multiplied together **only when the number of column in the first is equal to the number of rows in the second**.

Thus, if

$$A = \begin{bmatrix} 2 & 7 \\ 1 & 2 \\ 3 & 1 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 2 & 3 & 6 \\ 9 & 2 & 5 \end{bmatrix}$$

then
$$AB = \begin{bmatrix} 2 & 7 \\ 1 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 6 \\ 9 & 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times 2+7\times 9 & 2\times 3+7\times 2 & 2\times 6+7\times 5 \\ 1\times 2+2\times 9 & 1\times 3+2\times 2 & 1\times 6+2\times 5 \\ 3\times 2+1\times 9 & 3\times 3+1\times 2 & 3\times 6+1\times 5 \end{bmatrix}$$

$$= \begin{bmatrix} 67 & 20 & 47 \\ 20 & 7 & 16 \\ 15 & 11 & 23 \end{bmatrix}$$

In general AB ≠ BA. For example, $A = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$,

$AB = \begin{bmatrix} 9 & 8 \\ 1 & 2 \end{bmatrix}$ and $BA = \begin{bmatrix} 5 & 2 \\ 10 & 6 \end{bmatrix}$. Also $AI = IA = A$ where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Example 1.12

If,
$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$$

and
$$B = \begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 2 & -1 \end{bmatrix}$$

Obtain the product AB and explain why BA is not defined. Also show that $(AB)^T = B^T A^T$.

Solution

The number of columns of A is 3, and the number of rows in B is also 3, therefore the product AB is defined.

$$AB = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0\times 1+1\times(-1)+2\times 2 & 0\times(-2)+1\times 0+2\times(-1) \\ 1\times 1+2\times(-1)+3\times 2 & 1\times(-2)+2\times 0+3\times(-1) \\ 2\times 1+3\times(-1)+4\times 2 & 2\times(-2)+3\times 0+4\times(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 0-1+4 & 0+0-2 \\ 1-2+6 & -2+0-3 \\ 2-3+8 & -4+0-4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -2 \\ 5 & -5 \\ 7 & -8 \end{bmatrix}$$

Since the number of columns of B is 2, and the number of rows of A is 3, therefore, BA is not defined.

Example 1.13

Evaluate

$$A^2 - 3A + 9I \ ,$$

if I is the unit matrix of order 3 and

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$$

Solution

$$A^2 = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1-4-9 & -2-6+3 & 3+2+6 \\ 2+6+3 & -4+9-1 & 6-3-2 \\ -3+2-6 & 6+3+2 & -9-1+4 \end{bmatrix}$$

$$= \begin{bmatrix} -12 & -5 & 11 \\ 11 & 4 & 1 \\ -7 & 11 & -6 \end{bmatrix}$$

$$3A = 3\begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -6 & 9 \\ 6 & 9 & -3 \\ -9 & 3 & 6 \end{bmatrix}$$

$$9I = 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

By adding, we get

$$A^2 - 3A + 9I = \begin{bmatrix} -12 & -5 & 11 \\ 11 & 4 & 1 \\ -7 & 11 & -6 \end{bmatrix} - \begin{bmatrix} 3 & -6 & 9 \\ 6 & 9 & -3 \\ -9 & 3 & 6 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} -12-3+9 & -5+6+0 & 11-9+0 \\ 11-6+0 & 4-9+9 & 1+3+0 \\ -7+9+0 & 11-3+0 & -6-6+9 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & 1 & 2 \\ 5 & 4 & 4 \\ 2 & 8 & -3 \end{bmatrix}$$

SAQ 5

(a) Let $A(\theta)$ denotes the matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix};$$

Prove that

$$A[\theta_1].A[\theta_2] = A(\theta_1 + \theta_2)$$

(b) Let $f(x) = x^2 - 5x + 0$, find $f(A)$ if

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$$

(c) Express $(A + B)^2$ as a matrix, where

$$A = \begin{bmatrix} 2 & 3 \\ 6 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 7 \\ 4 & 0 \end{bmatrix}$$

and verify that

$$(A + B)^2 = A^2 + AB + BA + B^2$$

(d) Prove that the product of a matrix with its transpose is a symmetrical matrix.

(e) Prove that the product of two matrices

$$\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix} \text{ and } \begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$$

is a null matrix when θ and ϕ differ by an odd multiple of $\frac{\pi}{2}$.

(f) Matrix A has x rows and $x + 5$ columns. Matrix B has y rows and $11 - y$ columns. Both AB and BA exists. Find x and y.

1.6 DETERMINANTS

The expression

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

is called a determinants of the second order and stands for '$a_1 b_2 - a_2 b_1$'.

Similarly,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

is called a determinants of third order.

Minors and Cofactors

The minor of an element of a determinant is the determinant obtained by deleting the row and column which intersect in that element.

For example, in

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix},$$

minor of $a_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$ and minor of $a_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$.

The cofactor of any element in a determinant is its minor with the proper sign. The sign of an element in the ith row and j^{th} column's $(-1)^{i+j}$. The co-factor of an element is usually denoted by the corresponding capital letter.

Thus the cofactor of $a_{32} = (-1)^{3+2} \times$ minor of a_{32}.

Thus, $$C_{32} = -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

Similarly, the cofactor of $a_{23} = (-1)^{2+3} \times$ minor of a_{23}.

Thus, $$C_{23} = -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

Please note that

$$C_{ij} = (-1)^{i+j} \times M_{ij}$$

where C_{ij} is the cofactor of a_{ij}, and

M_{ij} is the minor of a_{ij}.

Laplace's Expansion

A determinant can be expanded in term of any row (or column) as follows.

Multiply each element of the row (or column) in terms of which we intend expansion with the determinant by its co-factor and then add up all these terms.

Let us try to find out the determinant of

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$\therefore$ Expanding by R_1 (i.e. 1st row)

$$\Delta = a_1\, C_{11} + b_1\, C_{12} + c_1\, C_{13} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

(Here C_{11}, C_{22} and C_{31} are the cofactors of a_1, b_1 and c_1 respectively.)

$$= a_1\,(b_2\, c_3 - b_3\, c_2) - b_1\,(a_2\, c_3 - a_3\, c_2) + c_1\,(a_2\, b_3 - a_3\, b_2).$$

Similarly, expanding by C_2 (i.e. 2nd column)

$$\Delta = b_1\, C_{12} + b_2\, C_{22} + b_3\, C_{32} = -\, b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

(Here C_{12}, C_{22} and C_{32} are the cofactors of b_1, b_2 and b_3 respectively.)

$$= -b_1 (a_2 c_3 - a_3 c_2) + b_2 (a_1 c_3 - a_3 c_1) - b_3 (a_1 c_2 - a_2 c_1)$$

and expanding by R_3 (i.e. 3rd row)

$$\Delta = a_3 C_{31} + b_3 C_{32} + c_3 C_{33}$$

(Here C_{31}, C_{32} and C_{33} are the cofactors of a_3, b_3 and c_3 respectively.)

$\therefore$ Thus Δ is the sum of the products of the elements of any row (or column) by the corresponding co-factors.

Such an expansion of a determinant is called Laplace's Expansion.

Properties of Determinants

Expanding a determinant in which the elements are large number can be a very tedious task. It is possible, however, by knowing something of the properties of determinants, to simplify the working. So here are some of the main properties.

(i) The value of a determinant remains unchanged if rows are changed to column and column to rows.

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

(ii) If two rows (or two columns) are interchanged, the sign of the determinant is changed

$$\begin{vmatrix} a_2 & b_2 \\ a_1 & b_1 \end{vmatrix} = -\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

(iii) If two rows (or two columns) are identical, the value of the determinant is zero.

$$\begin{vmatrix} a_1 & a_1 \\ a_2 & a_2 \end{vmatrix} = 0$$

(iv) If the elements of any one row (or column) are all multiplied by a common factor, the determinant is multiplied by that factors

$$\begin{vmatrix} k\,a_1 & k\,b_1 \\ a_2 & b_2 \end{vmatrix} = k\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

(v) If the elements of any row (or column) are increased (or decreased) by equal multiples of the corresponding elements of any other row (or column), the value of the determinant is unchanged.

$$\begin{vmatrix} a_1 + k\,b_1 & b_1 \\ a_2 + k\,b_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

Theorem

If A and B are $n \times n$ matrices, then det (AB) = det (A) det (B).

Cramer's Rule

If $AX = B$ is a system of n linear equations in n unknowns such that det (A) $\neq 0$, then the system has a unique solution.

$$(A = [a_{ij}]_{n \times n})$$

This solution is given by

$$X_j = \frac{\det A_j}{\det A}, \; j = 1, 2, \ldots, n$$

where A_j is the matrix obtained from A by replacing the entries in the jth column of A by this entries in the matrix $B = [b_1, b_2, \ldots, b_n]^T$.

Let us consider the following simultaneous equation

$$a_{11}\, x + a_{12}\, y + a_{13}\, z = b_1$$

$$a_{21}\, x + a_{22}\, y + a_{23}\, z = b_2$$

$$a_{31}\, x + a_{32}\, y + a_{33}\, z = b_3$$

Then by Cramer's rule

$$x = \frac{\Delta_x}{\Delta}, \; y = \frac{\Delta_y}{\Delta}, \; z = \frac{\Delta_z}{\Delta}$$

where
$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\Delta_x = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix},$$

$$\Delta_y = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}$$

and
$$\Delta_z = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

Example 1.14

Solve the following system by Cramer's rule :

$$2\,x - y + 3\,z = 2$$

$$x + 3\,y - z = 11$$

$$2\,x - 2\,y + 5\,z = 3$$

Solution

By Cramer's rule, we get

$$x = \frac{\Delta_x}{\Delta}, \; y = \frac{\Delta_y}{\Delta}, \; z = \frac{\Delta_z}{\Delta}$$

$$\Delta = \begin{vmatrix} 2 & -1 & 3 \\ 1 & 3 & -1 \\ 2 & -2 & 5 \end{vmatrix}$$

$$= 2\,(15 - 2) - 1\,(-2 - 5) + 3\,(-2 - 6)$$

$$= 9$$

$$\Delta_x = \begin{vmatrix} 2 & -1 & 3 \\ 11 & 3 & -1 \\ 3 & -2 & 5 \end{vmatrix}$$

$$= 2\,(15 - 2) - 1\,(-3 - 55) + 3\,(-22 - 9)$$

$$= 26 + 58 - 93$$

$$= -9$$

$$\Delta_y = \begin{vmatrix} 2 & 2 & 3 \\ 1 & 11 & -1 \\ 2 & 3 & 5 \end{vmatrix}$$

$$= 2\,(55 + 3) - 2\,(5 + 2) + 3\,(3 - 21)$$

$$= 116 - 14 - 57$$

$$= 45$$

$$\Delta_z = \begin{vmatrix} 2 & -1 & 2 \\ 1 & 3 & 11 \\ 2 & -2 & 3 \end{vmatrix}$$

$$= 2\,(9 + 22) + 1\,(3 - 22) + 2\,(-2 - 6)$$

$$= 62 - 19 - 16 = 27$$

$$x = \frac{\Delta_x}{\Delta} = \frac{-9}{9} = -1$$

$$y = \frac{\Delta_y}{\Delta} = \frac{45}{9} = 5$$

$$z = \frac{\Delta_z}{\Delta} = \frac{27}{9} = 3$$

Therefore, $x = -1,\ y = 5,\ z = 3$

SAQ 6

(a) Establish the following identity using the properties of the determinant.

$$\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ b+c & c+a & a+b \end{vmatrix} = (a - b)\,(b - c)\,(c - a)\,(a + b + c)$$

(b) Evaluate

$$\begin{vmatrix} -1 & 0 & 0 & a \\ 0 & -1 & 0 & b \\ 0 & 0 & -1 & c \\ x & y & z & -1 \end{vmatrix}$$

(c) Find the value of the determinant

$$\begin{vmatrix} x & x + 4\,y & 2\,y \\ 7\,y & 13\,y & 3\,y \\ 3\,z & 3\,z + 16\,x & 8\,x \end{vmatrix}$$

(d) Find the value of the determinant of the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 2 \\ -1 & 2 & 3 \end{bmatrix}$$

and hence using Cramer's rule or otherwise, solve the following system of equation.

$$x + 2y - z = 2$$

$$2x + 3y + 2z = 7$$

$$-x + 2y + 3z = 4$$

(e) If $a + b + c = 0$, solve

$$\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0$$

(f) If

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

choose α and β so that $(\alpha I + \beta A)^2 = A$.

1.7 ADJOINT OF A SQUARE MATRIX

Definition

If A is a square matrix, then the minor of entry a_{ij} is denoted by M_{ij} and is defined to be the determinant M_{ij} of the submatrix that remains after the ith row and j^{th} column are deleted from A. The number $(-1)^{i+j} M_{ij}$ is denoted by C_{ij} and is called the cofactor of entry a_{ij}.

The determinant of the square matrix

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

is

$$|A| = \Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

The matrix formed by the co-factors of the elements in A is matrix of co-factors from A

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

where

$$C_{11} = \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} = b_2\, c_3 - b_3\, c_2 \ , \ C_{12} = -\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} = -\, b_1\, c_3 + b_3\, c_1$$

$$C_{13} = \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} = b_1\, c_2 - b_2\, c_1 \ , \ C_{21} = -\begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} = -\, a_2\, c_3 + a_3\, c_2$$

$$C_{22} = \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} = a_1\, c_3 - a_3\, c_1 \ , \ C_{23} = -\begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix} = -\, a_1\, c_2 + a_2\, c_1$$

$$C_{31} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} = a_2\, b_3 - a_3\, b_2 \ , \ C_{32} = -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} = -\, a_1\, b_3 + a_3\, b_1$$

$$C_{33} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1\, b_2 - a_2\, b_1$$

Then the transpose of the matrix of co-factors

$$\text{Adjoint of } A = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

is called the adjoint of the matrix A and is written as Adj A.

1.8 INVERSE OF A SQUARE MATRIX

The adjoint of a square matrix is important, since it enables us to find the inverse of the matrix. If each element of the adjoint of A is divided by the value of the determinant of A, i.e. $|A|$ (provided $|A| \neq 0$, the matrix A is said to be non-singular), the resulting matrix is called the inverse of A and is denoted by A^{-1}, i.e. $A^{-1} = \dfrac{1}{|A|}\,\text{Adj}\, A$.

Let us take the matrix

$$A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$

Then $|A| = 3\,(-3+4) + 3\,(2-0) + 4\,(-2)$ (where $|A|$ is the determinant of the square matrix A)

$$= 3 + 6 - 8$$

$$= 1$$

The co-factors of elements of various rows of $|A|$ are

$$(-3+4) \quad (-2-0) \quad (-2-0) \quad \text{i.e.} \quad 1 \ \ -2 \ \ -2$$

$$(3-4) \quad (3-0) \quad (3-0) \quad \text{i.e.} \quad -1 \ \ 3 \ \ 3$$

$$(-12+12) \quad (-12+8) \quad (-9+6) \quad \text{i.e.} \quad 0 \ \ -4 \ \ -3$$

Therefore, the matrix formed by the co-factors of $|A|$ is

$$\begin{bmatrix} 1 & -2 & -2 \\ -1 & 3 & 3 \\ 0 & -4 & -3 \end{bmatrix}$$

$$\text{Adj. } A = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{ Adj. } A = \frac{1}{1}\begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

So, $$A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

It can be shown that $AA^{-1} = A^{-1}A = I_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Solutions of Simultaneous Linear Equations

Consider the following set of Linear equations.

$$\begin{array}{l} a_{11}\, x_1 + a_{12}\, x_2 + a_{13}\, x_3 + \ldots + a_{1n}\, x_n = b_1 \\ a_{21}\, x_1 + a_{22}\, x_2 + a_{23}\, x_3 + \ldots + a_{2n}\, x_n = b_2 \\ \ldots\ldots \\ \ldots\ldots \\ \ldots\ldots \\ a_{n1}\, x_1 + a_{n2}\, x_2 + a_{n3}\, x_3 + \ldots + a_{nn}\, x_n = b_n \end{array}$$

From the knowledge of matrix multipliction, this can be written in matrix form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & a_{n3} & \ldots & \ldots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \ldots \\ \ldots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \ldots \\ \ldots \\ b_n \end{bmatrix}$$

i.e., $AX = B$

where $$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & a_{n3} & \ldots & \ldots & a_{nn} \end{bmatrix}, \; X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \ldots \\ \ldots \\ x_n \end{bmatrix}, \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ b_2 \\ \ldots \\ \ldots \\ b_n \end{bmatrix}$$

If we multiply both sides of the matrix equation by the inverse of A, we have

$$A^{-1}AX = A^{-1}B$$

But as we know,

$$A^{-1} . A = I = A . A^{-1}$$

$$\therefore \quad IX = A^{-1} . B$$

i.e., $$X = A^{-1} B$$

Therefore, **if we form the inverse of the matrix of coefficients and pre-multiply right hand matrix by it, we shall determine the matrix of the solution, *X***.

Example 1.15

Solve the system of equation

$$x + y + z = 6$$

$$x - y + z = 2$$

$$2x + y - z = 1$$

Solution

The system of equation can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$$

We write it as $A\,X = B$

where $$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$$

$$\therefore \quad X = A^{-1} B$$

Now we should find A^{-1}.

Here $$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix} = (1-1) - 1\,(-1-2) + 1\,(1+2)$$

$$= 0 + 3 + 3$$

$$= 6 \neq 0$$

$\therefore \quad A^{-1}$ exists.

Now, $$C_{11} = \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} = 0, \; C_{12} = -\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = 3, \; C_{13} = \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = 3$$

$$C_{21} = -\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = 2, \; C_{22} = \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -3, \; C_{23} = -\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 1$$

$$C_{31} = \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2, \; C_{32} = -\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0, \; C_{33} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$$

$$\text{Adj. } A = \begin{bmatrix} 0 & 2 & 2 \\ 3 & -3 & 0 \\ 3 & 1 & -2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{6}\begin{bmatrix} 0 & 2 & 2 \\ 3 & -3 & 0 \\ 3 & 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{6} & -\frac{1}{3} \end{bmatrix}$$

$$\therefore \qquad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{6} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\therefore \quad x = 1,\ y = 2,\ z = 3$$

Example 1.16

Solve the system of equation

$$2x - y + 3z = 2$$

$$x + 3y - z = 11$$

$$2x - 2y + 5z = 3$$

Solution

The system of equation can be written as

$$\begin{bmatrix} 2 & -1 & 3 \\ 1 & 3 & -1 \\ 2 & -2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 11 \\ 3 \end{bmatrix}$$

We write it as $AX = B$,

where $\quad A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 3 & -1 \\ 2 & -2 & 5 \end{bmatrix}, \ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ 11 \\ 3 \end{bmatrix}$

$$\therefore \quad X = A^{-1} B$$

Now we shall find A^{-1}

Here $\begin{vmatrix} 2 & -1 & 3 \\ 1 & 3 & -1 \\ 2 & -2 & 5 \end{vmatrix} = 2\,(15-2) - (-1)\,(5+2) + 3\,(-2-6) = 9$

$$C_{11} = \begin{vmatrix} 3 & -1 \\ -2 & 5 \end{vmatrix} = 13, \quad C_{12} = -\begin{vmatrix} 1 & -1 \\ 2 & 5 \end{vmatrix} = -7, \; C_{13} = \begin{vmatrix} 1 & 3 \\ 2 & -2 \end{vmatrix} = -8$$

$$C_{21} = -\begin{vmatrix} -1 & 3 \\ -2 & 5 \end{vmatrix} = -1, \quad C_{22} = \begin{vmatrix} 2 & 3 \\ 2 & 5 \end{vmatrix} = 4, \quad C_{23} = -\begin{vmatrix} 2 & -1 \\ 2 & -2 \end{vmatrix} = 2$$

$$C_{31} = \begin{vmatrix} -1 & 3 \\ 3 & -1 \end{vmatrix} = -8, \quad C_{32} = -\begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} = 5, \quad C_{33} = \begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} = 7$$

$$\text{Adj. } A = \begin{bmatrix} 13 & -1 & -8 \\ -7 & 4 & 5 \\ -8 & 2 & 7 \end{bmatrix}$$

$$A^{-1} = \frac{1}{9}\begin{bmatrix} 13 & -1 & -8 \\ -7 & 4 & 5 \\ -8 & 2 & 7 \end{bmatrix}$$

$$X = A^{-1}\,B = \frac{1}{9}\begin{bmatrix} 13 & -1 & -8 \\ -7 & 4 & 5 \\ -8 & 2 & 7 \end{bmatrix} . \begin{bmatrix} 2 \\ 11 \\ 3 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} -9 \\ 45 \\ 27 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ 3 \end{bmatrix}$$

$\therefore \; x = -1, \; y = 5, \; z = 3.$

SAQ 7

(a) If $A = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 4 & 3 \\ 3 & 7 & 6 \end{bmatrix}$,

find A^{-1}.

(b) Solve the following system of equation by matrix method

$$3x + y + 2z = 3$$

$$2x - 3y - z = -3$$

$$x + 2y + z = 4$$

(c) If $A = \begin{bmatrix} 1 & -\tan x & 0 \\ \tan x & 1 & 0 \\ 0 & 0 & \sec x \end{bmatrix}$

show that $A^{-1} = \cos x \begin{bmatrix} \cos x & \sin x & 0 \\ -\sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(d) Show that

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & -\tan\frac{\theta}{2} \\ \tan\frac{\theta}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & \tan\frac{\theta}{2} \\ -\tan\frac{\theta}{2} & 1 \end{bmatrix}^{-1}$$

(e) Find the adjoint and inverse of the matrix

$$A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(f) How should the coefficient a, b and c be chosen so that the system

$$ax + by - 3z = -3$$

$$-2x - by + cz = -1$$

$$ax + 3y - cz = -3$$

has the solution $x = 1$, $y = -1$, and $z = 2$?

(g) Solve the following system of equation by matrix method

$$x + y + z = 6$$

$$x - 2y + 3z = 14$$

$$x + 4y + 9z = 36$$

1.9 RANK OF MATRIX

The rank of a matrix is said to be γ if

(i) It has at least one non-zero minor of order γ.

(ii) Every minor of A of order higher than γ is zero.

As a consequence of condition (ii), every minor of order greater than γ will be zero. In short, we say that the rank of a matrix is the largest order of a non-zero minor of the matrix. Clearly the rank of every n-rowed non-singular matrix A ($|A| \neq 0$, i.e. $\det(A) \neq 0$) is n.

Example 1.17

Find the rank of each of the following

(a) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 3 \\ 2 & 4 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 1 & 2 \\ -1 & 2 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 6 & 8 \\ -6 & -9 & -12 \end{bmatrix}$

Solution

(a) The rank of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 3 \\ 2 & 4 & 1 \end{bmatrix}$ is 3.

As $$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 3 \\ 2 & 4 & 1 \end{vmatrix}$$

$$= 1\,(5 - 12) - 2\,(4 - 6) + 3\,(16 - 10)$$

$$= -7 + 4 + 18$$

$$= 15 \neq 0.$$

(b) Rank of $A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 1 & 2 \\ -1 & 2 & 2 \end{bmatrix}$ is 2.

as $|\mathrm{A}| = 0$, and there is a non-zero minor of order 2, namely,

$$\begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} = -7 \neq 0$$

(c) For finally the rank of the matrix

$$A = \begin{vmatrix} 2 & 3 & 4 \\ 4 & 6 & 8 \\ -6 & -9 & -12 \end{vmatrix}$$

We see $|\mathrm{A}| = 0$, its rank $\neq 3$.

Also every minor of order 2 of A is also zero.

$\therefore$ Its rank $\neq 2$

As the matrix is a non-zero matrix, so its rank is 1.

Consistent System of Equations

A system of equation is said to be consistent if they have one or more solution in case of only one solution, the solution is said to be unique.

For example, the system of equations

$$x - 3y = 9$$

$$x + 3y = 13$$

has a unique solution as $x = 11$, $y = 2/3$.

The system of equations

$$4x + 3y = 9$$

$$12x + 9y = 2$$

have infinite solutions.

If a system of equation has no solution, it is said to be inconsistent.

For example,

$$x + 2y = 9$$
$$3x + 6y = 15$$

is a system of equations, which has no solution.

Consistency of a system of Linear Equation.

$$\begin{array}{c} a_{11}\ x_1 + a_{12}\ x_2 + \ldots + a_{1n}\ x_n = b_1 \\ a_{21}\ x_1 + a_{22}\ x_2 + \ldots + a_{2n}\ x_n = b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}\ x_1 + a_{m2}\ x_2 + \ldots + a_{mn}\ x_n = b_m \end{array}$$

which can be written as

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_m \end{bmatrix}$$

or $\qquad AX = B$

Here A is called coefficient matrix and $C = [A\ ,\ B]$

$$= \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{bmatrix}$$

is called the augmented matrix.

We now state a very important theorem.

Important Theorem

A necessary and sufficient condition for a system of linear equation $AX = B$ to be consistent is that the augmented matrix $[A : B]$ and the coefficient matrix of the system are of the same rank.

The proof of the above is beyond the scope of this unit, but we shall see its validity through the following solved examples.

Elementary Row Operations

The elementary row operations to reduce the

(i) Interchange two rows.

(ii) Add a multiple of one row to another row.

(iii) Multiply a row by any non-zero constant.

(These are similar to the operations with equations that resulted in equivalent system.)

Example 1.18

Show that the system of equations

$$2x - y + z = 1$$
$$x + y - z = 2$$
$$3x - y + z = 0$$

is inconsistent.

Solution

The augmented matrix

$$A = \begin{bmatrix} 2 & -1 & 1 & 1 \\ 1 & 1 & -1 & 2 \\ 3 & -1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & -1 & 2 \\ 2 & -1 & 1 & 1 \\ 3 & -1 & 1 & 0 \end{bmatrix} \quad [R_1 \leftrightarrow R_2]$$

$$= \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & -3 & 3 & -3 \\ 3 & -1 & 1 & 0 \end{bmatrix} \quad [R_2 \to R_2 - 2R_1]$$

$$= \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & -3 & 1 & -3 \\ 0 & -4 & 4 & -6 \end{bmatrix} \quad [R_3 \to R_3 - 3R_1]$$

$$= \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & -1 & 1 & -1 \\ 0 & -4 & 4 & -6 \end{bmatrix} \quad [R_2 \to \frac{1}{3}R_2]$$

$$= \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & -2 \end{bmatrix} \quad [R_3 \to R_3 - 4R_2]$$

Here we see that rank of coefficient matrix is 2 as

$$\begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} \neq 0$$

and also the rank of the augmented matrix is 3 because

$$\begin{vmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & -2 \end{vmatrix} \neq 0 .$$

Hence the given system of equation is inconsistent.

Thus system unit of consistent equations if Rank A = Rank C

The system has a unique solution if Rank A = Rank $C = n$

The system has infinite solution if Rank A = Rank $C = \gamma < n$.

The system consists of inconsistent equations if Rank $A \neq$ Rank C.

Example 1.19

Characterisation of solution of a linear system of equations ($n \times n$) coefficient matrix)

Solution

Reduced Form of the Augmented Matrix for *n* Equations in *n* Variables	Solution of the System
(1) Coefficient array is an identity matrix ($I_{n \times n}$)	Unique solution
(2) Coefficient array is not an identity matrix with either	
(a) A row of zeros in the coefficient array with non-zero entry in the argument	(a) No solution Example 10.8
(b) Or otherwise	(b) Infinitely many solution

Example for 2(a) : Solve for the lead variables in term of non-lead variables.

$$x + 2y - z = 3$$

$$3x + y = 4$$

$$2x - y + z = 2$$

Reduced form of [A / B]

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{5} & 1 \\ 0 & 1 & -\frac{3}{5} & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Last row gives the equation 0 = 1 – a contradiction.

Example for 1

$$2x + 5y + 4z = 4$$

$$x + 4y + 3z = 1$$

$$x - 3y - 2z = 5$$

equivalent system/reduced for n

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right] \Rightarrow \begin{array}{l} x = 3 \\ y = -2 \\ z = 2 \end{array}$$

Example for 2(b)

$$x - y + z = 3$$

$$3x + 2z = 7$$

$$x - 4y + 2z = 5$$

Reduced matrix is

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{2}{3} & \frac{7}{3} \\ 0 & 1 & -\frac{1}{3} & \frac{-2}{3} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \qquad x = \frac{7}{3} - \frac{2}{3} z$$

$$y = -\frac{2}{3} + \frac{1}{3} z$$

z = arbitrary $\Rightarrow$ Infinitely many solutions.

SAQ 8

(a) Find the rank of the matrix

$$\begin{bmatrix} 2 & -3 & 6 & -5 \\ 0 & 1 & -4 & 1 \\ 4 & -5 & 8 & -9 \end{bmatrix}$$

(b) Determine the rank of the matrix

$$A = \begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$$

(c) Find the values of γ and μ for which the system of equations

$$x + 2y + z = 6$$

$$x + 4y + 3z = 10$$

$$x + 4y + \lambda z = \mu$$

has a

(a) unique solution

(b) infinite number of solution

(c) no solution.

1.10 EIGENVALUES AND EIGENVECTORS

In many applications of matrices to engineering problem, involving complete oscillations and vibrations, equation of the form

$$AX = \lambda X$$

occurs, where $A = [a_{ij}]$ is a square matrix and λ is a number (scalar). Clearly, X = 0 is a solution for any value of λ and is not normally useful. For non-trivial solution, i.e. $X \neq 0$, the values of λ are called the eigenvalues or characteristics

values of the matrix A and the corresponding solutions of the given equations $A \,.\, X = \lambda\, X$ are called the eigenvectors or characteristics vectors of A (corresponding to the eigenvalue λ).

Expressed as a set of separate equations, we have

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix}$$

i.e.

$$\begin{aligned} a_{11}\, x_1 + a_{12}\, x_2 + \dots + a_{1n}\, x_n &= \lambda\, x_1 \\ a_{21}\, x_1 + a_{22}\, x_2 + \dots + a_{2n}\, x_n &= \lambda\, x_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1}\, x_1 + a_{n2}\, x_2 + \dots + a_{nn}\, x_n &= \lambda\, x_n \end{aligned}$$

Now, after rearranging, we get

$$\begin{aligned} (a_{11} - \lambda)\, x_1 + a_{12}\, x_2 + \dots + a_{1n}\, x_n &= 0 \\ a_{21}\, x_1 + (a_{22} - \lambda)\, x_2 + \dots + a_{2n}\, x_n &= 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1}\, x_1 + a_{n2}\, x_2 + \dots + (a_{nn} - \lambda)\, x_n &= 0 \end{aligned}$$

i.e.

$$\begin{bmatrix} (a_{11} - \lambda) & a_{12} & \dots & a_{1n} \\ a_{21} & (a_{22} - \lambda) & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & (a_{nn} - \lambda) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix} = \lambda \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix}$$

Thus $\quad A \,.\, X = \lambda\, X$ becomes $A \,.\, X - \lambda\, X = 0$

i.e. $\quad (A - \lambda\, I)\; X = 0$

Note that the unit matrix is introduced since we can subtract only a matrix from another matrix.

For this set of homogeneous linear equations (i.e. right-hand consists all zero) to have a non-trivial solution, $|A - \lambda\, I|$ must be zero.

$$\det|A-\lambda I| = \begin{vmatrix} (a_{11}-\lambda) & a_{12} & \dots & a_{1n} \\ a_{21} & (a_{22}-\lambda) & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & (a_{nn}-\lambda) \end{vmatrix} = 0$$

$|A-\lambda I|$ is called the characteristic determinant of A and $|A-\lambda I| = 0$ is the characteristic equation. The solution of the characteristic equation gives the values of λ, i.e. the eigen values of A. When expanded, the det $|A-\lambda I|$ is a polynomial in λ, called the characteristic polynomial of A.

Definition

If A is a $n \times n$ matrix, then a non-zero vector X in R^n is called an eigenvactor of A if AX is a scalar multiple of X, that is $AX = \lambda X$ for some scalar λ. The scalar λ is called an eigenvalue of A and X is called to be an eigenvactor of A corresponding to λ.

Example 1.20

Find the eigen values of the matrix $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$.

Solution

The characteristic equation is $|A-\lambda I| = 0$

i.e. $$\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

or $$\lambda^2 - 7\lambda + 6 = 0$$

or $$(\lambda-6)\ (\lambda-1) = 0, \quad \lambda = 6, 1$$

Thus the eigenvalues are 6 and 1. The characteristic polynomial is $\lambda^2 = 7\lambda - 6$ (say $P(\lambda)$).

Example 1.21

Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & 2 & -2 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}.$$

Solution

The characteristic equation is

$$|A-\lambda I| = 0$$

i.e. $$\begin{vmatrix} 2-\lambda & 2 & -2 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

or $$(2-\lambda)\left[(3-\lambda)(2-\lambda)-2\right] - 2\left[(2-\lambda)-1\right] - 2\left[2-(3-\lambda)\right] = 0$$

or $$(2-\lambda)\left[(\lambda^2 - 5\lambda + 4) + 2\lambda - 2 + 2 - 2\lambda\right]$$

or $(2 - \lambda)(\lambda - 1)(\lambda - 4) = 0$

or $\lambda = 1, 2, 4$

Thus the eigenvalues are 1, 2, 4.

Cayley-Hamilton Theorem

Statement : "Every square matrix satisfies its characteristic equation".

The proof of this theorem is beyond the scope of this unit. However, the Cayley-Hamilton theorem provides an efficient method for calculating powers of a matrix. For example, if A is a 2×2 matrix, with characteristic equation

$$C_0 + C_1 \lambda + \lambda^2 = 0,$$

then $$C_0 I + C_1 A + A^2 = 0,$$

so that $$A^2 = -C_1 A - C_0 I \qquad \ldots (1)$$

Multiplying Eq. (1) by A yields,

$$A^3 = -C_1 A^2 - C_0 A,$$

which expresses A^3 in terms of A^2 and A, and multiplying Eq. (1) by A^2, yields

$$A^4 = -C_1 A^3 - C_0 A^2,$$

which expresses A^4 in terms of A^3 and A^2. Continuing in this way, we can calculate successive powers of A by expressing them in term of lower power.

Example 1.22

Calculate A^2, A^3, A^4 and A^5 for $A = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}$

Solution

Here characteristic equation of A is $|A - \lambda I| = 0$

i.e. $$\begin{vmatrix} 3-\lambda & 6 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

i.e. $(3 - \lambda)(2 - \lambda) - 6 = 0$

i.e. $\lambda^2 - 5\lambda + 6 - 6 = 0$

i.e. $\lambda^2 = 5\lambda$

By Cayley-Hamilton theorem

$$A^2 = 5A = 5\begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 15 & 30 \\ 5 & 10 \end{bmatrix}$$

Now $$A^3 = A \,.\, A^2 = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 15 & 30 \\ 5 & 10 \end{bmatrix} = \begin{bmatrix} 75 & 150 \\ 25 & 50 \end{bmatrix}$$

Similarly, $$A^4 = \begin{bmatrix} 375 & 750 \\ 125 & 250 \end{bmatrix} \qquad A^5 = \begin{bmatrix} 1875 & 3750 \\ 625 & 1250 \end{bmatrix}$$

Eigenvectors

Each eigenvalue λ obtained has corresponding to it a solution of $AX = \lambda X$ called a eigenvector. In matrices, the term 'vector' indicates a row matrix or column matrix.

Example 1.23

Find eigenvectors for the matrix

$$A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}.$$

Solution

Find characteristics equation is $\begin{vmatrix} 4-\lambda & 1 \\ 3 & 2-\lambda \end{vmatrix} = 0$

$$\therefore \ (4-\lambda)(2-\lambda) - 3 = 0 \ \Rightarrow \ \lambda^2 - 6\lambda + 5 = 0$$

$$\therefore \ (\lambda - 1)(\lambda - 5) = 0 \ \therefore \ \lambda = 1 \text{ or } 5$$

$$\lambda_1 = 1; \ \lambda_2 = 5$$

For $\lambda = 1$, the equation $AX = \lambda X$ becomes

$$\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1 . \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\left.\begin{aligned} 4x_1 + x_2 &= x_1 \\ 3x_1 + 2x_2 &= x_2 \end{aligned}\right\}$$

Either of these gives $x_2 = -3x_1$

This result tells us that whatever value x_1 has, the value of x_2 is – 3 times of it. Therefore, the eigenvector

$$X_1 = \begin{bmatrix} x_1 \\ -3x_1 \end{bmatrix} \text{ where } X_1 \text{ is any arbitrary real number}$$

is the general form of an infinite number of such eigenvector. Eigenvector is therefore (take $x_1 = 1$).

$$X_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}.$$

For $\lambda_2 = 5$, the equation $A . X = \lambda X$ becomes

$$\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 5 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\left.\begin{aligned} 4x_1 + x_2 &= 5x_1 \\ 3x_1 + 2x_2 &= 5x_2 \end{aligned}\right\}$$

Either of these cases gives $x_2 = x_1$.

$$X_2 = \begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$$

Take $$x_1 = K, \quad X_2 = \begin{bmatrix} K \\ K \end{bmatrix}$$

is the general solution;

Eigen vector is therefore (take $K = 1$)

$$X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

is a solution.

Therefore $$X_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

is an eigenvector corresponding to $\lambda_1 = 1$ and

$$X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

is an eigenvector corresponding to $\lambda_2 = 5$.

SAQ 9

(a) Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 5 & 7 & -5 \\ 0 & 4 & -1 \\ 2 & 8 & -3 \end{bmatrix}$$

(b) Verify that the eigenvalues of

$$A = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 0 & 1 \\ -1 & -1 & 2 \end{bmatrix}$$

are $\lambda = 1, 1, 2$.

(c) Find the product of the eigenvalue of

$$\begin{bmatrix} 7 & 2 & 2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$$

(d) Find the characteristic equation of the matrix

$$A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}.$$

Show that the equation is satisfied by A and hence obtain the inverse of the given matrix.

(e) Show that a matrix is singular if and only if one of its eigenvalues is zero.

(f) You are given that

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 2 & -3 & 0 \\ 1 & 1 & -1 \end{bmatrix}.$$

Using the fact that the given matrix A satisfies

$$A^3 + 4\,A^2 = A + 12\,I$$

find the inverse of A.

(g) Verify Cayley-Hamilton theorem for the matrix A, and find its inverse.

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

(h) If

$$A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

show that

$$A^2 - 5\,A + 7\,I = 0$$

where I is a unit matrix of second order.

(i) Use Cayley-Hamilton theorem to compute B^3 and B^{-1} for the matrix

$$B = \begin{bmatrix} 1 & 2 \\ 8 & 7 \end{bmatrix}$$

1.11 COMPLEX MATRICES

So far, we have considered matrices whose elements were real numbers. The elements of a matrix can, however, be complex numbers also. For such matrices, we take up a few definitions.

(i) **Conjugate of a Matrix**

If the elements of a matrix $A = [a_{rs}]_{m\times n}$ are complex number, $\alpha_{rs} + i\,\beta_{rs}$, α_{rs} and β_{rs} being real, then the matrix

$$\overline{A} = [\overline{a}_{rs}] = [\alpha_{rs} - i\,\beta_{rs}]_{m\times n}$$

is called the conjugate matrix of A.

(ii) **Hermitian Matrix**

A square matrix A such that $A^T = \overline{A}$ is said to be a Hermitian matrix. The elements of the leading diagonal of a Hermitian matrix are evidently real, while every other element is the complex conjugate of the element in the transposed position. For example,

$$A = \begin{bmatrix} 2 & 4+9i & 6-5i \\ 4-9i & 3 & 7-2i \\ 6+5i & 7+2i & 4 \end{bmatrix}$$

is a Hermitian matrix, hence $A^T = \overline{A}$. Here $\overline{A}$ is the conjugate matrix of *A*.

Thus the leading diagonal elements of a Hermitian-matrix are all real.

(iii) **Skew-Hermitian Matrix**

A square matrix *A* such that $A^T = -\overline{A}$ is said to be a skew-Hermitian matrix.

This implies that the leading diagonal element of a skew-Hermitian matrix are either all zeros or all purely imaginary.

(iv) **Unitary Matrix**

If $U^{-1} = U^{-T}$, then U is unitary, *U* is orthogonal if $U^{-1} = U^T$.

$U^T U = I \Rightarrow U$ is orthogonal, *U* is invertible $U^{-1} = U^T$.

A square matrix *U* such that $U^T = U^{-1}$ is called a unitary matrix. This is a generalization of the orthogonal matrix to the complex field.

SAQ 10

(a) Prove that if *A* is a Hermitian matrix, then *iA* is a skew-Hermitian matrix.

(b) Find *k*, *l* and m to make *A* a Hermitian matrix.

$$A = \begin{bmatrix} -1 & k & -i \\ 3-5i & 0 & m \\ l & 2+4i & 2 \end{bmatrix}$$

(c) Given that

$$A = \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$$

show that

$$(I - A)(I + A)^{-1}$$

is a unitary matrix.

(d) Prove that

$$\frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$$

is a unitary matrix.

(e) Show that

$$\begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$$

is a Harmitian matrix.

1.12 INFINITE SERIES

In this section, you will be introduced to infinite series. Let $< x_n >$ be a sequence. Then the expression of the form $x_1 + x_2 + \ldots + x_n + \ldots$ is called an infinite series and is symbolically expressed as $\sum_{n=1}^{\alpha} x_n$ or simply $\Sigma\, x_n$. The members $x_1, x_2, \ldots, x_n \ldots$ are known as first, second, . . . n^{th}, . . . terms respectively of the infinite series.

Since $< x_n >$ is a sequence

$\Rightarrow$ the terms of the sequence $x_1, x_2, \ldots x_n \ldots$ are arranged and formed according to some definite law.

Thus, in the light of this, an infinite series can be defined as succession of terms which are formed according to some definite law.

1.12.1 Sequences

(a) If to each positive integer 1, 2, 3, 4 . . . corresponds a definite number x_n, then the numbers $x_1, x_2, x_3, \ldots x_n \ldots$ are said to form a sequence. A sequence, as defined above, may be denoted by $< x_n > = < x_1, x_2, \ldots, x_n \ldots >$.

If the number of terms is unlimited, then the sequences is said to be an *infinite sequence* and x_n is its *general term*.

For instance

(i) $1, 3, 5, 7, \ldots, (2n-1), \ldots,$

(ii) $1, 1/2, 1/3, \ldots, 1/n, \ldots,$

(iii) $1, -1, 1, -1, \ldots, (-1)^{n-1}, \ldots$ are infinite sequences.

(b) **Limit :** A sequence is said to tend to a limit l, if for every $\varepsilon > 0$, a value N of n can be found such that $| x_n - l | < \varepsilon$ for $n \geq N$.

We then write $\underset{n\to\infty}{\text{Lt}}\,(x_n) = l$ or simply $(a_n) \to l$ as $n \to \infty$.

(c) **Convergence :** *If a sequence* (x_n) *has a finite limit, it is called a* **convergent sequence**. If (x_n) is not convergent, it is said to be **divergent**.

In the above example, (ii) is convergent, while (i) and (iii) are divergent.

(d) **Bounded Sequence :** A sequence $< x_n >$ is said to be bounded if a constant K exists such that the inequality $| x_n | \leq K$ is satisfied for every n.

(e) **Monotonic Sequence :** "A sequence is said to be monotonic increasing if its elements are always increasing, i.e. $x_n < x_{n+1}\ \forall\ n \in N$." "A sequence is said to be monotonic decreasing if its elements are always decreasing, i.e. $x_n > x_{n+1}\ \forall\ n \in N$."

A monotonic sequence always tends to a limit, finite or infinite. Thus, a *sequence which is monotonic and bounded is* **convergent**.

1.12.2 Series

Definition

If $x_1, x_2, x_3, \ldots, x_n, \ldots$, be an infinite sequence of real numbers, then

$$x_1 + x_2 + x_3 + \ldots + x_n + \ldots \infty$$

is called an infinite series. An infinite series is denoted by $\Sigma\, x_n$ and the sum of its first n terms is denoted by s_n.

Convergence, Divergence and Oscillation of a Series

A series in which the sum (S_n) of n terms of the series tends to a definite value, as $n \to \propto$, is called convergent series. If S_n does not tend to a definite value as $n \to \propto$, the series is said to be divergent.

Consider the infinite series $\sum x_n = x_1 + x_2 + x_3 + \ldots + x_n + \ldots \infty$ and let the sum of the first n terms be $s_n = x_1 + x_2 + x_3 + \ldots + x_n$.

Clearly, s_n is a function of n and as n increases indefinitely three possibilities arise :

(a) If s_n tends to a finite limit as $n \to \infty$, the series $\Sigma\, x_n$ is said to be *convergent*.

(b) If s_n tends to $\pm\infty$ as $n \to \infty$, the series $\Sigma\, x_n$ is said to be *divergent*.

(c) If s_n does not tend to a unique limit as $n \to \infty$, then the series $\Sigma\, x_n$ is said to be *oscillatory or non-convergent*.

Example 1.24

Examine for convergence the series $1 + 2 + 3 + \ldots + n + \ldots \infty$.

Here $$s_n = 1 + 2 + 3 + \ldots + n = \frac{n\,(n+1)}{2}$$

$\therefore$ $$\underset{n\to\infty}{\text{Lt}}\ s_n = \frac{1}{2}\ \underset{n\to\infty}{\text{Lt}}\ n\,(n+1) \to \infty$$

Hence this series is *divergent*.

General Properties of Series

The truth of the following properties is self-evident and these may be regarded as axioms :

(a) The *convergence or divergence of an infinite series remains unaffected by the addition or removal of a finite number of its terms*; for the sum of these terms being the finite quantity does not on addition or removal alter the nature of its sum.

(b) *If a series in which all the terms are positive is convergent, the series remains convergent even when some or all of its terms are negative*; for the sum is clearly the greatest when all the terms are positive.

(c) *The convergence or divergence of an infinite series unaffected by multiplying each term by a finite number*.

Series of Positive Terms

Let $\Sigma\, x_n$ be an infinite series of positive terms and $< S_n >$ the sequence of its partial sum, so that

$$S_n = x_1 + x_2 + x_3 + \ldots + x_n > 0 \text{ for all } n.$$

$$\therefore \quad S_n - S_{n-1} = x_n > 0$$

$$\Rightarrow S_n > S_{n-1}$$

Thus the sequence $< S_n >$ of partial sum of a series of positive terms is a monotonic increasing sequence. Since a monotonic increasing sequence can either converge or diverges to + α, but cannot oscillate, hence a series of positive terms either converges or diverges to + α but it cannot oscillate.

Theorem

"A positive term series converges if, and, only if, the sequence of its partial sum is bounded above."

You know that the sequence of partial sum of a positive term series is a monotonic increasing sequence and a monotonic increasing sequence converges if and only if it is bounded above. Hence it follows that a positive term series converges if and only if its sequence of partial term is bounded above.

$$\therefore \quad s_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}}$$

$$> \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \ldots + \frac{1}{\sqrt{n}} = \frac{n}{\sqrt{n}} \text{ i.e. } \sqrt{n}$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\, s_n = \underset{n\to\infty}{\text{Lt}}\, \sqrt{n} \to \infty$$

Thus the series is divergent even though $\underset{n\to\infty}{\text{Lt}}\, u_n = \underset{n\to\infty}{\text{Lt}}\, \frac{1}{\sqrt{n}} = 0$.

Hence, $\underset{n\to\infty}{\text{Lt}}\, u_n = 0$ is a necessary but not sufficient condition for convergence of $\sum n_n$.

Obs. 1

The above result leads to a simple test for divergence : If $\underset{n\to\infty}{\text{Lt}}\, u_n \neq 0$, the series $\sum u_n$ must be divergent.

1.12.3 Comparison Tests

From our definition of convergence of an infinite series and fundamental principle, namely, Cauchy's General Principle of convergence of series, we shall deduce criteria which are more special, but are also easier to manipulate. This we shall do by the following "Comparison Test".

Comparison Test I(a)

"If $\Sigma\, u_n$ and $\Sigma\, v_n$ are two series of positive terms, such that $u_n < v_n$ and $\Sigma\, v_n$ is convergent, then $\Sigma\, u_n$ is also convergent."

Here if $\quad S_n = u_1 + u_2 + \ldots + u_n,\ t_n = v_1 + v_2 + \ldots + v_n$

then we have, $\quad S_n < t_n$

$\therefore \quad \underset{n\to\infty}{\text{Lt}}\ S_n \le \underset{n\to\infty}{\text{Lt}}\ t_n$

Hence $< S_n >$ converges $\Rightarrow \Sigma\, u_n$ is convergent.

Comparison Test I(b)

"If $u_n > v_n$ and $\Sigma\, v_n$ is divergent then $\Sigma\, u_n$ also diverges."

In this case $S_n > t_n$.

Since $\quad \underset{n\to\infty}{\text{Lt}}\ t_n \to \infty \quad \therefore \quad \underset{n\to\infty}{\text{Lt}}\ S_n \to \infty$

Hence $< S_n >$ diverges $\Rightarrow \Sigma\, u_n$ is divergent.

Comparison Test II(a)

"If $\text{Lt}\left(\dfrac{u_n}{v_n}\right)$ is finite and $\Sigma\, v_n$ is convergent, then $\Sigma\, u_n$ also converges."

If $\text{Lt}\left(\dfrac{u_n}{v_n}\right) = m$ and r is any positive number, $r > m$, then $\dfrac{u_n}{v_n} \angle r$ after a certain value of n.

$\therefore \quad u_n < r\, u_n$

Then by Test I(a), the result is proved.

Comparison Test II(b)

"If $\text{Lt}\left(\dfrac{u_n}{v_n}\right) > 0$ and $\Sigma\, v_n$ is divergent, then $\Sigma\, u_n$ is also diverges."

If $\text{Lt}\left(\dfrac{u_n}{v_n}\right) = l$ and $0 < r < l$, then we have $\dfrac{u_n}{v_n} > r$ after a certain value of n.

The result now follows from Test I(b).

A case of practical importance is when $\text{Lt}\left(\dfrac{u_n}{v_n}\right)$ exists. Then we have the following test.

Comparison Test II(c)

"If $\text{Lt}\left(\dfrac{u_n}{v_n}\right)$ is finite (but $\neq 0$ for divergence), then $\Sigma\, v_n$ and $\Sigma\, u_n$ either both converge or both diverge."

This follows from Tests II(a) and II(b).

In the above tests, we have compared the general term of the one series with the general term of the second series. There is a second type of comparison tests in which the ratio of two consecutive terms of one series will be

compared with the ratio of the corresponding consecutive terms of the second series.

Comparison Test III(a)

"If $\frac{u_{n+1}}{u_n} < \frac{v_{n+1}}{v_n}$ and $\Sigma\, v_n$ converges then $\Sigma\, u_n$ also converges."

Consider $\quad S_n = u_1 + u_2 + \ldots + u_n$

$$t_n = v_1 + v_2 + \ldots + v_n$$

Now
$$S_n = u_1 \left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_2} \cdot \frac{u_2}{u_1} + \frac{u_4}{u_3} \cdot \frac{u_3}{u_2} \cdot \frac{u_2}{u_1} + \ldots\right)$$

$$\angle u_1 \left(1 + \frac{v_2}{v_1} + \frac{v_3}{v_2} \cdot \frac{v_2}{v_1} + \frac{v_4}{v_3} \cdot \frac{v_3}{v_2} \cdot \frac{v_2}{v_1} + \ldots\right)$$

$$= \frac{u_1}{v_1} (v_1 + v_2 + v_3 + \ldots + v_n)$$

$\therefore \quad S_n < \frac{u_1}{v_1} t_n$

Because $\Sigma\, v_n$ converges, $< t_n >$ is bounded above.

Hence $< S_n >$ is bounded above $\Rightarrow \Sigma\, u_n$ is convergent.

Comparison Test III(b)

"If $\frac{u_{n+1}}{u_n} < \frac{v_{n+1}}{v_n}$ and $\Sigma\, v_n$ is divergent, then $\Sigma\, u_n$ also diverge."

The proof of III(b) is the same as far III(a) after replacing '<' by '>'.

We now take up some examples to illustrate the use of above tests.

Example 1.25

Test for convergence the series

$$\frac{1}{1.2.3} + \frac{3}{2.3.4} + \frac{5}{3.4.5} + \ldots \infty$$

Solution

We have
$$u_n = \frac{2n-1}{n\,(n+1)(n+2)} = \frac{1}{n^2} \frac{2 - \frac{1}{n}}{\left(1 + \frac{1}{n}\right)\left(1 + \frac{2}{n}\right)}$$

Take, $v_n = \frac{1}{n^2}$; then

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \frac{2 - \frac{1}{n}}{\left(1 + \frac{1}{n}\right)\left(1 + \frac{2}{n}\right)} = \frac{2-0}{(1+0)\,(1+0)}$$

$= 2$, which is finite and non-zero.

$\therefore \quad$ Both $\Sigma\, u_n$ and $\Sigma\, v_n$ converge or diverge together.

But $\Sigma v_n = \Sigma \frac{1}{n^2}$ is known to be convergent.

Hence Σu_n is also convergent.

Example 1.26

Test the convergence of the series :

(i) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n} + \sqrt{n+1}}$

(ii) $\sum_{n=1}^{\infty} \frac{1}{x^n + x^{-n}}$

Solution

(i) We have $u_n = \frac{\sqrt{n+1} - \sqrt{n}}{\left[\sqrt{(n+1)} + \sqrt{n}\right]\left[\sqrt{n+1} - \sqrt{n}\right]} = \sqrt{n+1} - \sqrt{n}$

$$= \sqrt{n}\left[\left(1 + \frac{1}{n}\right)^{1/2} - 1\right]$$

$$= \sqrt{n}\left\{\left(1 + \frac{1}{2n} - \frac{1}{8n^2} + \ldots\right) - 1\right\} = \sqrt{n}\left(\frac{1}{2n} - \frac{1}{8n^2} + \ldots\right) = \frac{1}{\sqrt{n}}\left(\frac{1}{2} - \frac{1}{8n} + \ldots\right)$$

Taking $v_n = \frac{1}{\sqrt{n}}$, we have

$$\underset{u \to \infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n \to \infty}{\text{Lt}} \left(\frac{1}{2} - \frac{1}{8n} + \ldots\right) = \frac{1}{2},$$

which is finite and non-zero.

$\therefore$ Both Σu_n and Σv_n converge or diverge together.

But $\Sigma v_n = \Sigma \frac{1}{\sqrt{n}}$ is known to be divergent. Hence Σu_n is also divergent.

(ii) When $x < 1$. comparing the given series Σu_n with $\Sigma v_n = \Sigma x^n$,

We get $\underset{n \to \infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n \to \infty}{\text{Lt}} \left(\frac{1}{x^n + x^{-n}} \cdot \frac{1}{x^n}\right) = \underset{n \to \infty}{\text{Lt}} \frac{1}{x^{2n} + 1} = 1$

[Θ $x^{2n} \to 0$ as $n \to \infty$]

But Σv_n is convergent, so Σu_n is also convergent.

When $x > 1$, comparing Σu_n with $\Sigma w_n = \Sigma x^{-n}$, we get

$$\underset{n \to \infty}{\text{Lt}} \frac{u_n}{w_n} = \underset{n \to \infty}{\text{Lt}} \left(\frac{1}{x^n + x^{-n}} \cdot x^n\right) = \underset{n \to \infty}{\text{Lt}} \frac{1}{1 + x^{-2n}} = 1$$

[Θ $x^{-2n} \to 0$ as $n \to \infty$]

But Σw_n is convergent, so Σu_n is also convergent.

When $x = 1$, $\sum u_n = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots \infty$ which is divergent.

Hence, $\sum u_n$ converges for $x < 1$ and $x > 1$ but diverges for $x = 1$.

D'Alembert's Ratio Test

If $\sum u_n$ is a series of positive terms and $\underset{n\to\infty}{\text{Lt}} \left(\frac{u_{n+1}}{u_n}\right) = \lambda$, then the series converges for $\lambda < 1$ and diverges for $\lambda > 1$.

Case I

When $\underset{n\to\infty}{\text{Lt}} \frac{u_{n+1}}{u_n} = \lambda < 1$.

By definition of a limit, we can find a positive number r (< 1) such that $\frac{u_{n+1}}{u_n} < r$ for all $n > m$.

Leaving out the first m terms, let the series be $u_1 + u_2 + u_3 + \ldots$ so that $\frac{u_2}{u_1} < r, \frac{u_3}{u_2} < r, \frac{u_4}{u_3} < r, \ldots$ and so on. Then $u_1 + u_2 + u_3 + \ldots \infty$.

$$= u_1 \left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_2} \cdot \frac{u_2}{u_1} + \frac{u_4}{u_3} \cdot \frac{u_3}{u_2} + \frac{u_2}{u_1} + \ldots \infty\right)$$

$$< u_1 (1 + r + r^2 + r^3 + \ldots \infty)$$

$= \frac{u_1}{1 - r}$, which is finite quantity. Hence $\sum u_n$ is convergent. [Θ $r < 1$]

Case II

When $\underset{n\to\infty}{\text{Lt}} \frac{u_{n+1}}{u_n} = \lambda > 1$.

By definition of limit, we can find m, such that $\frac{u_{n+1}}{u_n} \geq 1$ for all $n \geq m$.

Leaving out the first m terms, let the series be $u_1 + u_2 + u_3 + \ldots$ so that $\frac{u_2}{u_1} \geq 1, \frac{u_3}{u_2} \geq 1, \frac{u_4}{u_3} \geq 1$ and so on.

$$\therefore \quad u_1 + u_2 + u_3 + u_4 + \ldots + u_n = u_1 \left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_2} \cdot \frac{u_2}{u_1} + \ldots\right)$$

$$\geq u_1 (1 + 1 + 1 + \ldots \text{ to } n \text{ terms}) = nu_1$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}} (u_1 + u_2 + \ldots + u_n)$$

$$\geq \underset{n\to\infty}{\text{Lt}} (nu_1),$$

which tends to infinity. Hence $\sum u_n$ is divergent.

Example 1.27

Test for convergence the series.

$$\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^6}{5\sqrt{4}} + \ldots \infty.$$

Solution

We have $u_n = \dfrac{x^{2n-2}}{(n+1)\sqrt{n}}$ and $u_{n+1} = \dfrac{x^{2n}}{(n+2)\sqrt{n+1}}$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{\text{Lt}} \frac{x^{2n}}{(n+2)\sqrt{n+1}} \cdot \frac{(n+1)\sqrt{n}}{x^{2n-2}} = \underset{n\to\infty}{\text{Lt}} \left[\frac{n+1}{n+2}\left(\frac{n}{n+1}\right)^{1/2}\right] x^2$$

$= \underset{n\to\infty}{\text{Lt}} \left[\dfrac{1+\frac{1}{n}}{1+\frac{2}{n}} \cdot \dfrac{1}{\sqrt{1+\frac{1}{n}}}\right] x^2 = x^2$. Hence $\Sigma\, u_n$ converges if $x^2 < 1$ and diverges if $x^2 > 1$.

If $x^2 = 1$, then $u_n = \dfrac{1}{(n+1)\sqrt{n}} = \dfrac{1}{n^{3/2}} \cdot \dfrac{1}{1+\frac{1}{n}}$.

Taking $v_n = \dfrac{1}{n^{3/2}}$, we get $\underset{n\to\infty}{\text{Lt}} \dfrac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \dfrac{1}{1+\frac{1}{n}} = 1$, a finite quantity.

$\therefore$ Both $\Sigma\, u_n$ and $\Sigma\, v_n$ converge or diverge together. But $\Sigma\, v_n = \Sigma \dfrac{1}{n^{3/2}}$ is a convergent series.

$\therefore$ $\Sigma\, u_n$ is also convergent. Hence the given series converges if $x^2 \le 1$ and diverges if $x^2 > 1$.

SAQ 11

Discuss the convergence of the series.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{3^n}$

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

1.13 SUMMARY

In the field of Engineering and Technology, matrices and determinants play a very important role in solving field problems. In this unit, we have covered the following points.

- A number of the type $z = x + i\,y$ where x and y are real numbers and $i = \sqrt{-1}$ is called a complex number.
- x = real part of z = Re (z)

y = imaginary part of $z = \text{Im}\,(z)$

- $z = x + i\,y$ is represented by a point $P\,(x, y)$ in XOY plane (Argands plane) and $|\,OP\,|$ is called the modulus of z and is denoted by $|\,z\,|$.

$$|z| = \sqrt{x^2 + y^2} = r \qquad \text{(Say)}$$

Then $x = r\cos\theta,\ y = r\sin\theta$ where $XOP! = \theta$ and is called the argument of z.

- $x - i\,y$ is called the complex conjugate of the complex number $z = x + i\,y$ and is denoted by $\bar{z}$, i.e. $\bar{z} = x - i\,y$.

- (i) $\bar{\bar{z}} = z$

  (ii) $|z_1 z_2| = |z_1||z_2|$ and $\overline{z_1 z_2} = \bar{z}_1\,\bar{z}_2$

  (iii) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|},\ z_2 \neq 0$ and $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$

  (iv) $|z|^2 = z\,\bar{z}$

  (v) $|z_1 + z_2| \leq |z_1| + |z_2|$

  (vi) $|z_1 - z_2| \leq |z_1| + |z_2|$

  and $\geq ||z_1| - |z_2||$

  (vii) $\text{amp}\,(z_1 z_2) = \text{amp}\,(z_1) + \text{amp}\,(z_2),\ z_1, z_2 \neq 0$

  (viii) $\text{amp}\left(\dfrac{z_1}{z_2}\right) = \text{amp}\,(z_1) - \text{amp}\,(z_2),\ z_1, z_2 \neq 0$

- De Moivres theorem

  (i) If n is any integer, then $(\cos\theta + i\sin\theta)^n = \cos n\,\theta + i\sin n\,\theta$.

  (ii) If n is a rational number, then $\cos n\,\theta + i\sin n\,\theta$ is one of the values of $(\cos\theta + i\sin\theta)^n$.

- A system of mn numbers arranged in a rectangular formation along m rows and n columns and bounded by the brackets [ ] is called an m by n matrix : when $m = n$, it is called a square matrix.

  To locate any particular element of a matrix, the elements are denoted by a letter followed by two suffixes, which respectively specify the rows and the columns. This a_{ij} is the element in the ith row and j^{th} column of the matrix A. In the notation, the matrix A is denoted by $[a_{ij}]$.

- Two matrices can be multiplied only when the number of column in the first is equal to the number of row in the second. Thus matrices $A = [a_{ij}]$ and $B = [b_{jk}]$ are compatible for multiplication and $C = [c_{ik}]$ is their multiplication is not commutative, that is $AB \neq BA$ in general.

  If $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{n\times p}$, then

$$c_{ij} = a_{i1}\,b_{ij} + a_{i2}\,b_{2j} + \ldots + a_{in}\,b_{nj}$$

If $\qquad A = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$,

then $AB = \begin{bmatrix} 9 & 8 \\ 1 & 2 \end{bmatrix}$ and $BA = \begin{bmatrix} 5 & 2 \\ 10 & 6 \end{bmatrix}$.

So $AB \neq BA$.

- If A be any matrix, then a matrix B, if it exists, such that $AB = BA = I$ is called the inverse of A.
- The expression $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is called a determinant of the second order and stands for '$a_1 b_2 - a_2 b_1$'.

  Similarly, $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

  is called a determinant of the third order.
- Inverse of a square matrix A

$$A^{-1} = \frac{\text{Adj } A}{|A|}$$

- Product of a square matrix and its inverse

$$AA^{-1} = A^{-1} A = I$$

- Solution of a set of linear equations

$$A \,.\, X = B \text{ is } X = A^{-1} B.$$

- Infinite series : $S_n = u_1 + u_2 + u_3 + \ldots + u_n + \ldots$ If $\lim\limits_{n \to \infty} S_n$ is a definite value, series is convergent. If $\lim\limits_{n \to \infty} S_n$ is not a definite value, series is divergent.
- D' Alembert's ratio test : If $\Sigma\, u_n$ is a series of positive terms and

$$\operatorname*{Lt}_{n \to \infty} \left(\frac{u_{n+1}}{u_n} \right) = \lambda$$

  then the series converges if $\lambda < 1$, diverges for $\lambda > 1$ but test fails if $\lambda = 1$.

1.14 KEY WORDS

Matrix	:	A rectangular array of numbers (elements).
Row Matrix	:	One row only.
Column Matrix	:	One column only.
Equal Matrices	:	Corresponding elements equal.
Diagonal Matrix	:	All elements zero except those on the leading diagonal.
Null Matrix	:	All elements zero.
Eigen Values	:	Values of λ for which $A \,.\, X = \lambda X$.

1.15 ANSWERS TO SAQs

SAQ 1

(a) (i) $\frac{13}{2} + \frac{9}{2} i$

(ii) $\frac{-7}{2} + \frac{1}{2} i$

(b) (i) $\sqrt{2}, \frac{3\pi}{4}$

(ii) $2, \frac{-5\pi}{6}$

(c) (i) $2\left(\cos\frac{\pi}{3} + i \sin\frac{\pi}{3}\right)$

(ii) $r = \frac{\sqrt{290}}{5}, \theta = \tan^{-1}\left(-\frac{1}{17}\right)$

SAQ 2

(a) (i) $2^{\frac{1}{10}}\left[\cos\frac{1}{5}\left(2n\,\pi + \frac{\pi}{4}\right) + i \sin\frac{1}{5}\left(2n\,\pi + \frac{\pi}{4}\right)\right], n = 0, 1, 2, 3, 4$

(ii) $2^{\frac{3}{4}}\left[\cos\left(\frac{3n\pi}{2} + \frac{\pi}{4}\right) + i \sin\left(\frac{3n\pi}{2} + \frac{\pi}{4}\right)\right], n = 0, 1, 2$

(b) (i) $\cos(2r+1)\frac{\pi}{4} + i \sin(2r+1)\frac{\pi}{4}, r = 0, 1, 2, 3, 4$

(ii) $x = \cos(6n+1)\frac{\pi}{15} \pm i \sin(6n+1)\frac{\pi}{15}, n = 0, 1, 2, 3, 4$

(iii) $x = \cos(2n+1)\frac{\pi}{7} + i \sin(2n+1)\frac{\pi}{7}, n = 0, 1, 2, 3, 4, 5, 6$

SAQ 3

(i) '3 × 2' or '3 by 2' matrix

(ii) '2 × 4' or '2 by 4' matrix

SAQ 4

(i) $\begin{bmatrix} -2 & -1 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 8 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} -2x & -y \\ -3y & 4y \end{bmatrix} + \begin{bmatrix} 8 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

So, we get

$-2x - y + 8 = 0$

$-3x + 4y + 1 = 0$

After rearranging

$2x + y = 8$

$3x - 4y = 1$

Solving these equations, we get

$$x = 3,\ y = 2$$

(ii) $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Now, $A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

So, $A\,A^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \cos^2\theta + \sin^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So, matrix A is orthogonal.

(iii) For orthogonal matrix , $A\,A^T = I$

Now, $A\,A^T = \frac{1}{3}\begin{bmatrix} x & 2 & 2 \\ 2 & 1 & y \\ 2 & z & 1 \end{bmatrix} \times \frac{1}{3}\begin{bmatrix} x & 2 & 2 \\ 2 & 1 & z \\ 2 & y & 1 \end{bmatrix}$

$$= \frac{1}{9}\begin{bmatrix} x^2 + 8 & 2x + 2y + 2 & 2x + 2z + 2 \\ 2x + 2y + 2 & y^2 + 5 & y + z + 4 \\ 2x + 2z + 2 & y + z + 4 & z^2 + 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So, $2x + 2y + 2 = 0 \quad \Rightarrow x + y = -1$

$2x + 2z + 2 = 0 \quad \Rightarrow x + z = -1$

$y + z + 4 = 0 \quad \Rightarrow y + z = -4$

After solving, we get $x = 1,\ y = -2,\ z = -2$.

(iv) $A = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$

$$A^T = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

So the matrix A is orthogonal.

(v) Let us take the symmetric and a skew-symmetric matrices as

$$\begin{bmatrix} g & f \\ f & k \end{bmatrix} \text{ and } \begin{bmatrix} 0 & f \\ -f & 0 \end{bmatrix} \text{ respectively.}$$

Now $$\begin{bmatrix} g & f \\ f & k \end{bmatrix} \text{ and } \begin{bmatrix} 0 & f \\ -f & 0 \end{bmatrix} = \begin{bmatrix} g & 2f \\ 0 & k \end{bmatrix}$$

We can write

$$\begin{bmatrix} g & 2f \\ 0 & k \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 0 & 4 \end{bmatrix}$$

So, $g = 2, \quad f = -1.5, k = 4$

So, the matrices are

$$\begin{bmatrix} 2 & -1.5 \\ -1.5 & 4 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & -1.5 \\ 1.5 & 0 \end{bmatrix}$$

SAQ 5

(i) Now, $A(\theta_1)\, A(\theta_2) = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix}$

$$= \begin{bmatrix} \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 & -\sin\theta_1 \cos\theta_2 - \sin\theta_2 \cos\theta_1 \\ \sin\theta_1 \cos\theta_2 + \sin\theta_2 \cos\theta_1 & \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta_1 + \theta_2) & -\sin(\theta_1 + \theta_2) \\ \sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) \end{bmatrix}$$

$= A(\theta_1 + \theta_2)$ Proved.

(ii) $f(A) = A^2 - 5A + 6I$

$$A^2 = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} \times \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4+0+1 & 0+0-1 & 2+0+0 \\ 4+2+3 & 0+1-3 & 2+3+0 \\ 2-2+0 & 0-1+0 & 1-3+0 \end{bmatrix} = \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix}$$

$$A^2 - 5A + 6I = \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} - \begin{bmatrix} 10 & 0 & 5 \\ 10 & 5 & 15 \\ 5 & -5 & 0 \end{bmatrix} + \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{bmatrix}$$

(iii) $A + B = \begin{bmatrix} 5 & 10 \\ 10 & 5 \end{bmatrix}$

$$(A+B)^2 = \begin{bmatrix} 5 & 10 \\ 10 & 5 \end{bmatrix} \begin{bmatrix} 5 & 10 \\ 10 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 125 & 100 \\ 100 & 125 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 2 & 3 \\ 6 & 5 \end{bmatrix} \times \begin{bmatrix} 2 & 3 \\ 6 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & 21 \\ 42 & 43 \end{bmatrix}$$

$$A\,B = \begin{bmatrix} 2 & 3 \\ 6 & 5 \end{bmatrix} \times \begin{bmatrix} 3 & 7 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 18 & 14 \\ 38 & 42 \end{bmatrix}$$

$$B\,A = \begin{bmatrix} 3 & 7 \\ 4 & 0 \end{bmatrix} \times \begin{bmatrix} 2 & 3 \\ 6 & 5 \end{bmatrix} = \begin{bmatrix} 48 & 44 \\ 8 & 12 \end{bmatrix}$$

$$B^2 = \begin{bmatrix} 3 & 7 \\ 4 & 0 \end{bmatrix} \times \begin{bmatrix} 3 & 7 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 37 & 21 \\ 12 & 28 \end{bmatrix}$$

$$A^2 + A\,B + B\,A + B^2 = \begin{bmatrix} 125 & 100 \\ 100 & 125 \end{bmatrix}$$

So, $(A + B)^2 = A^2 + A\,B + B\,A + B^2$ (Proved)

(iv) Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$A\,A^T = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}^2 + a_{12}^2 + a_{13}^2 & a_{11}\,a_{21} + a_{12}\,a_{22} + a_{13}\,a_{33} & a_{11}\,a_{31} + a_{12}\,a_{32} + a_{13}\,a_{33} \\ a_{11}\,a_{21} + a_{12}\,a_{22} + a_{13}\,a_{33} & a_{21}^2 + a_{22}^2 + a_{23}^2 & a_{21}\,a_{31} + a_{22}\,a_{32} + a_{23}\,a_{33} \\ a_{11}\,a_{31} + a_{12}\,a_{32} + a_{13}\,a_{33} & a_{21}\,a_{31} + a_{22}\,a_{32} + a_{23}\,a_{33} & a_{31}^2 + a_{32}^2 + a_{33}^2 \end{bmatrix}$$

It is a symmetric matrix.

(v) Now,

$$\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix} \times \begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta\cos^2\phi + \cos\theta\sin\theta\cos\phi\sin\phi & \cos^2\theta\cos\phi\sin\phi + \cos\theta\sin\theta\sin^2\phi \\ \cos\theta\sin\theta\cos^2\phi + \sin^2\theta\cos\phi\sin\phi & \cos\theta\sin\theta\cos\phi\sin\phi + \sin^2\theta\sin^2\phi \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta\cos\phi\ \cos(\theta - \phi) & \cos\theta\sin\phi\ \cos(\theta - \phi) \\ \sin\theta\cos\phi\ \cos(\theta - \phi) & \sin\theta\sin\phi\ \cos(\theta - \phi) \end{bmatrix}$$

So, the product of two matrices is a null matrix when θ and ϕ differ by an odd multiple of $\frac{\pi}{2}$ because $\cos(2n + 1)\frac{\pi}{2} = 0$,

when $n = 0, 1, 2, 3, \ldots$

(vi) From the condition for multiplication of two matrices, we get

$$x + 5 = y$$

$$11 - y = x$$

After solving $x = 3$, $y = 8$.

SAQ 6

(i) $$\begin{vmatrix} a-b & b-c & c \\ a^2-b^2 & b^2-c^2 & c^2 \\ b-a & c-b & a+b \end{vmatrix} \quad \begin{matrix} C_1 \to C_1 - C_2 \\ C_2 \to C_2 - C_3 \end{matrix}$$

$$= (a-b)(b-c)\begin{vmatrix} 1 & 1 & c \\ a+b & b+c & c^2 \\ -1 & -1 & a+b \end{vmatrix}$$

$$= (a-b)(b-c)\begin{vmatrix} 0 & 0 & a+b+c \\ a+b & b+c & c^2 \\ -1 & -1 & a+b \end{vmatrix} \quad (R_1 \to R_1 + R_3)$$

$$= (a-b)(b-c)(c-a)(a+b+c)$$

(ii) $$\begin{vmatrix} -1 & 0 & 0 & a \\ 0 & -1 & 0 & b \\ 0 & 0 & -1 & c \\ x & y & z & -1 \end{vmatrix}$$

$$= -\begin{vmatrix} -1 & 0 & b \\ 0 & -1 & c \\ y & z & -1 \end{vmatrix} - a\begin{vmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ x & y & z \end{vmatrix}$$

$$= -[-(1 - c\,z) + b\,y] - a\,x$$

$$= 1 - a\,x - b\,y - c\,z$$

(iii) $$\begin{vmatrix} x & x+4y & 2y \\ 7y & 13y & 3y \\ 3z & 3z+16x & 8x \end{vmatrix}$$

$$= \begin{vmatrix} x & x & 2y \\ 7y & 7y & 3y \\ 3z & 3z & 8x \end{vmatrix} \quad C_2 \to C_2 - 2C_3$$

(Since C_1 and C_2 are identical)

$$= 0$$

(iv) $|A| = 1(9-4) - 2(6+2) - 1(4+3) = -18$

$$\Delta_x = \begin{vmatrix} 2 & 2 & -1 \\ 7 & 3 & 2 \\ 4 & 2 & 3 \end{vmatrix} \begin{matrix} = 2(9-4) - 2(21-8) - 1(14-12) \\ = -18 \end{matrix}$$

$$\Delta_y = \begin{vmatrix} 1 & 2 & -1 \\ 2 & 7 & 2 \\ -1 & 4 & 3 \end{vmatrix} = 1(21-8) - 2(6+2) - 1(8+7) = -18$$

$$\Delta_z = \begin{vmatrix} 1 & 2 & 2 \\ 2 & 3 & 7 \\ -1 & 2 & 4 \end{vmatrix} = 1\,(12-14) - 2\,(8+7) + 2\,(4+3) = -18$$

By Cramer's Rule

$$x = \frac{\Delta_x}{\Delta} = 1,\ y = \frac{\Delta_y}{\Delta} = 1,\ z = \frac{\Delta_y}{\Delta} = 1$$

So, $\quad x = 1,\ y = 1,\ z = 1$

(v) $$\begin{bmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} a+b+c-x & c & b \\ a+b+c-x & b-x & a \\ a+b+c-x & a & c-x \end{bmatrix} = 0 \quad (\Theta\ C_1 \to C_1 + C_2 + C_3)$$

$$\Rightarrow (a+b+c-x)\begin{vmatrix} 1 & c & b \\ 1 & b-x & a \\ 1 & a & c-x \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 1 & c & b \\ 1 & b-x & a \\ 1 & a & c-x \end{vmatrix} = 0$$

$$\begin{vmatrix} 0 & c-b+x & b-a \\ 0 & b-x-a & a-c+x \\ 1 & a & c-x \end{vmatrix} = 0 \quad \begin{pmatrix} R_1 \to R_1 - R_2 \\ R_2 \to R_2 - R_3 \end{pmatrix}$$

$$\Rightarrow x^2 = a^2 + b^2 + c^2 - ab - bc - ca$$

$$\Rightarrow x = \pm\sqrt{a^2 + b^2 + c^2 - ab - bc - ca}$$

(vi) Now, $\quad \alpha I + \beta A = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} + \begin{bmatrix} 0 & \beta \\ -\beta & 0 \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$

Now, $\quad (\alpha I + \beta A)^2 = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \times \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} = \begin{bmatrix} \alpha^2 - \beta^2 & 2\alpha\beta \\ -2\alpha\beta & \alpha^2 - \beta^2 \end{bmatrix}$

Comparison

$$\alpha^2 - \beta^2 = 0, \qquad 2\alpha\beta = 1$$

After solving, we get

$$\alpha = \beta = \pm\frac{1}{\sqrt{2}}.$$

SAQ 7

(i) $\quad A^{-1} = \dfrac{\text{Adj. } A}{|A|}$

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 4 & 3 \\ 3 & 7 & 6 \end{bmatrix} \quad \Rightarrow \ |A| = 1$$

$$\text{Adj. } A = \begin{bmatrix} 3 & 8 & -5 \\ -3 & -6 & 4 \\ 2 & 3 & -2 \end{bmatrix} \quad \Rightarrow \quad A^{-1} = \begin{bmatrix} 3 & 8 & -5 \\ -3 & -6 & 4 \\ 2 & 3 & -2 \end{bmatrix}$$

(ii) (a) Solving by Cramer's rule

Here $$\Delta = \begin{vmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 8$$

$$\Delta_x = \begin{vmatrix} 3 & 1 & 2 \\ -3 & -3 & -1 \\ 4 & 2 & 1 \end{vmatrix} = 8$$

$$\Delta_y = \begin{vmatrix} 3 & 3 & 2 \\ 2 & -3 & -1 \\ 1 & 4 & 1 \end{vmatrix} = 16$$

$$\Delta_z = \begin{vmatrix} 3 & 1 & 3 \\ 2 & -3 & -3 \\ 1 & 2 & 4 \end{vmatrix} = -8$$

$$x = \frac{\Delta x}{\Delta} = 1,\ y = \frac{\Delta y}{\Delta} = 2,\ z = \frac{\Delta z}{\Delta} = -1$$

So, $x = 1,\ y = 2,\ z = -1$

(b) Solution by matrix inverse method

Here $$\Delta = \begin{vmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_1 & c_1 \\ a_3 & b_1 & c_1 \end{vmatrix} \text{ (say)}$$

Then
$$c_{11} = -1 \quad c_{12} = -3 \quad c_{13} = 7$$
$$c_{21} = 3 \quad c_{22} = 1 \quad c_{23} = -5$$
$$c_{31} = 5 \quad c_{32} = 7 \quad c_{33} = -11$$

Also $\Delta = a_1\, C_{11} + b_1\, C_{12} + C_1\, C_{13} = 8$

Now $AX = B$, so $X = A^{-1}\, B$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{\Delta} \begin{vmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{vmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} -1 & 3 & 5 \\ -3 & 1 & 7 \\ 7 & -5 & -11 \end{bmatrix} \times \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Hence $x = 1,\ y = 2,\ z = -1$

(iii) $|\mathrm{A}| = \sec^3 x$

$$\text{Adj. } A = \begin{bmatrix} \sec x & \tan x \sec x & 0 \\ -\tan x \sec x & \sec x & 0 \\ 0 & 0 & \sec^2 x \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{ Adj. } A$$

$$= \cos x \begin{bmatrix} \cos x & \sin x & 0 \\ -\sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(iv) $$|A| = \begin{bmatrix} 1 & \tan\frac{\theta}{2} \\ -\tan\frac{\theta}{2} & 1 \end{bmatrix} = 1 + \tan^2\frac{\theta}{2}$$

$$\text{R.H.S.} = \begin{bmatrix} 1 & -\tan\frac{\theta}{2} \\ \tan\frac{\theta}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & \tan\frac{\theta}{2} \\ -\tan\frac{\theta}{2} & 1 \end{bmatrix}^{-1}$$

$$= \frac{1}{1+\tan^2\frac{\theta}{2}} \begin{bmatrix} 1 & -\tan\frac{\theta}{2} \\ \tan\frac{\theta}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & -\tan\frac{\theta}{2} \\ \tan\frac{\theta}{2} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} & -\dfrac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} \\ \dfrac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} & \dfrac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

= L. H. S. (Proved)

(v) $|A| = \cos^3\theta + \sin^3\theta$

$$\text{Adj. } A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}^T$$

$$= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{ Adj. } A$$

$$= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(vi) $a x + b y - 3 z = -3$

$-2 x - b y + c z = -1$

$$a x + 3 y - c z = -3$$

Putting the value of $x = 1$, $y = -1$, and $z = 2$;

$$a - b - 6 = -3$$

$$a - b = 3$$

$$-2 + b + 2c = -1$$

$$b + 2c = 1$$

$$a - 3 - 2c = -3$$

$$a = 2c,$$

After solving $a = 2$, $b = -1$, $c = 1$.

(vii) $|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = 1 \times (18 - 12) - 1 \times (9 - 3) + 1 \times (4 - 2) = 2$

$$\text{Adj. } A = \begin{bmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{bmatrix}^T \Rightarrow \text{Adj. } A = \begin{bmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{Adj. } A = \frac{1}{2} \begin{bmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{bmatrix}$$

Now, $X = A^{-1} B$

$$= \frac{1}{2} \begin{bmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 14 \\ 36 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Therefore, $x = 1$, $y = 2$, $z = 3$.

SAQ 8

(i) Let us take (3×3) minor

$$\begin{vmatrix} 2 & -3 & 6 \\ 0 & 1 & -4 \\ 4 & -5 & 8 \end{vmatrix} = 2\,(8 - 20) - (-3)\,(16) + 6\,(-4) = 0$$

So, the rank of the matrix is less than 3.

Now, we take a (2×2) minor

$$\begin{vmatrix} 2 & -3 \\ 0 & 1 \end{vmatrix} = 2 \neq 0$$

So, the rank of the matrix is 2.

(ii) Now,

$$|A| = \begin{vmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{vmatrix} = 3\,(4 - 4) - (-1)\,(-12 + 12) + 2\,(-6 + 6) = 0$$

So, rank of the matrix is less than 3.

Now take any (2 × 2) minor.

$$\begin{vmatrix} 3 & -1 \\ -6 & 2 \end{vmatrix} = 6 - 6 = 0$$

So, the rank of the matrix is less than 2.

So, the rank of the matrix is one.

(iii) We have

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 4 & 3 \\ 1 & 4 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

The system admits of unique solution if and only if, the coefficient matrix is of rank 3. This requires that

$$\begin{vmatrix} 1 & 2 & 1 \\ 1 & 4 & 3 \\ 1 & 4 & \lambda \end{vmatrix} \begin{matrix} = 1\,(4\,\lambda - 12) - 2\,(\lambda - 3) + (4 + 4) \\ = 2\,(\lambda - 3) \end{matrix}$$

Thus for a unique solution $\lambda \neq 3$ and μ may have any value. If $\lambda = 3$, the system matrices

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 4 & 3 \\ 1 & 4 & 3 \end{bmatrix} \text{ and } K = \begin{bmatrix} 1 & 2 & 1 & 6 \\ 1 & 4 & 3 & 10 \\ 1 & 4 & 3 & \mu \end{bmatrix}$$

are not of the same rank.

But A is of rank 2 and K is not of rank 2, unless $\mu = 10$.

Thus if $\lambda = 3$, and $\mu \neq 10$, the system will have no solution. If $\lambda = 3$, and
$\mu = 10$, the system will have an infinite number of solution.

SAQ 9

(i) Now, $|A - \lambda\, I| = 0$

$$\begin{vmatrix} 5-\lambda & 7 & -5 \\ 0 & 4-\lambda & -1 \\ 2 & 8 & -3-\lambda \end{vmatrix} = 0$$

or, $(5 - \lambda)\,[(4 - \lambda)\,(-3 - \lambda) + 8] - 7\,(2) - 5\,[-2\,(4 - \lambda)] = 0$

or, $-\lambda^3 + 6\,\lambda^2 - 11\,\lambda + 6 = 0$

or, $\lambda^3 - 6\,\lambda^2 + 11\,\lambda - 6 = 0$

or, $\lambda = 1, 2, 3$

(ii) Now, $|A - \lambda\, I| = 0$

$$\begin{vmatrix} 2-\lambda & 1 & -1 \\ -1 & 0-\lambda & 1 \\ -1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)[-\lambda(2-\lambda)+1] - 1[-(2-\lambda)+1] - (1-\lambda) = 0$$

$$\Rightarrow (2-\lambda)(\lambda^2 - 2\lambda + 1) = 0$$

$$\Rightarrow \lambda = 1, 1, 2$$

(iii) We know that the product of the eigen values is equal to the determinant of the matrix.

$$\Delta = \begin{vmatrix} 7 & 2 & 2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{vmatrix}$$

$$= 7(1-4) - 2(6-12) + 2(-12+6)$$

$$= -21 + 12 - 12$$

$$= -21$$

So, product of the eigen value is – 21.

(iv) $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} \Rightarrow A - \lambda I = \begin{bmatrix} 1-\lambda & 3 & 7 \\ 4 & 2-\lambda & 3 \\ 1 & 2 & 1-\lambda \end{bmatrix} = 0$

The characteristic equation is

$$\lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0$$

For matrix;

$$A^3 - 4A^2 - 20A - 35I = 0$$

$$A^2 = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix}$$

$$A^2 = A^2 A = \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix}$$

$$4A^2 + 20A + 35I = \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix} \qquad \text{(Proved)}$$

$$A^2 - 4A - 20I - 35A^{-1} = 0$$

$$\Rightarrow A^{-1} = \frac{1}{35}[A^2 - 4A - 20I]$$

Now, $A^2 - 4A - 20I = \begin{bmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & -10 \end{bmatrix}$

$$A^{-1} = \frac{1}{35}\begin{bmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & -10 \end{bmatrix}$$

(v) Constant term in the characteristic polynomial of an $n \times n$ matrix A is $(-1)^n$ det A and also equal to $(-1)^n \lambda_1 \lambda_2 \lambda_3 \ldots \lambda_n$ (from theory of equation) where $\det(\lambda I - A) = \lambda^n + C_1 \lambda^{n-1} + \ldots + C_{n-1} \lambda + C_n$.

We know, the determinant of the matrix is equal to the product of the eigen values, so the determinant of the matrix will be zero, if one of its eigen values is zero. We can conclude that the matrix is singular if and only if, one of its eigen values is zero.

(vi) Now, $|A - \lambda I| = 0$

$$\begin{vmatrix} -\lambda & 1 & 2 \\ 2 & -3-\lambda & 0 \\ 1 & -1 & -1-\lambda \end{vmatrix} = 0$$

or, $-\lambda[(-3-\lambda)(-1-\lambda) - 0] - 1[2(-1-\lambda) - 0) + 2[2 - (-3-\lambda)] = 0$

or, $\lambda^3 + 4\lambda^2 - \lambda - 12 = 0$

For Cayley-Hamilton, find

$$A^3 + 4A^2 - A - 12I = 0$$

$$\Rightarrow A^3 + 4A^2 = A + 12I$$

or, $A^2 + 4A = I + 12A^{-1}$

or, $A^{-1} = \frac{1}{12}[A^2 + 4A - I]$

(vii) $|A - \lambda I| = 0$,

$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

or, $\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$

Cayley-Hamilton equation

$$A^3 - 6A^2 + 9A - 4I = 0$$

$$A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

Now, $A^3 - 6A^2 + 9A - 4I = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - \begin{bmatrix} 36 & -30 & 30 \\ -30 & 36 & -30 \\ 30 & -30 & 36 \end{bmatrix}$

$$+ \begin{bmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Proved.

(viii) Now, $|A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)(2-\lambda) + 1 = 0$$

$$\Rightarrow \lambda^2 - 5\lambda + 7 = 0$$

From Cayley-Hamilton equation

$$A^2 - 5A + 7I = 0$$

(ix) $B = \begin{bmatrix} 1 & 2 \\ 8 & 7 \end{bmatrix}$

$B - \lambda I = 0$

$$\begin{vmatrix} 1-\lambda & 2 \\ 8 & 7-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - 8\lambda + 7 - 16 = 0$$

$$\Rightarrow \lambda^2 - 8\lambda - 9 = 0$$

Cayley-Hamilton equation

$$B^2 - 8B - 9I = 0$$

Multiplying B^{-1}, we get

$$B - 8I - 9B^{-1} = 0$$

$$\Rightarrow B^{-1} = \frac{1}{9}(B - 8I)$$

$$= \frac{1}{9}\begin{bmatrix} -7 & 2 \\ 8 & -1 \end{bmatrix}$$

Now from Cayley-Hamilton equation, we get $B^3 - 8B^2 - 9B = 0$

$$B^3 = 8B^2 + 9B \Rightarrow B^3 = \begin{bmatrix} 145 & 146 \\ 584 & 583 \end{bmatrix}$$

SAQ 10

(i) Let us take a Hermitian matrix

$$A = \begin{bmatrix} a_{11} & a_{12} + i\alpha_{12} & a_{13} + i\alpha_{13} \\ a_{12} - i\alpha_{12} & a_{22} & a_{23} + i\alpha_{23} \\ a_{13} - i\alpha_{13} & a_{23} - i\alpha_{23} & a_{33} \end{bmatrix}$$

Now, $$i\,A = \begin{bmatrix} i\,a_{11} & i\,a_{12} - \alpha_{12} & i\,a_{13} - \alpha_{13} \\ i\,a_{12} + \alpha_{12} & i\,a_{22} & i\,a_{23} - \alpha_{23} \\ i\,a_{13} + \alpha_{13} & i\,a_{23} + \alpha_{23} & i\,a_{33} \end{bmatrix}$$

Now, if A is a Hermitian matrix, $i\,A$ is a Skew-Hermitian matrix.

(ii) A is a Hermitian Matrix;

$$A = \begin{bmatrix} -1 & k & -i \\ 3-5i & 0 & m \\ l & 2+4i & 2 \end{bmatrix} \Rightarrow \begin{matrix} k = 3 + 5\,i \\ m = 2 - 4\,i \\ l = i \end{matrix}$$

(iii) $$I + A = \begin{bmatrix} 1 & 1 + 2\,i \\ -1 + 2\,i & 1 \end{bmatrix}$$

$$|I + A| = 1 - (-1 - 4) = 6$$

$$(I + A)^{-1} = \frac{1}{6}\begin{bmatrix} 1 & -1 - 2\,i \\ 1 - 2\,i & 1 \end{bmatrix}.$$

Also, $$I - A = \begin{bmatrix} 1 & -1 - 2\,i \\ 1 + 2\,i & 1 \end{bmatrix}$$

$$\therefore\ (I - A)\,(I + A)^{-1} = \frac{1}{6}\begin{bmatrix} 1 & -1 - 2\,i \\ 1 - 2\,i & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 - 2\,i \\ 1 - 2\,i & 1 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} -4 & -2 - 4\,i \\ 2 - 4\,i & -4 \end{bmatrix} \qquad \ldots$$

(i)

Its conjugate-transpose

$$= \frac{1}{6}\begin{bmatrix} -4 & 2 + 4\,i \\ -2 + 4\,i & -4 \end{bmatrix} \qquad \ldots \text{(ii)}$$

Product of Eqs. (i) and (ii)

$$= \frac{1}{36}\begin{bmatrix} -4 & -2 - 4\,i \\ 2 - 4\,i & -4 \end{bmatrix}\begin{bmatrix} -4 & 2 + 4\,i \\ -2 + 4\,i & -4 \end{bmatrix} = \frac{1}{36}\begin{bmatrix} 36 & 0 \\ 0 & 36 \end{bmatrix} = I$$

Hence the result.

(iv) $$U = \frac{1}{2}\begin{bmatrix} 1 + i & -1 + i \\ 1 + i & 1 - i \end{bmatrix}$$

Its conjugate transpose,

$$= \frac{1}{2}\begin{bmatrix} 1 - i & 1 - i \\ -1 - i & 1 + i \end{bmatrix}$$

Product

$$= \frac{1}{4}\begin{bmatrix} 1 - i^2 - i^2 + 1 & 1 - i^2 + i^2 - 1 \\ 1 - i^2 + i^2 + 1 & 1 - i^2 + 1 - i^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(v) It satisfies the condition $A^T = \overline{A}$. The elements of the leading diagonal of the matrix are real, while every other element is the complex conjugate of the element in the transposed position.

SAQ 11

(i) We have $u_n = \dfrac{n^2}{3^n}$ and $u_{n+1} = \dfrac{(n+1)^2}{3^{n+1}}$

$$\therefore \operatorname*{Lt}_{n\to\infty} \frac{u_{n+1}}{u_n} = \operatorname*{Lt}_{n\to\infty} \frac{(n+1)^2}{3^{n+1}} \times \frac{3^n}{n^2} = \operatorname*{Lt}_{n\to\infty} \frac{1}{3}\left(1+\frac{1}{n}\right)^2 = \frac{1}{3} \text{ which is } < 1.$$

Hence the given series is convergent.

(ii) Here $\dfrac{u_{n+1}}{u_n} = \dfrac{(n+1)!}{(n+1)^{n+1}} \cdot \dfrac{n^n}{n!} = \dfrac{n^n}{(n+1)^n} = \dfrac{1}{\left(1+\dfrac{1}{n}\right)^n}$

$$\therefore \operatorname*{Lt}_{n\to\infty} \frac{u_{n+1}}{u_n} = 1 \div \left\{ \operatorname*{Lt}_{n\to\infty} \left(1+\frac{1}{n}\right)^n \right\} = \frac{1}{e}, \text{ which is } < 1.$$

Hence the given series is convergent.