TEST CODE: MIII (Objective type) 2012

SYLLABUS

Algebra — Permutations and combinations. Binomial theorem. Theory of equations. Inequalities. Complex numbers and De Moivre's theorem. Elementary set theory. Functions and relations. Divisibility and congruences. Algebra of matrices. Determinant, rank and inverse of a matrix. Solutions of linear equations. Eigenvalues and eigenvectors of matrices. Simple properties of a group.

Coordinate geometry — Straight lines, circles, parabolas, ellipses and hyperbolas. Elements of three dimensional coordinate geometry — straight lines, planes and spheres.

Calculus — Sequences and series. Limits and continuity of functions of one variable. Differentiation and integration of functions of one variable with applications. Power series. Taylor and Maclaurin series. Definite integrals. Areas using integrals. Definite integrals as limits of Riemann sums. Maxima and minima. Functions of several variables - limits, continuity, differentiability. Double integrals and their applications. Ordinary linear differential equations.

SAMPLE QUESTIONS

Note: *For each question there are four suggested answers of which only one is correct.*

1. Let $\{f_n(x)\}$ be a sequence of polynomials defined inductively as

$$\begin{aligned} f_1(x) &= (x-2)^2 \\ f_{n+1}(x) &= (f_n(x)-2)^2, \quad n \geq 1. \end{aligned}$$

Let a_n and b_n respectively denote the constant term and the coefficient of x in $f_n(x)$. Then

(A) $a_n = 4$, $b_n = -4^n$ (B) $a_n = 4$, $b_n = -4n^2$

(C) $a_n = 4^{(n-1)!}$, $b_n = -4^n$ (D) $a_n = 4^{(n-1)!}$, $b_n = -4n^2$.

2. If a, b are positive real variables whose sum is a constant λ, then the minimum value of $\sqrt{(1+1/a)(1+1/b)}$ is

(A) $\lambda - 1/\lambda$ (B) $\lambda + 2/\lambda$ (C) $\lambda + 1/\lambda$ (D) none of the above.

3. Let x be a positive real number. Then

(A) $x^2 + \pi^2 + x^{2\pi} > x\pi + (\pi + x)x^\pi$

(B) $x^\pi + \pi^x > x^{2\pi} + \pi^{2x}$

(C) $\pi x + (\pi + x)x^\pi > x^2 + \pi^2 + x^{2\pi}$

(D) none of the above.

4. Suppose in a competition 11 matches are to be played, each having one of 3 distinct outcomes as possibilities. The number of ways one can predict the outcomes of all 11 matches such that exactly 6 of the predictions turn out to be correct is

(A) $\binom{11}{6} \times 2^5$ (B) $\binom{11}{6}$ (C) 3^6 (D) none of the above.

5. A set contains $2n+1$ elements. The number of subsets of the set which contain at most n elements is

(A) 2^n (B) 2^{n+1} (C) 2^{n-1} (D) 2^{2n}.

6. A club with x members is organized into four committees such that

(a) each member is in exactly two committees,

(b) any two committees have exactly one member in common.

Then x has

(A) exactly two values both between 4 and 8

(B) exactly one value and this lies between 4 and 8

(C) exactly two values both between 8 and 16

(D) exactly one value and this lies between 8 and 16.

7. Let X be the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Define the set $\mathcal{R}$ by

$\mathcal{R} = \{(x, y) \in X \times X : x \text{ and } y \text{ have the same remainder when divided by } 3\}$.

Then the number of elements in $\mathcal{R}$ is

(A) 40 (B) 36 (C) 34 (D) 33.

8. Let A be a set of n elements. The number of ways, we can choose an ordered pair (B, C), where B, C are disjoint subsets of A, equals

(A) n^2 (B) n^3 (C) 2^n (D) 3^n.

9. Let $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n$, n being a positive integer. The value of

$$\left(1+\frac{C_0}{C_1}\right)\left(1+\frac{C_1}{C_2}\right)\cdots\left(1+\frac{C_{n-1}}{C_n}\right)$$

is

(A) $\left(\frac{n+1}{n+2}\right)^n$ (B) $\frac{n^n}{n!}$ (C) $\left(\frac{n}{n+1}\right)^n$ (D) $\frac{(n+1)^n}{n!}$.

10. The value of the infinite product

$$P = \frac{7}{9} \times \frac{26}{28} \times \frac{63}{65} \times \cdots \times \frac{n^3-1}{n^3+1} \times \cdots$$

is

(A) 1 (B) 2/3 (C) 7/3 (D) none of the above.

11. The number of positive integers which are less than or equal to 1000 and are divisible by none of 17, 19 and 23 equals

(A) 854 (B) 153 (C) 160 (D) none of the above.

12. Consider the polynomial $x^5 + ax^4 + bx^3 + cx^2 + dx + 4$ where a, b, c, d are real numbers. If $(1+2i)$ and $(3-2i)$ are two roots of this polynomial then the value of a is

(A) $-524/65$ (B) $524/65$ (C) $-1/65$ (D) $1/65$.

13. The number of real roots of the equation

$$2\cos\left(\frac{x^2+x}{6}\right) = 2^x + 2^{-x}$$

is

(A) 0 (B) 1 (C) 2 (D) infinitely many.

14. Consider the following system of equivalences of integers.

$$\begin{aligned} x &\equiv 2 \bmod 15 \\ x &\equiv 4 \bmod 21. \end{aligned}$$

The number of solutions in x, where $1 \le x \le 315$, to the above system of equivalences is

(A) 0 (B) 1 (C) 2 (D) 3.

15. The number of real solutions of the equation $(9/10)^x = -3 + x - x^2$ is

(A) 2 (B) 0 (C) 1 (D) none of the above.

16. If two real polynomials $f(x)$ and $g(x)$ of degrees m (≥ 2) and n (≥ 1) respectively, satisfy

$$f(x^2+1) = f(x)g(x),$$

for every $x \in \mathbb{R}$, then

(A) f has exactly one real root x_0 such that $f'(x_0) \neq 0$

(B) f has exactly one real root x_0 such that $f'(x_0) = 0$

(C) f has m distinct real roots

(D) f has no real root.

17. Let $X = \dfrac{1}{1001} + \dfrac{1}{1002} + \dfrac{1}{1003} + \cdots + \dfrac{1}{3001}$. Then,

(A) $X < 1$ (B) $X > 3/2$

(C) $1 < X < 3/2$ (D) none of the above holds.

18. The set of complex numbers z satisfying the equation

$$(3+7i)z + (10-2i)\overline{z} + 100 = 0$$

represents, in the complex plane,

(A) a straight line

(B) a pair of intersecting straight lines

(C) a point

(D) a pair of distinct parallel straight lines.

19. The limit $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \left| e^{\frac{2\pi ik}{n}} - e^{\frac{2\pi i(k-1)}{n}} \right|$ is

(A) 2 (B) $2e$ (C) 2π (D) $2i$.

20. The limit $\displaystyle\lim_{n\to\infty} \left(1 - \frac{1}{n^2}\right)^n$ equals

(A) e^{-1} (B) $e^{-1/2}$ (C) e^{-2} (D) 1.

21. Let ω denote a complex fifth root of unity. Define

$$b_k = \sum_{j=0}^{4} j\omega^{-kj},$$

for $0 \leq k \leq 4$. Then $\sum_{k=0}^{4} b_k \omega^k$ is equal to

(A) 5 (B) 5ω (C) $5(1+\omega)$ (D) 0.

22. Let $a_n = \left(1 - \frac{1}{\sqrt{2}}\right) \cdots \left(1 - \frac{1}{\sqrt{n+1}}\right), \quad n \geq 1$. Then $\lim_{n\to\infty} a_n$

(A) equals 1 (B) does not exist (C) equals $\frac{1}{\sqrt{\pi}}$ (D) equals 0.

23. Let X be a nonempty set and let $\mathcal{P}(X)$ denote the collection of all subsets of X. Define $f : X \times \mathcal{P}(X) \to \mathbb{R}$ by

$$f(x, A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Then $f(x, A \cup B)$ equals

(A) $f(x, A) + f(x, B)$

(B) $f(x, A) + f(x, B) - 1$

(C) $f(x, A) + f(x, B) - f(x, A) \cdot f(x, B)$

(D) $f(x, A) + |f(x, A) - f(x, B)|$

24. The series $\sum_{k=2}^{\infty} \frac{1}{k(k-1)}$ converges to

(A) -1 (B) 1 (C) 0 (D) does not converge.

25. The limit $\lim_{x\to\infty} \left(\frac{3x-1}{3x+1}\right)^{4x}$ equals

(A) 1 (B) 0 (C) $e^{-8/3}$ (D) $e^{4/9}$

26. $\lim_{n\to\infty} \frac{1}{n}\left(\frac{n}{n+1} + \frac{n}{n+2} + \cdots + \frac{n}{2n}\right)$ is equal to

(A) ∞ (B) 0 (C) $\log_e 2$ (D) 1

27. Let $\cos^6\theta = a_6\cos 6\theta + a_5\cos 5\theta + a_4\cos 4\theta + a_3\cos 3\theta + a_2\cos 2\theta + a_1\cos\theta + a_0$. Then a_0 is

(A) 0 (B) 1/32. (C) 15/32. (D) 10/32.

28. In a triangle ABC, AD is the median. If length of AB is 7, length of AC is 15 and length of BC is 10 then length of AD equals

(A) $\sqrt{125}$ (B) 69/5 (C) $\sqrt{112}$ (D) $\sqrt{864}/5$.

29. The set $\{x : \left|x + \frac{1}{x}\right| > 6\}$ equals the set

(A) $(0, 3-2\sqrt{2}) \cup (3+2\sqrt{2}, \infty)$
(B) $(-\infty, -3-2\sqrt{2}) \cup (-3+2\sqrt{2}, \infty)$
(C) $(-\infty, 3-2\sqrt{2}) \cup (3+2\sqrt{2}, \infty)$
(D) $(-\infty, -3-2\sqrt{2}) \cup (-3+2\sqrt{2}, 3-2\sqrt{2}) \cup (3+2\sqrt{2}, \infty)$

30. Suppose that a function f defined on $\mathbb{R}^2$ satisfies the following conditions:

$$\begin{aligned} f(x+t, y) &= f(x,y) + ty, \\ f(x, t+y) &= f(x,y) + tx \text{ and} \\ f(0,0) &= K, \text{ a constant.} \end{aligned}$$

Then for all $x, y \in \mathbb{R}$, $f(x,y)$ is equal to

(A) $K(x+y)$. (B) $K - xy$. (C) $K + xy$. (D) none of the above.

31. Consider the sets defined by the real solutions of the inequalities

$$A = \{(x,y) : x^2 + y^4 \le 1\} \qquad B = \{(x,y) : x^4 + y^6 \le 1\}.$$

Then

(A) $B \subseteq A$
(B) $A \subseteq B$
(C) Each of the sets $A - B$, $B - A$ and $A \cap B$ is non-empty
(D) none of the above.

32. If a square of side a and an equilateral triangle of side b are inscribed in a circle then a/b equals

(A) $\sqrt{2/3}$ (B) $\sqrt{3/2}$ (C) $3/\sqrt{2}$ (D) $\sqrt{2}/3$.

33. If $f(x)$ is a real valued function such that

$$2f(x) + 3f(-x) = 15 - 4x,$$

for every $x \in \mathbb{R}$, then $f(2)$ is

(A) -15 (B) 22 (C) 11 (D) 0.

34. If $f(x) = \dfrac{\sqrt{3}\sin x}{2 + \cos x}$, then the range of $f(x)$ is

(A) the interval $[-1, \sqrt{3}/2]$ (B) the interval $[-\sqrt{3}/2, 1]$

(C) the interval $[-1, 1]$ (D) none of the above.

35. If $f(x) = x^2$ and $g(x) = x\sin x + \cos x$ then

(A) f and g agree at no points

(B) f and g agree at exactly one point

(C) f and g agree at exactly two points

(D) f and g agree at more than two points.

36. For non-negative integers m, n define a function as follows

$$f(m,n) = \begin{cases} n+1 & \text{if } m = 0 \\ f(m-1, 1) & \text{if } m \neq 0, n = 0 \\ f(m-1, f(m, n-1)) & \text{if } m \neq 0, n \neq 0 \end{cases}$$

Then the value of $f(1,1)$ is

(A) 4 (B) 3 (C) 2 (D) 1.

37. Let a be a nonzero real number. Define

$$f(x) = \begin{vmatrix} x & a & a & a \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix}$$

for $x \in \mathbb{R}$. Then, the number of distinct real roots of $f(x) = 0$ is

(A) 1 (B) 2 (C) 3 (D) 4.

38. A real 2×2 matrix M such that

$$M^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1-\varepsilon \end{pmatrix}$$

(A) exists for all $\varepsilon > 0$

(B) does not exist for any $\varepsilon > 0$

(C) exists for some $\varepsilon > 0$

(D) none of the above is true

39. The eigenvalues of the matrix $X = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ are

(A) 1, 1, 4 (B) 1, 4, 4 (C) 0, 1, 4 (D) 0, 4, 4.

40. Let $x_1, x_2, x_3, x_4, y_1, y_2, y_3$ and y_4 be fixed real numbers, not all of them equal to zero. Define a 4×4 matrix $\mathbf{A}$ by

$$\mathbf{A} = \begin{pmatrix} x_1^2 + y_1^2 & x_1x_2 + y_1y_2 & x_1x_3 + y_1y_3 & x_1x_4 + y_1y_4 \\ x_2x_1 + y_2y_1 & x_2^2 + y_2^2 & x_2x_3 + y_2y_3 & x_2x_4 + y_2y_4 \\ x_3x_1 + y_3y_1 & x_3x_2 + y_3y_2 & x_3^2 + y_3^2 & x_3x_4 + y_3y_4 \\ x_4x_1 + y_4y_1 & x_4x_2 + y_4y_2 & x_4x_3 + y_4y_3 & x_4^2 + y_4^2 \end{pmatrix}.$$

Then rank($\mathbf{A}$) equals

(A) 1 or 2. (B) 0. (C) 4. (D) 2 or 3.

41. If M is a 3×3 matrix such that

$$[0 \quad 1 \quad 2]M = [1 \quad 0 \quad 0] \quad \text{and} \quad [3 \quad 4 \quad 5]M = [0 \quad 1 \quad 0]$$

then $[6 \quad 7 \quad 8]M$ is equal to

(A) $[2 \quad 1 \quad -2]$ (B) $[0 \quad 0 \quad 1]$ (C) $[-1 \quad 2 \quad 0]$ (D) $[9 \quad 10 \quad 8]$.

42. Let $\lambda_1, \lambda_2, \lambda_3$ denote the eigenvalues of the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{pmatrix}.$$

If $\lambda_1 + \lambda_2 + \lambda_3 = \sqrt{2} + 1$, then the set of possible values of t, $-\pi \le t < \pi$, is

(A) Empty set (B) $\left\{\frac{\pi}{4}\right\}$ (C) $\left\{-\frac{\pi}{4}, \frac{\pi}{4}\right\}$ (D) $\left\{-\frac{\pi}{3}, \frac{\pi}{3}\right\}$.

43. The values of η for which the following system of equations

$$\begin{array}{rcrcrcl} x & + & y & + & z & = & 1 \\ x & + & 2y & + & 4z & = & \eta \\ x & + & 4y & + & 10z & = & \eta^2 \end{array}$$

has a solution are

(A) $\eta = 1, -2$ (B) $\eta = -1, -2$ (C) $\eta = 3, -3$ (D) $\eta = 1, 2$.

44. Let P_1, P_2 and P_3 denote, respectively, the planes defined by

$$\begin{aligned} a_1x + b_1y + c_1z &= \alpha_1 \\ a_2x + b_2y + c_2z &= \alpha_2 \\ a_3x + b_3y + c_3z &= \alpha_3. \end{aligned}$$

It is given that P_1, P_2 and P_3 intersect exactly at one point when $\alpha_1 = \alpha_2 = \alpha_3 = 1$. If now $\alpha_1 = 2, \alpha_2 = 3$ and $\alpha_3 = 4$ then the planes

(A) do not have any common point of intersection
(B) intersect at a unique point
(C) intersect along a straight line
(D) intersect along a plane.

45. Angles between any pair of 4 main diagonals of a cube are

(A) $\cos^{-1} 1/\sqrt{3}, \pi - \cos^{-1} 1/\sqrt{3}$ (B) $\cos^{-1} 1/3, \pi - \cos^{-1} 1/3$
(C) $\pi/2$ (D) none of the above.

46. If the tangent at the point P with co-ordinates (h, k) on the curve $y^2 = 2x^3$ is perpendicular to the straight line $4x = 3y$, then

(A) $(h, k) = (0, 0)$

(B) $(h, k) = (1/8, -1/16)$

(C) $(h, k) = (0, 0)$ or $(h, k) = (1/8, -1/16)$

(D) no such point (h, k) exists.

47. Consider the family $\mathcal{F}$ of curves in the plane given by $x = cy^2$, where c is a real parameter. Let $\mathcal{G}$ be the family of curves having the following property: every member of $\mathcal{G}$ intersects each member of $\mathcal{F}$ orthogonally. Then $\mathcal{G}$ is given by

(A) $xy = k$

(B) $x^2 + y^2 = k^2$

(C) $y^2 + 2x^2 = k^2$

(D) $x^2 - y^2 + 2yk = k^2$

48. Suppose the circle with equation $x^2 + y^2 + 2fx + 2gy + c = 0$ cuts the parabola $y^2 = 4ax$, $(a > 0)$ at four distinct points. If d denotes the sum of ordinates of these four points, then the set of possible values of d is

(A) $\{0\}$ (B) $(-4a, 4a)$ (C) $(-a, a)$ (D) $(-\infty, \infty)$.

49. The polar equation $r = a\cos\theta$ represents

(A) a spiral (B) a parabola (C) a circle (D) none of the above.

50. Let

$$\begin{aligned} V_1 &= \frac{7^2 + 8^2 + 15^2 + 23^2}{4} - \left(\frac{7 + 8 + 15 + 23}{4}\right)^2, \\ V_2 &= \frac{6^2 + 8^2 + 15^2 + 24^2}{4} - \left(\frac{6 + 8 + 15 + 24}{4}\right)^2, \\ V_3 &= \frac{5^2 + 8^2 + 15^2 + 25^2}{4} - \left(\frac{5 + 8 + 15 + 25}{4}\right)^2. \end{aligned}$$

Then

(A) $V_3 < V_2 < V_1$

(B) $V_3 < V_1 < V_2$

(C) $V_1 < V_2 < V_3$

(D) $V_2 < V_3 < V_1$.

51. If a sphere of radius r passes through the origin and cuts the three co-ordinate axes at points A, B, C respectively, then the centroid of the triangle ABC lies on a sphere of radius

(A) r (B) $\frac{r}{\sqrt{3}}$ (C) $\sqrt{\frac{2}{3}}\,r$ (D) $\frac{2r}{3}$.

52. Consider the tangent plane $\mathcal{T}$ at the point $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$ to the sphere $x^2 + y^2 + z^2 = 1$. If P is an arbitrary point on the plane

$$\frac{x}{\sqrt{3}} + \frac{y}{\sqrt{3}} + \frac{z}{\sqrt{3}} = -2,$$

then the minimum distance of P from the tangent plane, $\mathcal{T}$, is always

(A) $\sqrt{5}$ (B) 3 (C) 1 (D) none of these.

53. Let S_1 denote a sphere of unit radius and C_1 a cube inscribed in S_1. Inductively define spheres S_n and cubes C_n such that S_{n+1} is inscribed in C_n and C_{n+1} is inscribed in S_{n+1}. Let v_n denote the sum of the volumes of the first n spheres. Then $\lim_{n\to\infty} v_n$ is

(A) 2π. (B) $\frac{8\pi}{3}$. (C) $\frac{2\pi}{13}(9+\sqrt{3})$. (D) $\frac{6+2\sqrt{3}}{3}\pi$.

54. If $0 < x < 1$, then the sum of the infinite series $\frac{1}{2}x^2 + \frac{2}{3}x^3 + \frac{3}{4}x^4 + \cdots$ is

(A) $\log\frac{1+x}{1-x}$ (B) $\frac{x}{1-x} + \log(1+x)$

(C) $\frac{1}{1-x} + \log(1-x)$ (D) $\frac{x}{1-x} + \log(1-x)$.

55. Let $\{a_n\}$ be a sequence of real numbers. Then $\lim_{n\to\infty} a_n$ exists if and only if

(A) $\lim_{n\to\infty} a_{2n}$ and $\lim_{n\to\infty} a_{2n+2}$ exists

(B) $\lim_{n\to\infty} a_{2n}$ and $\lim_{n\to\infty} a_{2n+1}$ exist

(C) $\lim_{n\to\infty} a_{2n}$, $\lim_{n\to\infty} a_{2n+1}$ and $\lim_{n\to\infty} a_{3n}$ exist

(D) none of the above.

56. Let $\{a_n\}$ be a sequence of non-negative real numbers such that the series $\sum_{n=1}^{\infty} a_n$ is convergent. If p is a real number such that the series $\sum \frac{\sqrt{a_n}}{n^p}$ diverges, then

(A) p must be strictly less than $\frac{1}{2}$

(B) p must be strictly less than or equal to $\frac{1}{2}$

(C) p must be strictly less than or equal to 1 but can be greater than $\frac{1}{2}$

(D) p must be strictly less than 1 but can be greater than or equal to $\frac{1}{2}$.

57. Suppose $a > 0$. Consider the sequence

$$a_n = n\{\sqrt[n]{ea} - \sqrt[n]{a}\}, \quad n \geq 1.$$

Then

(A) $\lim_{n\to\infty} a_n$ does not exist

(B) $\lim_{n\to\infty} a_n = e$

(C) $\lim_{n\to\infty} a_n = 0$

(D) none of the above.

58. Let $\{a_n\}$, $n \geq 1$, be a sequence of real numbers satisfying $|a_n| \leq 1$ for all n. Define

$$A_n = \frac{1}{n}(a_1 + a_2 + \cdots + a_n),$$

for $n \geq 1$. Then $\lim_{n\to\infty} \sqrt{n}(A_{n+1} - A_n)$ is equal to

(A) 0 (B) -1 (C) 1 (D) none of these.

59. In the Taylor expansion of the function $f(x) = e^{x/2}$ about $x = 3$, the coefficient of $(x-3)^5$ is

(A) $e^{3/2}\frac{1}{5!}$ (B) $e^{3/2}\frac{1}{2^5 5!}$ (C) $e^{-3/2}\frac{1}{2^5 5!}$ (D) none of the above.

60. Let σ be the permutation:

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 5 & 6 & 2 & 4 & 9 & 8 & 7 & 1, \end{matrix}$$

I be the identity permutation and m be the order of σ i.e.

$$m = \min\{\text{positive integers } n : \sigma^n = I\}.$$

Then m is

(A) 8 (B) 12 (C) 360 (D) 2520.

61. Let

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

Then

(A) there exists a matrix C such that $A = BC = CB$

(B) there is no matrix C such that $A = BC$

(C) there exists a matrix C such that $A = BC$, but $A \neq CB$

(D) there is no matrix C such that $A = CB$.

62. If the matrix

$$A = \begin{bmatrix} a & 1 \\ 2 & 3 \end{bmatrix}$$

has 1 as an eigenvalue, then $trace(A)$ is

(A) 4 (B) 5 (C) 6 (D) 7.

63. Let $\theta = 2\pi/67$. Now consider the matrix

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Then the matrix A^{2010} is

(A) $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ (B) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(C) $\begin{pmatrix} \cos^{30}\theta & \sin^{30}\theta \\ -\sin^{30}\theta & \cos^{30}\theta \end{pmatrix}$ (D) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

64. A rod of length 10 feet slides with its two ends on the coordinate axes. If the end on the x-axis moves with a constant velocity of 2 feet per minute then the magnitude of the velocity per minute of the middle point of the rod at the instant the rod makes an angle of 30° with the x-axis is

(A) $19/2$ (B) 2 (C) $4/\sqrt{19}$ (D) $2/19$.

65. Let the position of a particle in three dimensional space at time t be $(t, \cos t, \sin t)$. Then the length of the path traversed by the particle between the times $t = 0$ and $t = 2\pi$ is

(A) 2π. (B) $2\sqrt{2}\pi$. (C) $\sqrt{2}\pi$ (D) none of the above.

66. Let n be a positive real number and p be a positive integer. Which of the following inequalities is true?

(A) $n^p > \dfrac{(n+1)^{p+1} - n^{p+1}}{p+1}$ (B) $n^p < \dfrac{(n+1)^{p+1} - n^{p+1}}{p+1}$

(C) $(n+1)^p < \dfrac{(n+1)^{p+1} - n^{p+1}}{p+1}$ (D) none of the above.

67. The smallest positive number K for which the inequality

$$|\sin^2 x - \sin^2 y| \leq K|x - y|$$

holds for all x and y is

(A) 2 (B) 1 (C) $\dfrac{\pi}{2}$ (D) there is no smallest positive value of K; any $K > 0$ will make the inequality hold.

68. Given two real numbers $a < b$, let

$$d(x, [a,b]) = \min\{|x - y| : a \leq y \leq b\} \quad \text{for } -\infty < x < \infty.$$

Then the function

$$f(x) = \frac{d(x, [0,1])}{d(x, [0,1]) + d(x, [2,3])}$$

satisfies

(A) $0 \leq f(x) < \frac{1}{2}$ for every x

(B) $0 < f(x) < 1$ for every x

(C) $f(x) = 0$ if $2 \leq x \leq 3$ and $f(x) = 1$ if $0 \leq x \leq 1$

(D) $f(x) = 0$ if $0 \leq x \leq 1$ and $f(x) = 1$ if $2 \leq x \leq 3$.

69. Let

$$f(x,y) = \begin{cases} e^{-1/(x^2+y^2)} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Then $f(x,y)$ is

(A) not continuous at $(0,0)$

(B) continuous at $(0,0)$ but does not have first order partial derivatives

(C) continuous at $(0,0)$ and has first order partial derivatives, but not differentiable at $(0,0)$

(D) differentiable at $(0,0)$

70. Consider the function

$$f(x) = \begin{cases} \displaystyle\int_0^x \{5 + |1-y|\}dy & \text{if } x > 2 \\ 5x+2 & \text{if } x \leq 2 \end{cases}$$

Then

(A) f is not continuous at $x = 2$

(B) f is continuous and differentiable everywhere

(C) f is continuous everywhere but not differentiable at $x = 1$

(D) f is continuous everywhere but not differentiable at $x = 2$.

71. Let $w = \log(u^2 + v^2)$ where $u = \mathrm{e}^{(x^2+y)}$ and $v = \mathrm{e}^{(x+y^2)}$. Then

$$\left.\frac{\partial w}{\partial x}\right|_{x=0,y=0}$$

is

(A) 0 (B) 1 (C) 2 (D) 4

72. Let $p > 1$ and for $x > 0$, define $f(x) = (x^p - 1) - p(x-1)$. Then

(A) $f(x)$ is an increasing function of x on $(0, \infty)$

(B) $f(x)$ is a decreasing function of x on $(0, \infty)$

(C) $f(x) \geq 0$ for all $x > 0$

(D) $f(x)$ takes both positive and negative values for $x \in (0, \infty)$.

73. The map $f(x) = a_0 \cos|x| + a_1 \sin|x| + a_2|x|^3$ is differentiable at $x = 0$ if and only if

(A) $a_1 = 0$ and $a_2 = 0$ (B) $a_0 = 0$ and $a_1 = 0$

(C) $a_1 = 0$ (D) a_0, a_1, a_2 can take any real value.

74. $f(x)$ is a differentiable function on the real line such that $\lim_{x\to\infty} f(x) = 1$ and $\lim_{x\to\infty} f'(x) = \alpha$. Then

(A) α must be 0 (B) α need not be 0, but $|\alpha| < 1$

(C) $\alpha > 1$ (D) $\alpha < -1$.

75. Let f and g be two differentiable functions such that $f'(x) \leq g'(x)$ for all $x < 1$ and $f'(x) \geq g'(x)$ for all $x > 1$. Then

(A) if $f(1) \geq g(1)$, then $f(x) \geq g(x)$ for all x

(B) if $f(1) \leq g(1)$, then $f(x) \leq g(x)$ for all x

(C) $f(1) \leq g(1)$

(D) $f(1) \geq g(1)$.

76. The length of the curve $x = t^3$, $y = 3t^2$ from $t = 0$ to $t = 4$ is

(A) $5\sqrt{5} + 1$ (B) $8(5\sqrt{5} + 1)$

(C) $5\sqrt{5} - 1$ (D) $8(5\sqrt{5} - 1)$.

77. Given that $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$, the value of

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+xy+y^2)}\, dxdy$$

is

(A) $\sqrt{\pi/3}$ (B) $\pi/\sqrt{3}$ (C) $\sqrt{2\pi/3}$ (D) $2\pi/\sqrt{3}$.

78. Let

$$I = \int_1^2\int_1^2\int_1^2\int_1^2 \frac{x_1 + x_2 + x_3 - x_4}{x_1 + x_2 + x_3 + x_4}\, dx_1dx_2dx_3dx_4.$$

Then I equals

(A) 1/2 (B) 1/3 (C) 1/4 (D) 1.

79. Let $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. The value of the double integral

$$\int\int_D (x^2 + y^2)\, dxdy$$

is

(A) π (B) $\frac{\pi}{2}$ (C) 2π (D) π^2

80. Let $g(x, y) = \max\{12 - x, 8 - y\}$. Then the minimum value of $g(x, y)$ as (x, y) varies over the line $x + y = 10$ is

(A) 5 (B) 7 (C) 1 (D) 3.

81. Let $0 < \alpha < \beta < 1$. Then

$$\sum_{k=1}^{\infty} \int_{1/(k+\beta)}^{1/(k+\alpha)} \frac{1}{1+x}\, dx$$

is equal to

(A) $\log_e \frac{\beta}{\alpha}$ (B) $\log_e \frac{1+\beta}{1+\alpha}$ (C) $\log_e \frac{1+\alpha}{1+\beta}$ (D) ∞.

82. If f is continuous in [0, 1] then

$$\lim_{n\to\infty} \sum_{j=0}^{[n/2]} \frac{1}{n} f\left(\frac{j}{n}\right)$$

(where $[y]$ is the largest integer less than or equal to y)

(A) does not exist
(B) exists and is equal to $\frac{1}{2}\int_0^1 f(x)\, dx$
(C) exists and is equal to $\int_0^1 f(x)\, dx$
(D) exists and is equal to $\int_0^{1/2} f(x)\, dx$.

83. The volume of the solid, generated by revolving about the horizontal line $y = 2$ the region bounded by $y^2 \leq 2x$, $x \leq 8$ and $y \geq 2$, is

(A) $2\sqrt{2\pi}$ (B) $28\pi/3$ (C) 84π (D) none of the above.

84. If α, β are complex numbers then the maximum value of $\frac{\alpha\bar{\beta} + \bar{\alpha}\beta}{|\alpha\beta|}$ is

(A) 2
(B) 1
(C) the expression may not always be a real number and hence maximum does not make sense
(D) none of the above.

85. For positive real numbers $a_1, a_2, \ldots, a_{100}$, let

$$p = \sum_{i=1}^{100} a_i \quad \text{and} \quad q = \sum_{1 \leq i < j \leq 100} a_i a_j.$$

Then

(A) $q = \frac{p^2}{2}$ (B) $q^2 \geq \frac{p^2}{2}$ (C) $q < \frac{p^2}{2}$ (D) none of the above.

86. The differential equation of all the ellipses centred at the origin is

(A) $y^2 + x(y')^2 - yy' = 0$ (B) $xyy'' + x(y')^2 - yy' = 0$

(C) $yy'' + x(y')^2 - xy' = 0$ (D) none of these.

87. The coordinates of a moving point P satisfy the equations

$$\frac{dx}{dt} = \tan x, \quad \frac{dy}{dt} = -\sin^2 x, \quad t \geq 0.$$

If the curve passes through the point $(\pi/2, 0)$ when $t = 0$, then the equation of the curve in rectangular co-ordinates is

(A) $y = 1/2\cos^2 x$ (B) $y = \sin 2x$

(C) $y = \cos 2x + 1$ (D) $y = \sin^2 x - 1$.

88. Let y be a function of x satisfying

$$\frac{dy}{dx} = 2x^3\sqrt{y} - 4xy$$

If $y(0) = 0$ then $y(1)$ equals

(A) $1/4e^2$ (B) $1/e$ (C) $e^{1/2}$ (D) $e^{3/2}$.

89. Let $f(x)$ be a given differentiable function. Consider the following differential equation in y

$$f(x)\frac{dy}{dx} = yf'(x) - y^2.$$

The general solution of this equation is given by

(A) $y = -\frac{x+c}{f(x)}$ (B) $y^2 = \frac{f(x)}{x+c}$

(C) $y = \frac{f(x)}{x+c}$ (D) $y = \frac{[f(x)]^2}{x+c}$.

90. Let $y(x)$ be a non-trivial solution of the second order linear differential equation

$$\frac{d^2y}{dx^2} + 2c\frac{dy}{dx} + ky = 0,$$

where $c < 0$, $k > 0$ and $c^2 > k$. Then

(A) $|y(x)| \to \infty$ as $x \to \infty$

(B) $|y(x)| \to 0$ as $x \to \infty$

(C) $\lim_{x\to\pm\infty} |y(x)|$ exists and is finite

(D) none of the above is true.

91. The differential equation of the system of circles touching the y-axis at the origin is

(A) $x^2 + y^2 - 2xy\frac{dy}{dx} = 0$ (B) $x^2 + y^2 + 2xy\frac{dy}{dx} = 0$

(C) $x^2 - y^2 - 2xy\frac{dy}{dx} = 0$ (D) $x^2 - y^2 + 2xy\frac{dy}{dx} = 0$.

92. Suppose a solution of the differential equation

$$(xy^3 + x^2y^7)\frac{dy}{dx} = 1,$$

satisfies the initial condition $y(1/4) = 1$. Then the value of $\frac{dy}{dx}$ when $y = -1$ is

(A) $\frac{4}{3}$ (B) $-\frac{4}{3}$ (C) $\frac{16}{5}$ (D) $-\frac{16}{5}$.

93. Consider the group

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a, b \in \mathbb{R}, a > 0 \right\}$$

with usual matrix multiplication. Let

$$N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{R} \right\}.$$

Then,

(A) N is not a subgroup of G

(B) N is a subgroup of G but not a normal subgroup

(C) N is a normal subgroup and the quotient group G/N is of finite order

(D) N is a normal subgroup and the quotient group is isomorphic to $\mathbb{R}^+$ (the group of positive reals with multiplication).

94. Let G be a group with identity element e. If x and y are elements in G satisfying $x^5y^3 = x^8y^5 = e$, then which of the following conditions is true?

(A) $x = e, y = e$

(B) $x \neq e, y = e$

(C) $x = e, y \neq e$

(D) $x \neq e, y \neq e$

95. Let G be the group $\{\pm 1, \pm i\}$ with multiplication of complex numbers as composition. Let H be the quotient group $\mathbb{Z}/4\mathbb{Z}$. Then the number of nontrivial group homomorphisms from H to G is

(A) 4 (B) 1 (C) 2 (D) 3.