$F\left(x_0, y_0 + \frac{p}{2}\right)$,

Coordinates of the vertex

$$x_0 = -\frac{b}{2a},\ y_0 = ax_0^2 + bx_0 + c = \frac{4ac - b^2}{4a}.$$

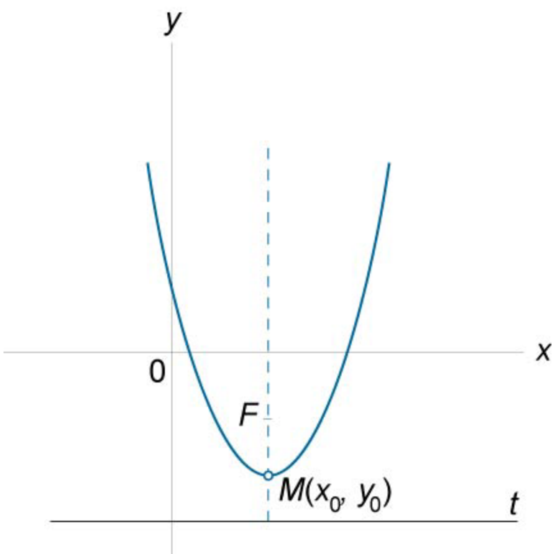


Figure 122.

7.8 Three-Dimensional Coordinate System

Point coordinates: x_0, y_0, z_0, x_1, y_1, z_1, ...

Real number: λ

Distance between two points: d

Area: S

Volume: V

670. **Distance Between Two Points**

$$d = AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

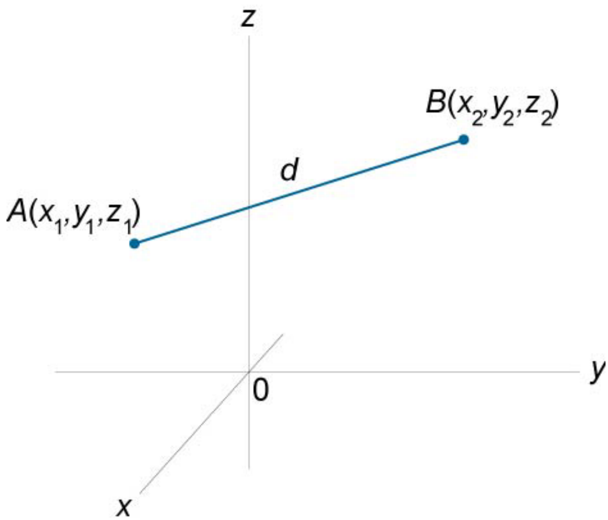


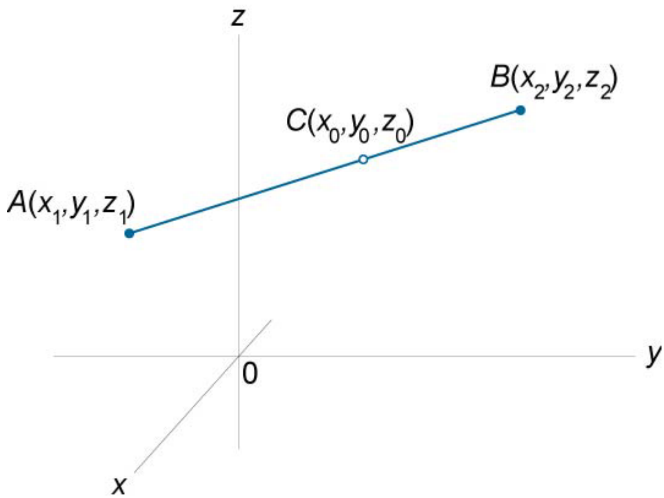
Figure 123.

671. **Dividing a Line Segment in the Ratio λ**

$$x_0 = \frac{x_1 + \lambda x_2}{1+\lambda},\ y_0 = \frac{y_1 + \lambda y_2}{1+\lambda},\ z_0 = \frac{z_1 + \lambda z_2}{1+\lambda},$$

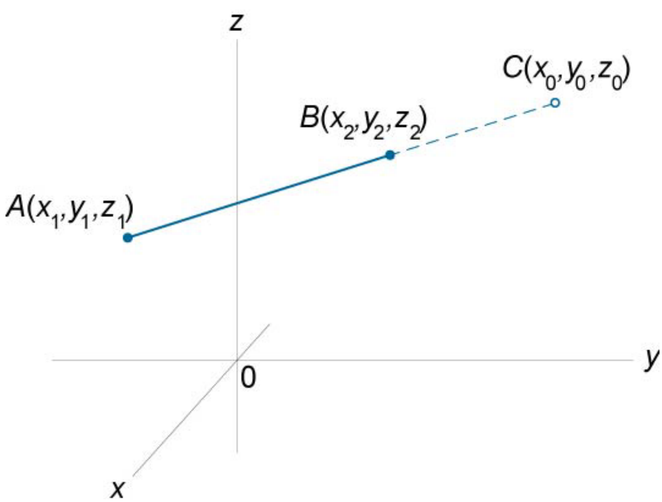
where

$$\lambda = \frac{AC}{CB},\ \lambda \neq -1.$$



$\lambda > 0$

Figure 124.



$\lambda < 0$

Figure 125.

672. Midpoint of a Line Segment

$$x_0 = \frac{x_1 + x_2}{2},\ y_0 = \frac{y_1 + y_2}{2},\ z_0 = \frac{z_1 + z_2}{2},\ \lambda = 1.$$

673. Area of a Triangle

The area of a triangle with vertices $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, and $P_3(x_3, y_3, z_3)$ is given by

$$S = \frac{1}{2}\sqrt{\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}^2 + \begin{vmatrix} z_1 & x_1 & 1 \\ z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \end{vmatrix}^2 + \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2}.$$

674. Volume of a Tetrahedron

The volume of a tetrahedron with vertices $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, $P_3(x_3, y_3, z_3)$, and $P_4(x_4, y_4, z_4)$ is given by

$$V = \pm\frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix},$$

or

$$V = \pm\frac{1}{6}\begin{vmatrix} x_1 - x_4 & y_1 - y_4 & z_1 - z_4 \\ x_2 - x_4 & y_2 - y_4 & z_2 - z_4 \\ x_3 - x_4 & y_3 - y_4 & z_3 - z_4 \end{vmatrix}.$$

Note: We choose the sign (+) or (–) so that to get a positive answer for volume.

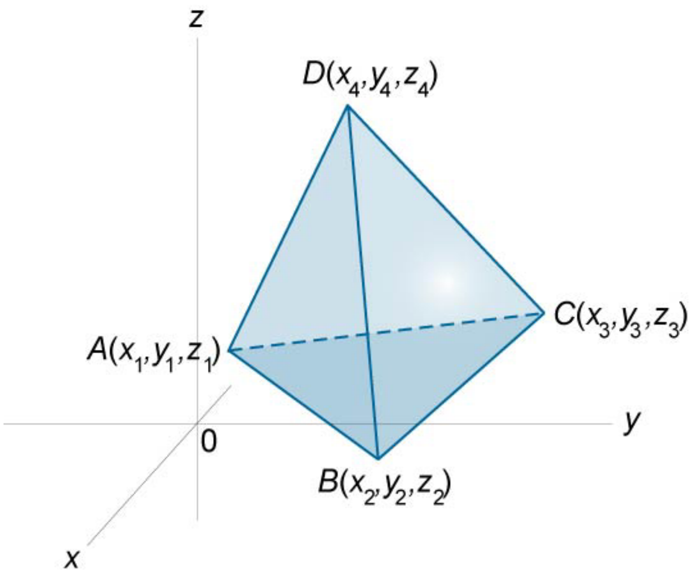


Figure 126.

7.9 Plane

Point coordinates: x, y, z, x_0, y_0, z_0, x_1, y_1, z_1, …
Real numbers: A, B, C, D, A_1, A_2, a, b, c, a_1, a_2, λ, p, t, …
Normal vectors: $\vec{n}$, $\vec{n}_1$, $\vec{n}_2$
Direction cosines: $\cos\alpha$, $\cos\beta$, $\cos\gamma$
Distance from point to plane: d

675. General Equation of a Plane
$$Ax + By + Cz + D = 0$$

676. Normal Vector to a Plane
The vector $\vec{n}(A, B, C)$ is normal to the plane
$Ax + By + Cz + D = 0$.

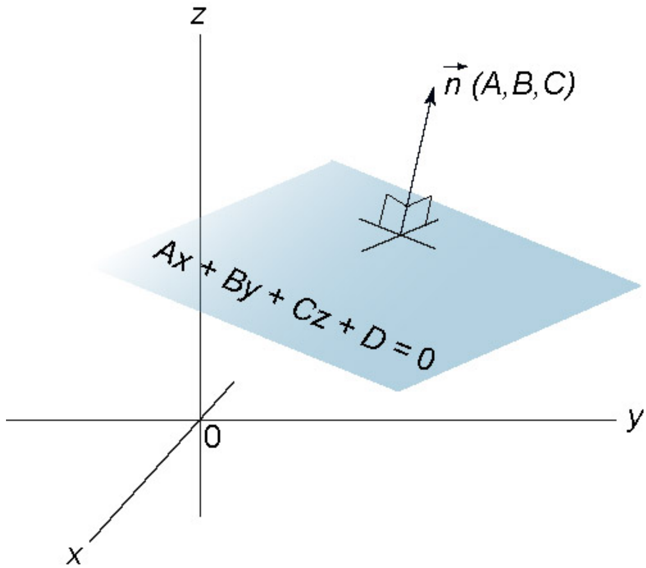


Figure 127.

677. Particular Cases of the Equation of a Plane
$Ax + By + Cz + D = 0$

If $A = 0$, the plane is parallel to the x-axis.
If $B = 0$, the plane is parallel to the y-axis.
If $C = 0$, the plane is parallel to the z-axis.
If $D = 0$, the plane lies on the origin.

If $A = B = 0$, the plane is parallel to the xy-plane.
If $B = C = 0$, the plane is parallel to the yz-plane.
If $A = C = 0$, the plane is parallel to the xz-plane.

678. Point Direction Form

$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$,

where the point $P(x_0, y_0, z_0)$ lies in the plane, and the vector (A, B, C) is normal to the plane.

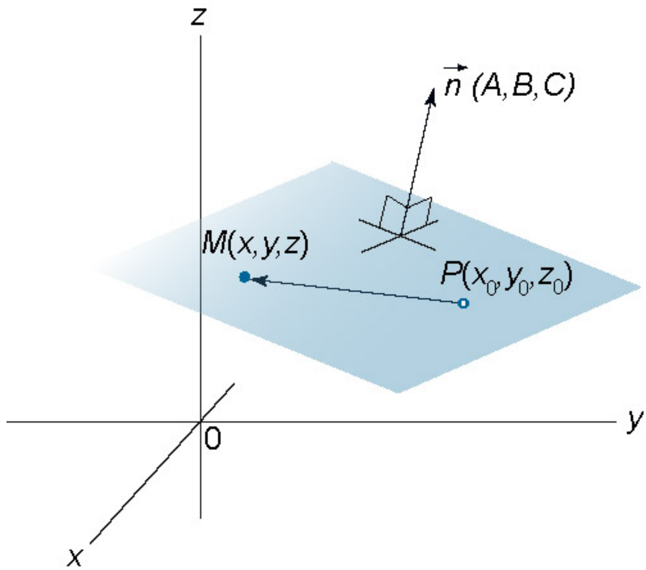


Figure 128.

679. Intercept Form

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

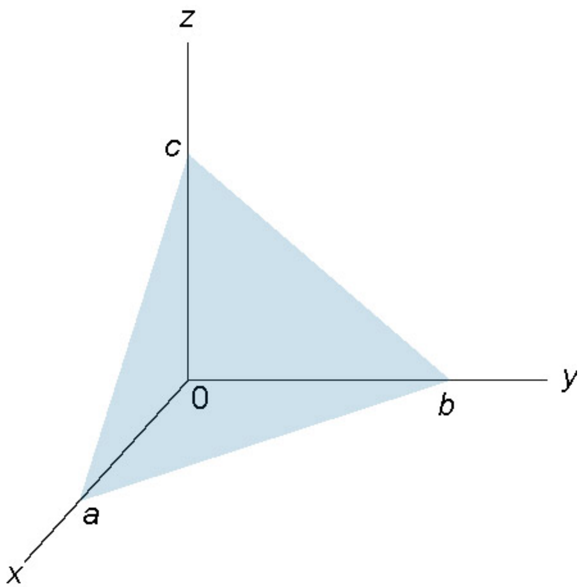


Figure 129.

680. Three Point Form

$$\begin{vmatrix} x - x_3 & y - y_3 & z - z_3 \\ x_1 - x_3 & y_1 - y_3 & z_1 - z_3 \\ x_2 - x_3 & y_2 - y_3 & z_2 - z_3 \end{vmatrix} = 0,$$

or

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0.$$

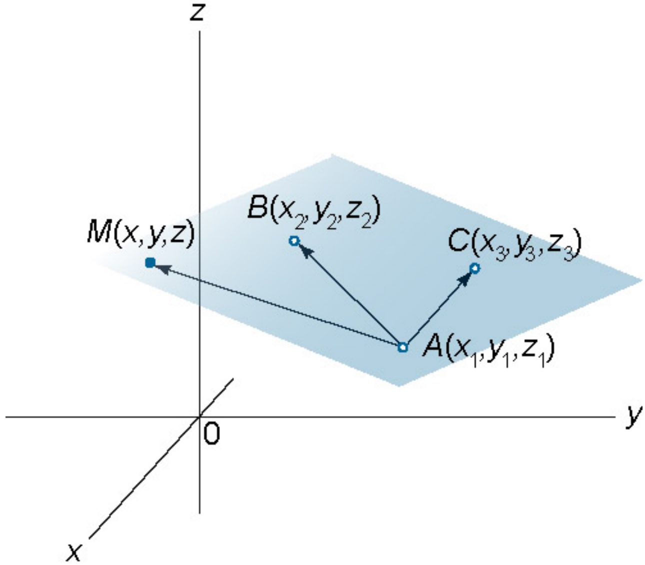


Figure 130.

681. Normal Form

$x\cos\alpha + y\cos\beta + z\cos\gamma - p = 0,$

where p is the perpendicular distance from the origin to the plane , and $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of any line normal to the plane.

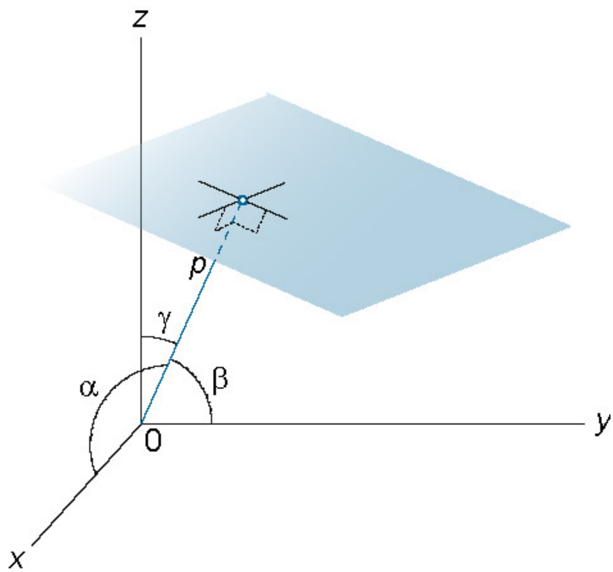


Figure 131.

682. Parametric Form

$$\begin{cases} x = x_1 + a_1 s + a_2 t \\ y = y_1 + b_1 s + b_2 t, \\ z = z_1 + c_1 s + c_2 t \end{cases}$$

where (x, y, z) are the coordinates of any unknown point on the line, the point $P(x_1, y_1, z_1)$ lies in the plane, the vectors (a_1, b_1, c_1) and (a_2, b_2, c_2) are parallel to the plane.

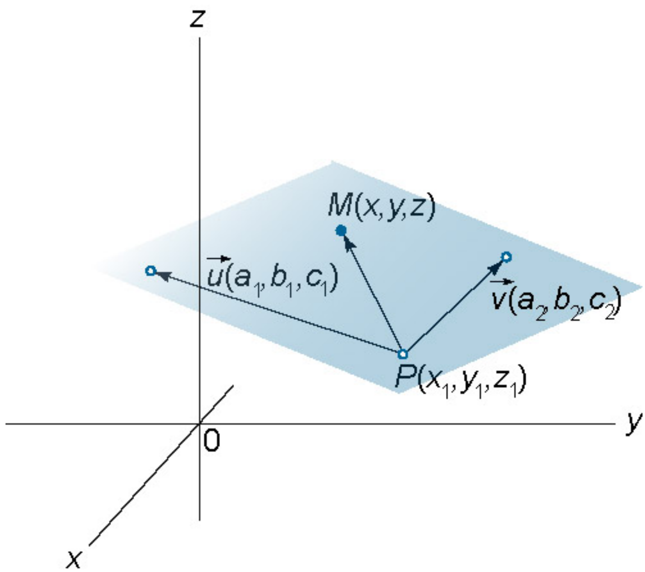


Figure 132.

683. Dihedral Angle Between Two Planes

If the planes are given by

$A_1x + B_1y + C_1z + D_1 = 0$,

$A_2x + B_2y + C_2z + D_2 = 0$,

then the dihedral angle between them is

$$\cos\varphi = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| \cdot |\vec{n}_2|} = \frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2} \cdot \sqrt{A_2^2 + B_2^2 + C_2^2}}.$$

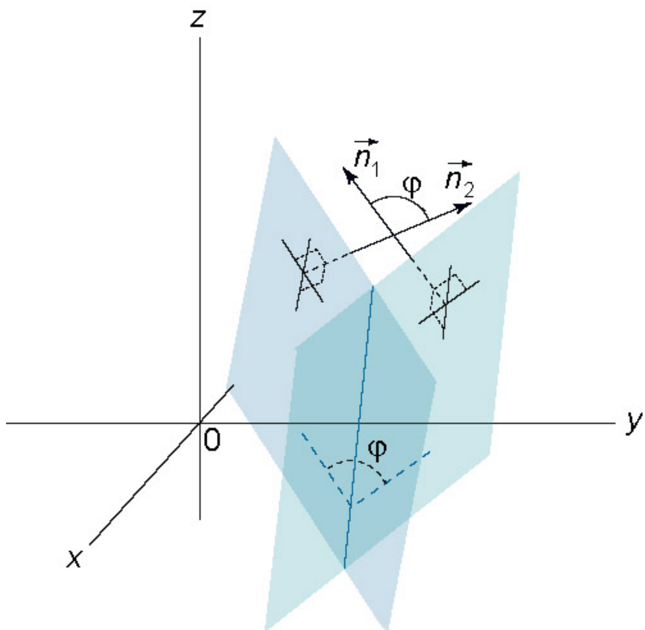


Figure 133.

684. Parallel Planes
Two planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$ are parallel if

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}.$$

685. Perpendicular Planes
Two planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$ are perpendicular if

$$A_1A_2 + B_1B_2 + C_1C_2 = 0.$$

686. Equation of a Plane Through $P(x_1, y_1, z_1)$ and Parallel To the Vectors (a_1, b_1, c_1) and (a_2, b_2, c_2) (Fig.132)

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

687. **Equation of a Plane Through $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, and Parallel To the Vector (a, b, c)**

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a & b & c \end{vmatrix} = 0$$

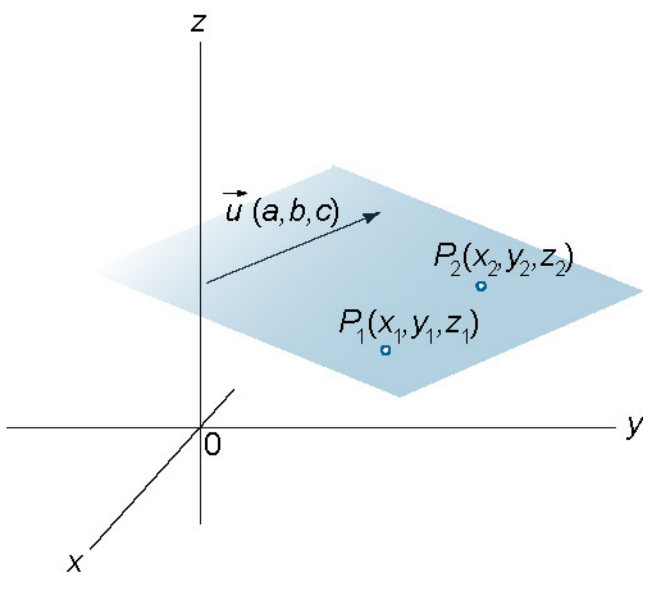


Figure 134.

688. **Distance From a Point To a Plane**

The distance from the point $P_1(x_1, y_1, z_1)$ to the plane $Ax + By + Cz + D = 0$ is

$$d = \left| \frac{Ax_1 + By_1 + Cz_1 + D}{\sqrt{A^2 + B^2 + C^2}} \right|.$$

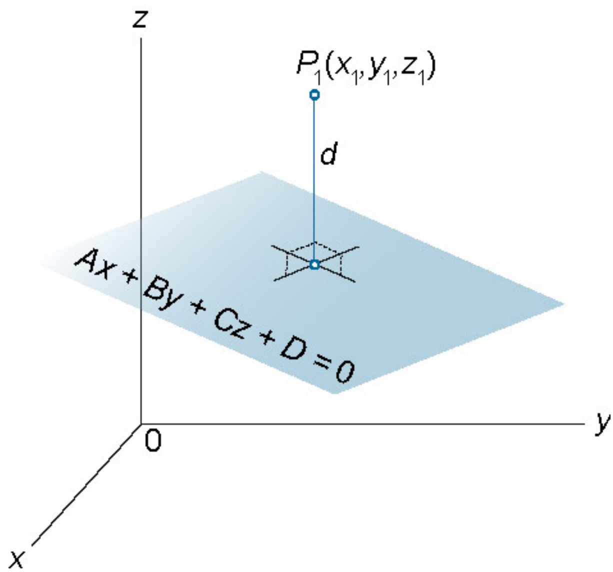


Figure 135.

689. Intersection of Two Planes

If two planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$ intersect, the intersection straight line is given by

$$\begin{cases} x = x_1 + at \\ y = y_1 + bt, \\ z = z_1 + ct \end{cases}$$

or

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c},$$

where

$$a = \begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix}, \; b = \begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix}, \; c = \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix},$$

$$x_1 = \frac{b\begin{vmatrix} D_1 & C_1 \\ D_2 & C_2 \end{vmatrix} - c\begin{vmatrix} D_1 & B_1 \\ D_2 & B_2 \end{vmatrix}}{a^2 + b^2 + c^2},$$

$$y_1 = \frac{c\begin{vmatrix} D_1 & A_1 \\ D_2 & A_2 \end{vmatrix} - a\begin{vmatrix} D_1 & C_1 \\ D_2 & C_2 \end{vmatrix}}{a^2 + b^2 + c^2},$$

$$z_1 = \frac{a\begin{vmatrix} D_1 & B_1 \\ D_2 & B_2 \end{vmatrix} - b\begin{vmatrix} D_1 & A_1 \\ D_2 & A_2 \end{vmatrix}}{a^2 + b^2 + c^2}.$$

7.10 Straight Line in Space

Point coordinates: x, y, z, x_1, y_1, z_1, …
Direction cosines: $\cos\alpha$, $\cos\beta$, $\cos\gamma$
Real numbers: A, B, C, D, a, b, c, a_1, a_2, t, …
Direction vectors of a line: $\vec{s}$, $\vec{s}_1$, $\vec{s}_2$
Normal vector to a plane: $\vec{n}$
Angle between two lines: φ

690. Point Direction Form of the Equation of a Line

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c},$$

where the point $P_1(x_1, y_1, z_1)$ lies on the line, and (a, b, c) is the direction vector of the line.

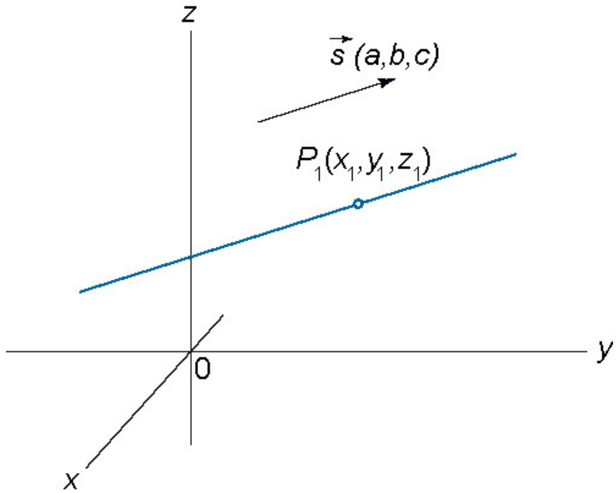


Figure 136.

691. Two Point Form

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

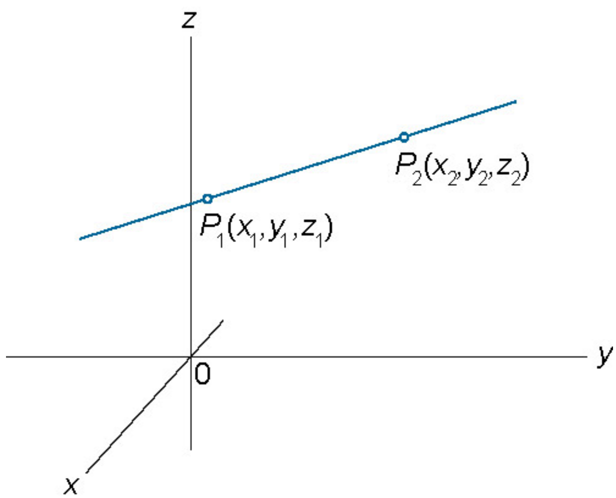


Figure 137.

692. Parametric Form

$$\begin{cases} x = x_1 + t\cos\alpha \\ y = y_1 + t\cos\beta \\ z = z_1 + t\cos\gamma \end{cases},$$

where the point $P_1(x_1, y_1, z_1)$ lies on the straight line, $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of the direction vector of the line, the parameter t is any real number.

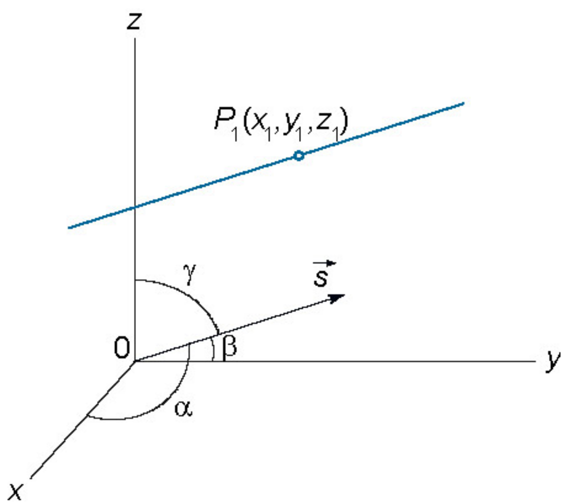


Figure 138.

693. Angle Between Two Straight Lines

$$\cos\varphi = \frac{\vec{s}_1 \cdot \vec{s}_2}{|\vec{s}_1| \cdot |\vec{s}_2|} = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \cdot \sqrt{a_2^2 + b_2^2 + c_2^2}}$$

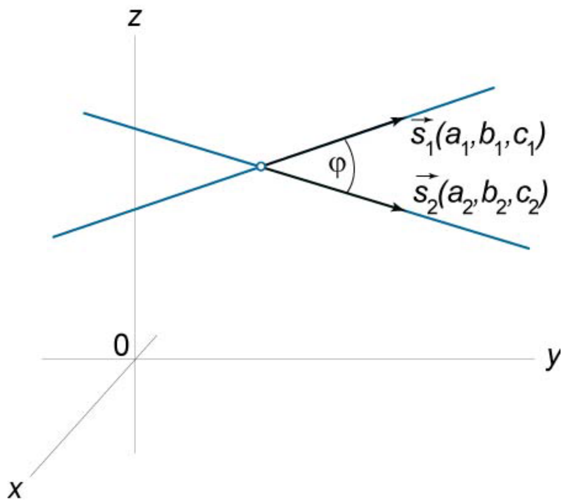


Figure 139.

694. Parallel Lines

Two lines are parallel if

$\vec{s}_1 \parallel \vec{s}_2$,

or

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}.$$

695. Perpendicular Lines

Two lines are parallel if

$\vec{s}_1 \cdot \vec{s}_2 = 0$,

or

$$a_1a_2 + b_1b_2 + c_1c_2 = 0.$$

696. Intersection of Two Lines

Two lines $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ and

$$\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$$ intersect if

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0.$$

697. Parallel Line and Plane

The straight line $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$ and the plane $Ax+By+Cz+D=0$ are parallel if

$\vec{n}\cdot\vec{s}=0$,

or

$Aa+Bb+Cc=0$.

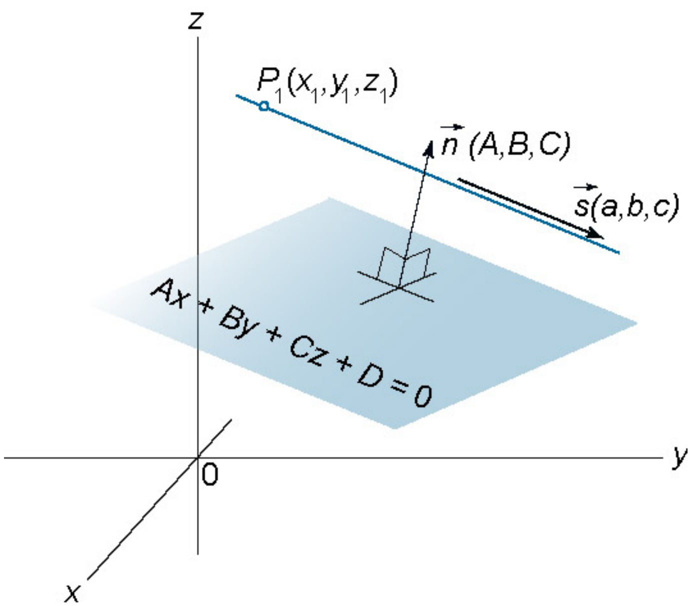


Figure 140.

698. Perpendicular Line and Plane

The straight line $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$ and the plane $Ax+By+Cz+D=0$ are perpendicular if

$\vec{n}\,||\,\vec{s}$,

or

$\frac{A}{a}=\frac{B}{b}=\frac{C}{c}$.

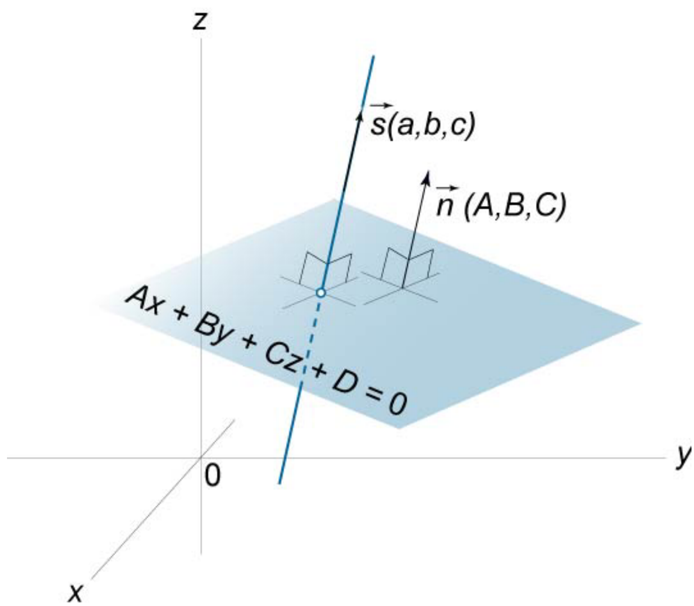


Figure 141.

7.11 Quadric Surfaces

Point coordinates of the quadric surfaces: x, y, z
Real numbers: A, B, C, a, b, c, k_1, k_2, k_3, …

699. General Quadratic Equation

$$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy + 2Px + 2Qy + 2Rz + D = 0$$

700. Classification of Quadric Surfaces

Case	Rank(e)	Rank(E)	Δ	k signs	Type of Surface
1	3	4	<0	Same	Real Ellipsoid
2	3	4	>0	Same	Imaginary Ellipsoid
3	3	4	>0	Different	Hyperboloid of 1 Sheet
4	3	4	<0	Different	Hyperboloid of 2 Sheets
5	3	3		Different	Real Quadric Cone
6	3	3		Same	Imaginary Quadric Cone
7	2	4	<0	Same	Elliptic Paraboloid
8	2	4	>0	Different	Hyperbolic Paraboloid
9	2	3		Same	Real Elliptic Cylinder
10	2	3		Same	Imaginary Elliptic Cylinder
11	2	3		Different	Hyperbolic Cylinder
12	2	2		Different	Real Intersecting Planes
13	2	2		Same	Imaginary Intersecting Planes
14	1	3			Parabolic Cylinder
15	1	2			Real Parallel Planes
16	1	2			Imaginary Parallel Planes
17	1	1			Coincident Planes

Here

$$e = \begin{pmatrix} A & H & G \\ H & B & F \\ G & F & C \end{pmatrix}, \quad E = \begin{pmatrix} A & H & Q & P \\ H & B & F & Q \\ G & F & C & R \\ P & Q & R & D \end{pmatrix}, \quad \Delta = \det(E),$$

k_1, k_2, k_3 are the roots of the equation,

$$\begin{vmatrix} A-x & H & G \\ H & B-x & F \\ G & F & C-x \end{vmatrix} = 0.$$

701. Real Ellipsoid (Case 1)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

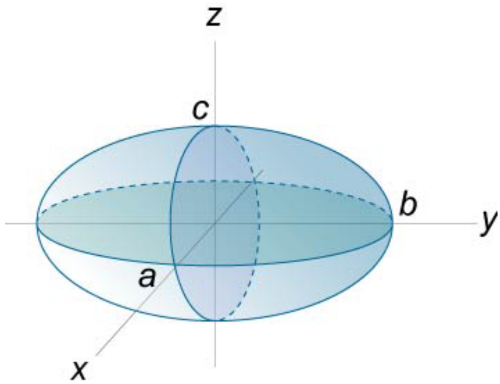


Figure 142.

702. Imaginary Ellipsoid (Case 2)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=-1$$

703. Hyperboloid of 1 Sheet (Case 3)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

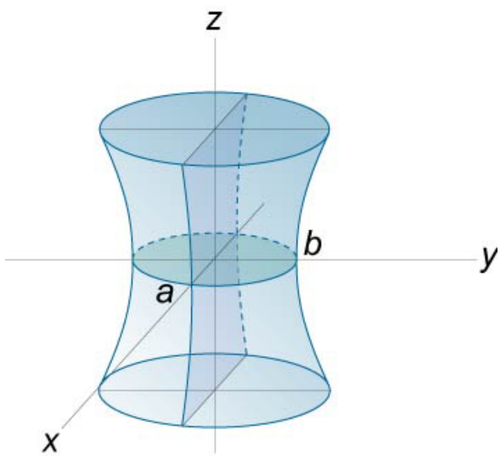


Figure 143.

704. Hyperboloid of 2 Sheets (Case 4)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$$

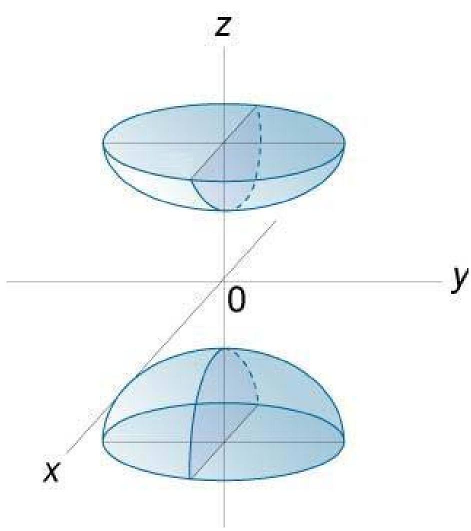


Figure 144.

705. Real Quadric Cone (Case 5)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$$

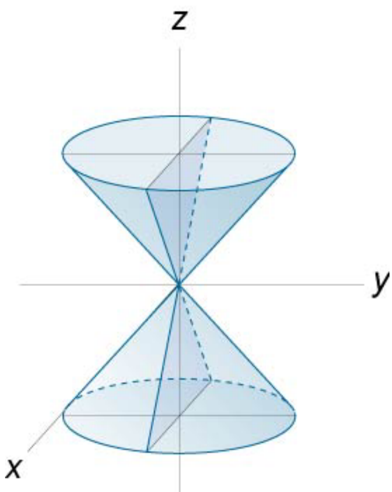


Figure 145.

706. Imaginary Quadric Cone (Case 6)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=0$$

707. Elliptic Paraboloid (Case 7)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-z=0$$

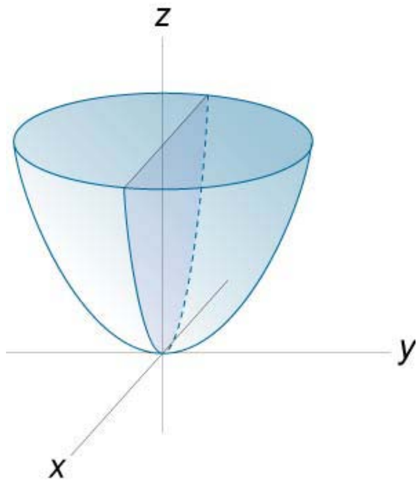


Figure 146.

708. Hyperbolic Paraboloid (Case 8)

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-z=0$$

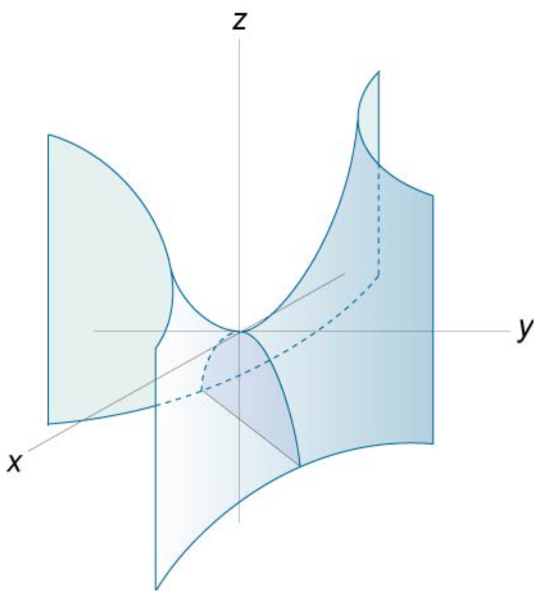


Figure 147.

709. Real Elliptic Cylinder (Case 9)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

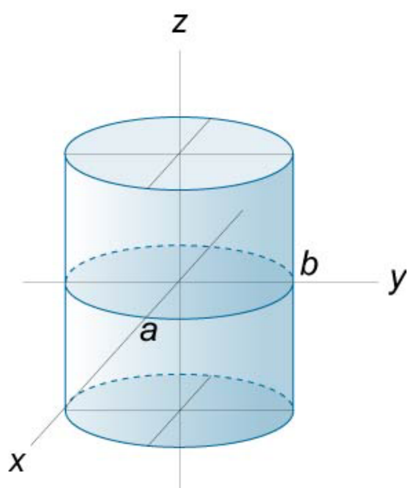


Figure 148.

710. Imaginary Elliptic Cylinder (Case 10)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=-1$$

711. Hyperbolic Cylinder (Case 11)

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

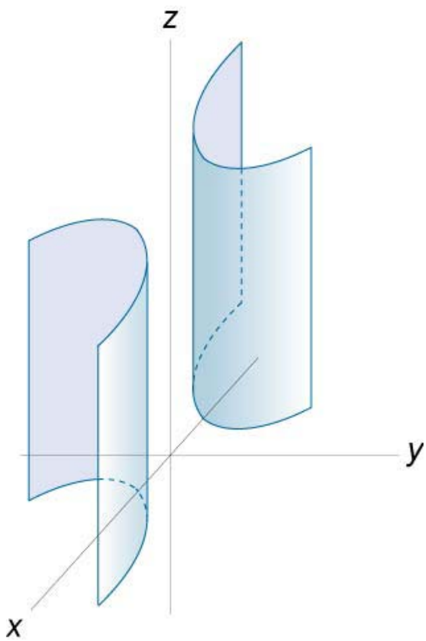


Figure 149.

712. Real Intersecting Planes (Case 12)

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$$

713. Imaginary Intersecting Planes (Case 13)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$$

714. Parabolic Cylinder (Case 14)

$$\frac{x^2}{a^2} - y = 0$$

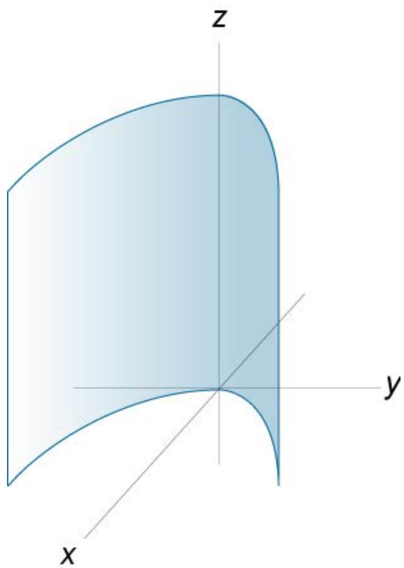


Figure 150.

715. Real Parallel Planes (Case 15)

$$\frac{x^2}{a^2}=1$$

716. Imaginary Parallel Planes (Case 16)

$$\frac{x^2}{a^2}=-1$$

717. Coincident Planes (Case 17)

$$x^2=0$$

7.12 Sphere

Radius of a sphere: R
Point coordinates: $x, y, z, x_1, y_1, z_1, \ldots$
Center of a sphere: (a, b, c)
Real numbers: A, D, E, F, M

718. Equation of a Sphere Centered at the Origin (Standard Form)
$$x^2 + y^2 + z^2 = R^2$$

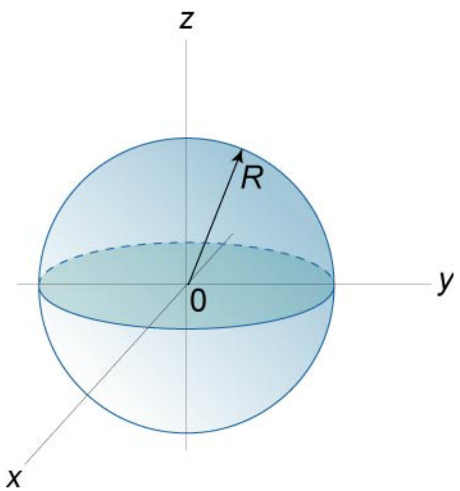


Figure 151.

719. Equation of a Circle Centered at Any Point (a, b, c)
$$(x-a)^2 + (y-b)^2 + (z-c)^2 = R^2$$

720. Diameter Form
$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0,$$

where
$P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$ are the ends of a diameter.

721. Four Point Form

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ x_1^2+y_1^2+x_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2+y_2^2+x_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2+y_3^2+x_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2+y_4^2+x_4^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

722. General Form

$Ax^2 + Ay^2 + Az^2 + Dx + Ey + Fz + M = 0$ (A is nonzero).
The center of the sphere has coordinates (a, b, c), where

$$a = -\frac{D}{2A}, \quad b = -\frac{E}{2A}, \quad c = -\frac{F}{2A}.$$

The radius of the sphere is

$$R = \frac{\sqrt{D^2 + E^2 + F^2 - 4A^2M}}{2A}.$$

Chapter 8

Differential Calculus

Functions: f, g, y, u, v
Argument (independent variable): x
Real numbers: a, b, c, d
Natural number: n
Angle: α
Inverse function: f^{-1}

8.1 Functions and Their Graphs

723. Even Function
$f(-x) = f(x)$

724. Odd Function
$f(-x) = -f(x)$

725. Periodic Function
$f(x + nT) = f(x)$

726. Inverse Function
$y = f(x)$ is any function, $x = g(y)$ or $y = f^{-1}(x)$ is its inverse function.

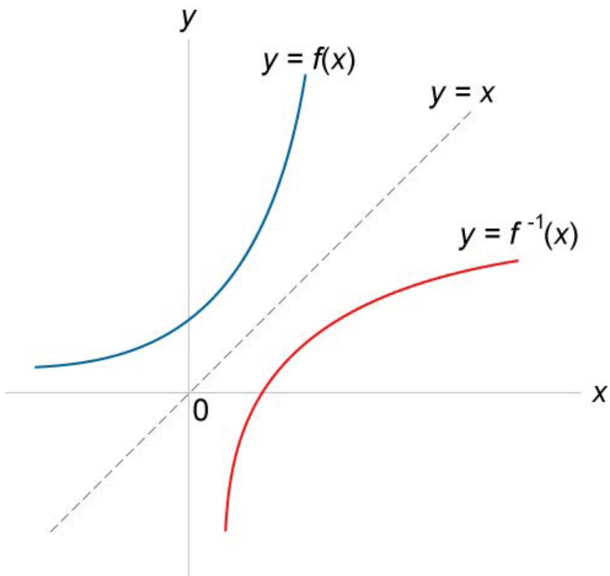


Figure 152.

727. Composite Function
$y = f(u)$, $u = g(x)$, $y = f(g(x))$ is a composite function.

728. Linear Function
$y = ax + b$, $x \in R$, $a = \tan\alpha$ is the slope of the line, b is the y-intercept.

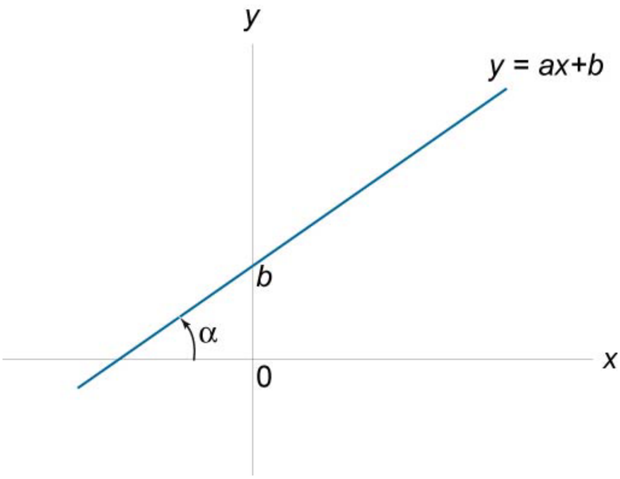


Figure 153.

729. Quadratic Function

$y = x^2, \ x \in R$.

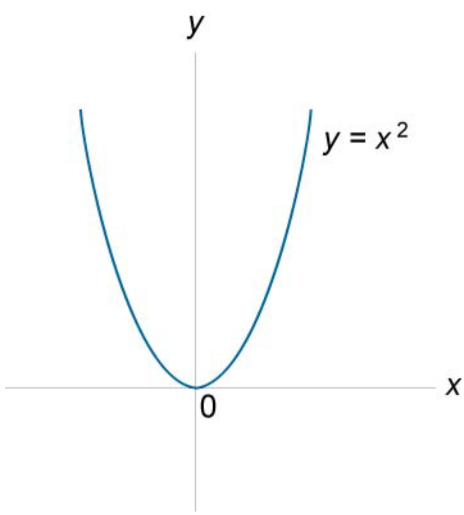


Figure 154.

730. $y = ax^2 + bx + c, \ x \in R.$

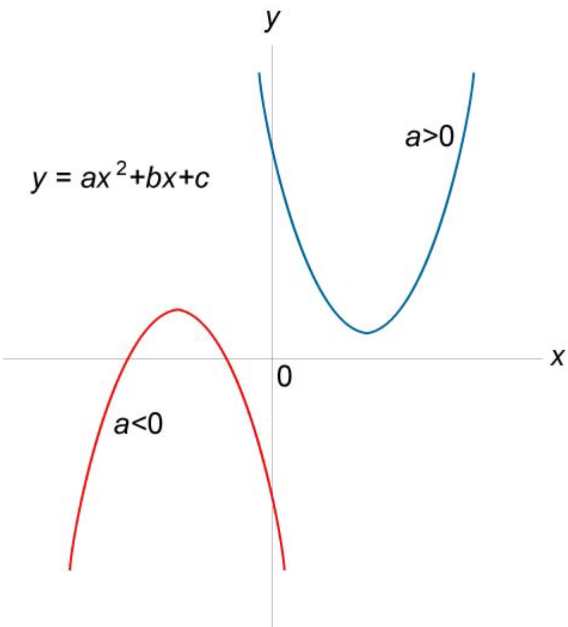


Figure 155.

731. Cubic Function
$y = x^3, \ x \in R.$

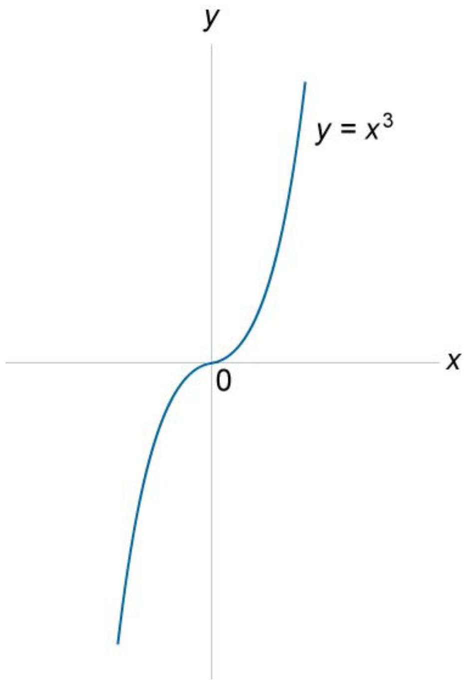


Figure 156.

732. $y = ax^3 + bx^2 + cx + d,\ x \in R.$

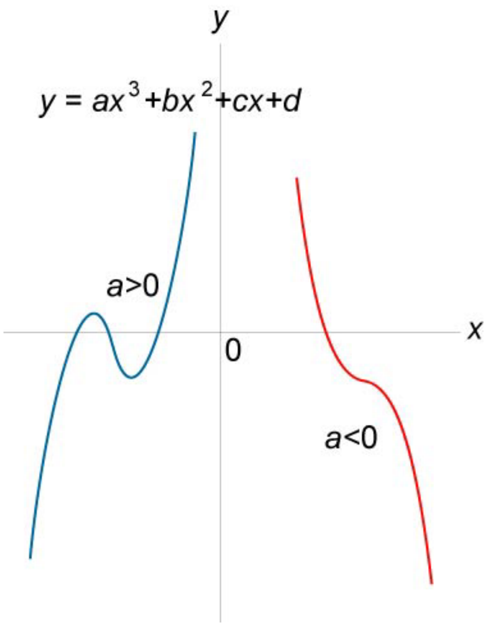


Figure 157.

733. Power Function

$y = x^n,\ n \in N$.

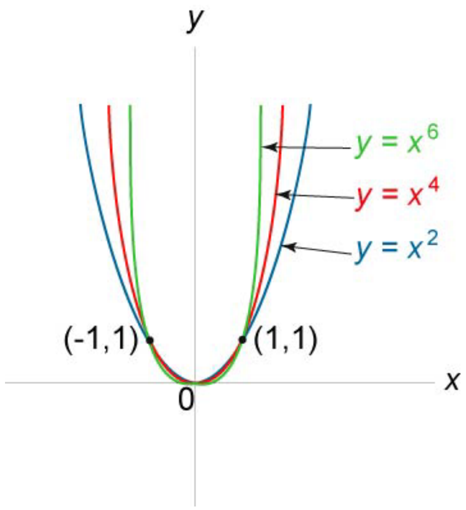


Figure 158.

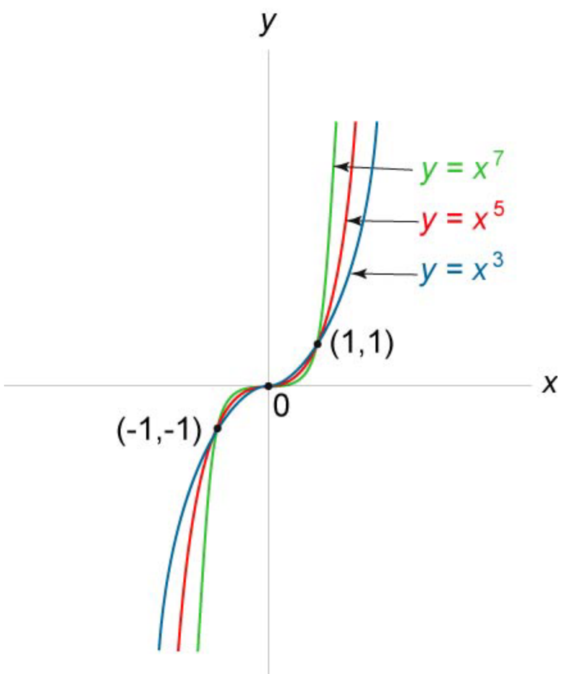


Figure 159.

734. Square Root Function

$y = \sqrt{x}$, $x \in [0, \infty)$.

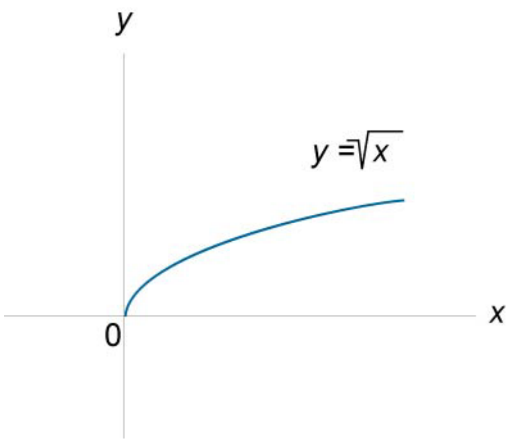


Figure 160.

735. Exponential Functions

$y = a^x$, $a > 0$, $a \neq 1$,

$y = e^x$ if $a = e$, $e = 2.71828182846\ldots$

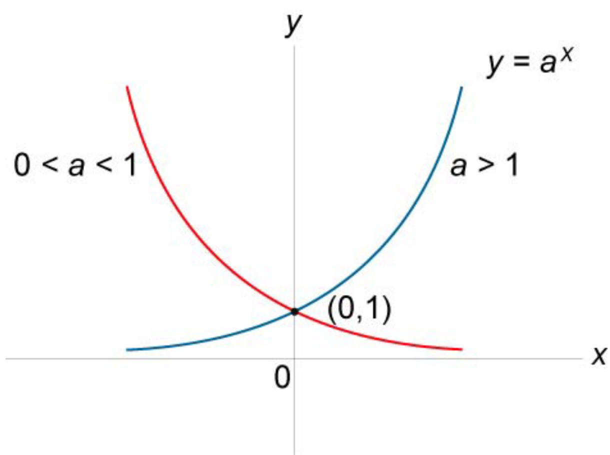


Figure 161.

736. Logarithmic Functions

$y = \log_a x$, $x \in (0, \infty)$, $a > 0$, $a \neq 1$,

$y = \ln x$ if $a = e$, $x > 0$.

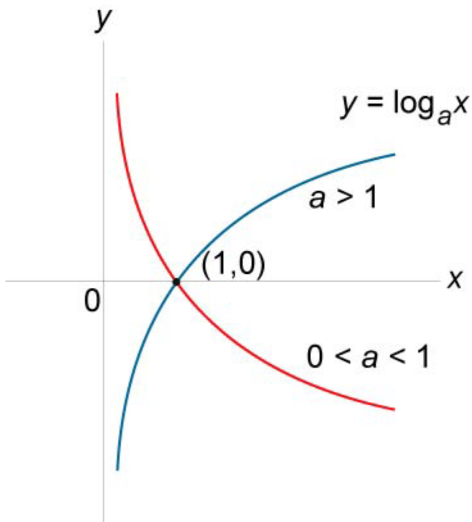


Figure 162.

737. Hyperbolic Sine Function

$$y = \sinh x, \ \sinh x = \frac{e^x - e^{-x}}{2}, \ x \in R.$$

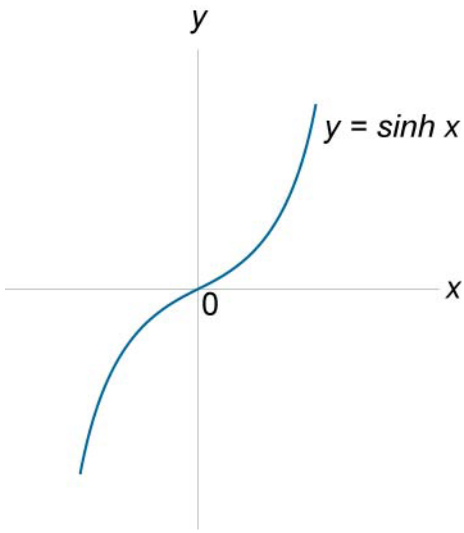


Figure 163.

738. Hyperbolic Cosine Function

$$y = \cosh x, \ \cosh x = \frac{e^x + e^{-x}}{2}, \ x \in R.$$

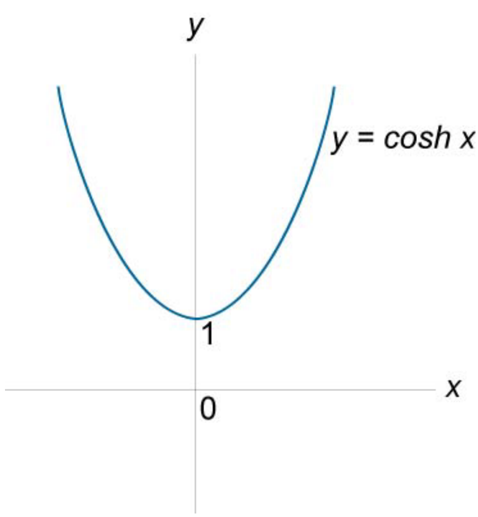


Figure 164.

739. Hyperbolic Tangent Function

$$y = \tanh x, \; y = \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \; x \in R.$$

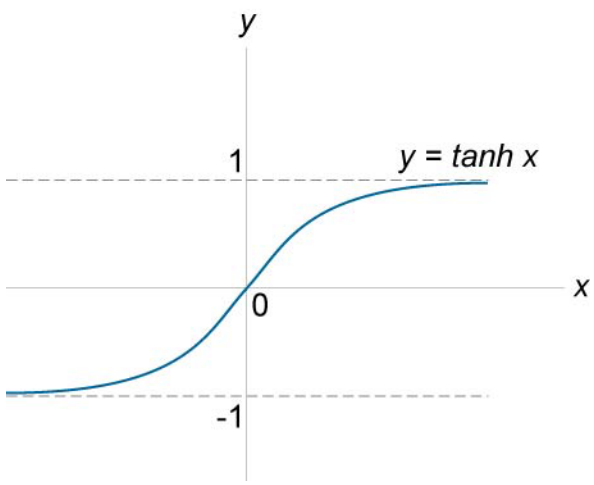


Figure 165.

740. Hyperbolic Cotangent Function

$$y = \coth x, \; y = \coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}, \; x \in R, \; x \neq 0.$$

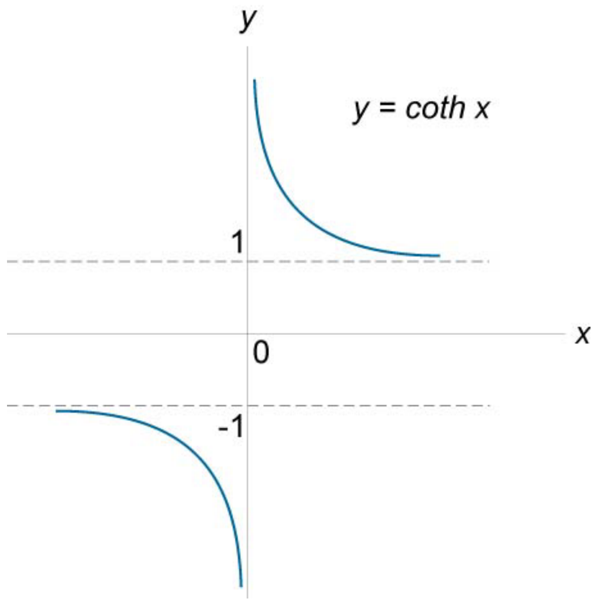


Figure 166.

741. Hyperbolic Secant Function

$$y = \operatorname{sech} x, \quad y = \operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}, \quad x \in R.$$

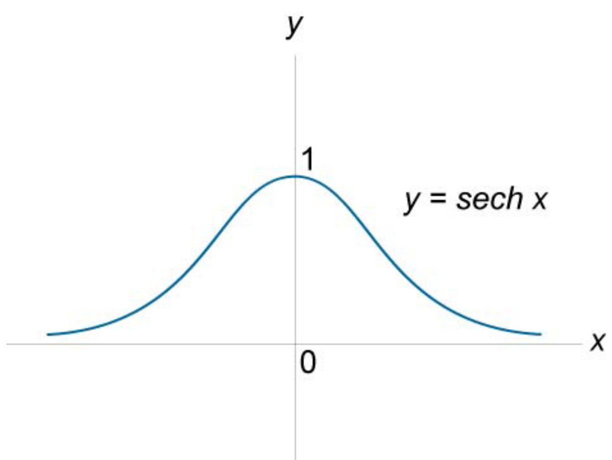


Figure 167.

742. Hyperbolic Cosecant Function

$$y = \operatorname{csch} x,\; y = \operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^{x} - e^{-x}},\; x \in R,\; x \neq 0.$$

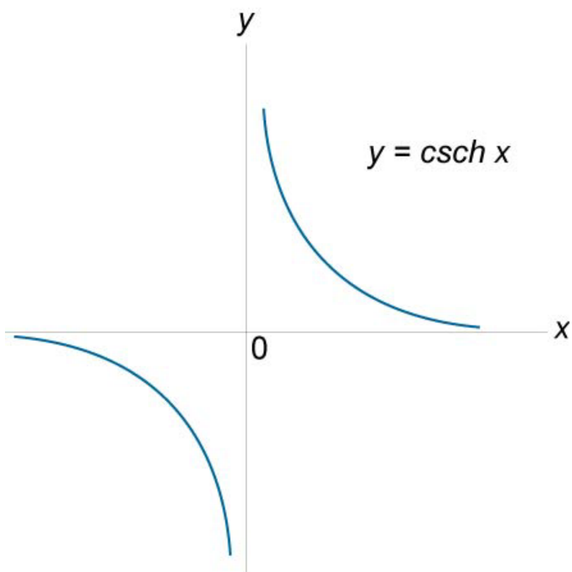


Figure 168.

743. Inverse Hyperbolic Sine Function

$$y = \operatorname{arcsinh} x,\; x \in R.$$

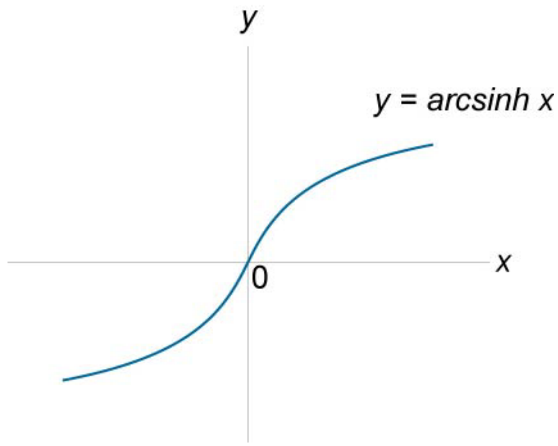


Figure 169.

744. Inverse Hyperbolic Cosine Function
$y = \operatorname{arccosh} x,\ x \in [1, \infty)$.

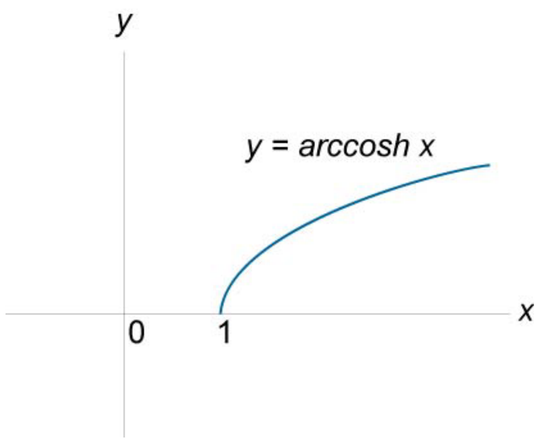


Figure 170.

745. Inverse Hyperbolic Tangent Function
$y = \operatorname{arctanh} x,\ x \in (-1, 1)$.

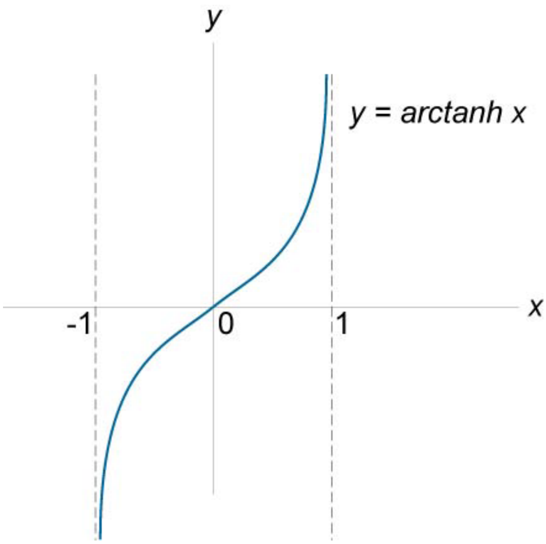


Figure 171.

746. **Inverse Hyperbolic Cotangent Function**
$y = \text{arccoth}\, x$, $x \in (-\infty, -1) \cup (1, \infty)$.

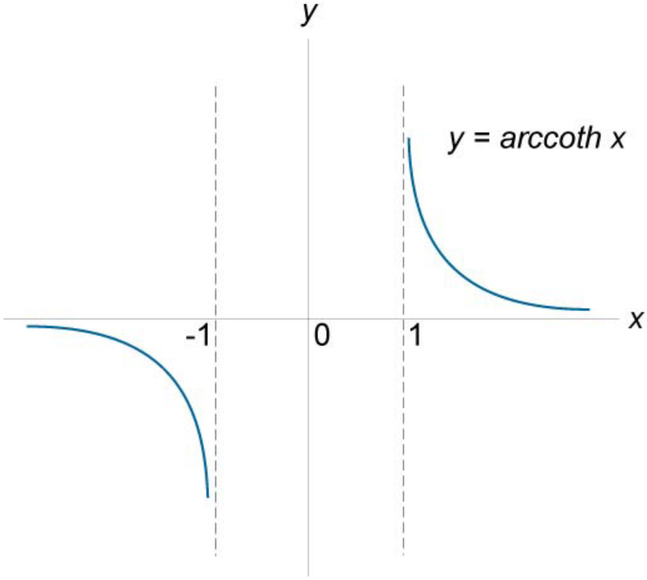


Figure 172.

747. **Inverse Hyperbolic Secant Function**
$y = \text{arcsech}\, x$, $x \in (0, 1]$.

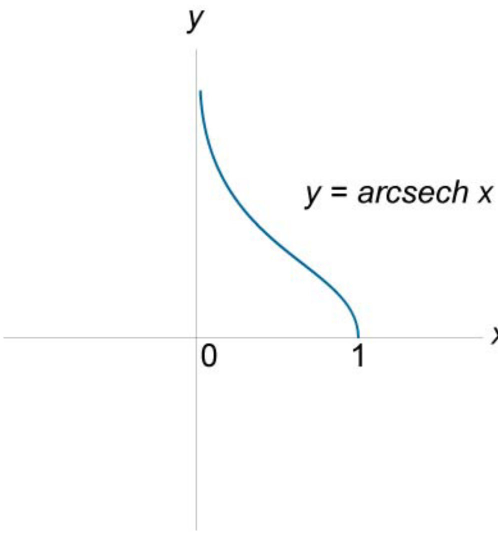


Figure 173.

748. Inverse Hyperbolic Cosecant Function

$y = \text{arccsch}\, x,\ x \in R,\ x \neq 0.$

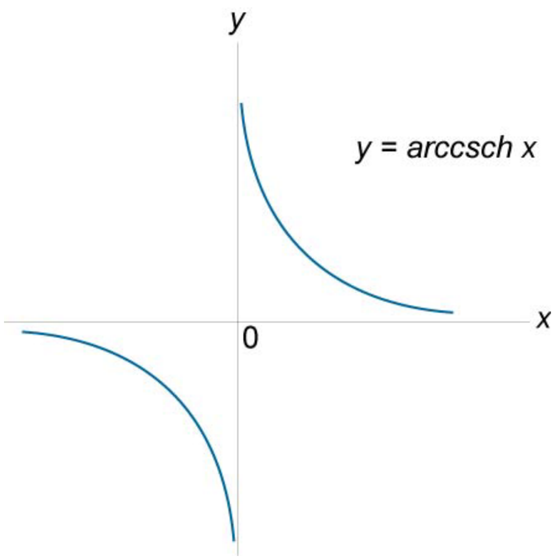


Figure 174.

8.2 Limits of Functions

Functions: $f(x)$, $g(x)$
Argument: x
Real constants: a, k

749. $\lim_{x\to a}[f(x)+g(x)]=\lim_{x\to a}f(x)+\lim_{x\to a}g(x)$

750. $\lim_{x\to a}[f(x)-g(x)]=\lim_{x\to a}f(x)-\lim_{x\to a}g(x)$

751. $\lim_{x\to a}[f(x)\cdot g(x)]=\lim_{x\to a}f(x)\cdot\lim_{x\to a}g(x)$

752. $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a}f(x)}{\lim_{x\to a}g(x)}$, if $\lim_{x\to a}g(x)\neq 0$.

753. $\lim_{x\to a}[kf(x)]=k\lim_{x\to a}f(x)$

754. $\lim_{x\to a}f(g(x))=f(\lim_{x\to a}g(x))$

755. $\lim_{x\to a}f(x)=f(a)$, if the function $f(x)$ is continuous at $x=a$.

756. $\lim_{x\to 0}\frac{\sin x}{x}=1$

757. $\lim_{x\to 0}\frac{\tan x}{x}=1$

758. $\lim_{x\to 0}\frac{\sin^{-1}x}{x}=1$

759. $\lim_{x \to 0} \frac{\tan^{-1} x}{x} = 1$

760. $\lim_{x \to 0} \frac{\ln(1+x)}{x} = 1$

761. $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$

762. $\lim_{x \to \infty} \left(1 + \frac{k}{x}\right)^x = e^k$

763. $\lim_{x \to 0} a^x = 1$

8.3 Definition and Properties of the Derivative

Functions: f, g, y, u, v
Independent variable: x
Real constant: k
Angle: α

764. $y'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$

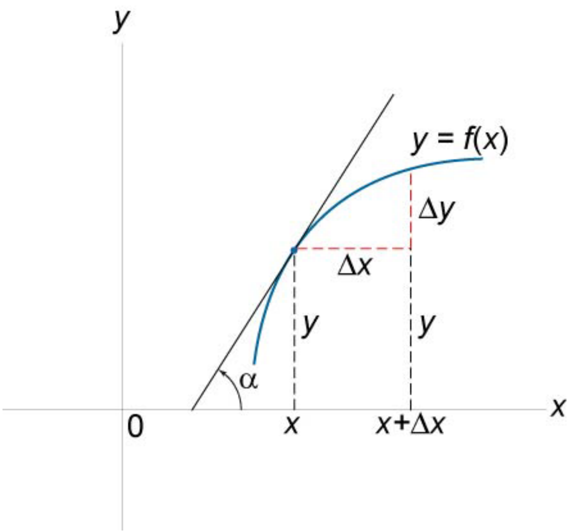


Figure 175.

765. $\frac{dy}{dx} = \tan\alpha$

766. $\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

767. $\frac{d(u-v)}{dx} = \frac{du}{dx} - \frac{dv}{dx}$

768. $\frac{d(ku)}{dx} = k\frac{du}{dx}$

769. Product Rule

$$\frac{d(u \cdot v)}{dx} = \frac{du}{dx} \cdot v + u \cdot \frac{dv}{dx}$$

770. Quotient Rule

$$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{\frac{du}{dx}\cdot v-u\cdot\frac{dv}{dx}}{v^2}$$

771. Chain Rule

$y=f(g(x)),\ u=g(x),$

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}.$$

772. Derivative of Inverse Function

$$\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}},$$

where $x(y)$ is the inverse function of $y(x)$.

773. Reciprocal Rule

$$\frac{d}{dx}\left(\frac{1}{y}\right)=-\frac{\frac{dy}{dx}}{y^2}$$

774. Logarithmic Differentiation

$y=f(x),\ \ln y=\ln f(x),$

$$\frac{dy}{dx}=f(x)\cdot\frac{d}{dx}[\ln f(x)].$$

8.4 Table of Derivatives

Independent variable: x
Real constants: C, a, b, c
Natural number: n

775. $\frac{d}{dx}(C)=0$

776. $\frac{d}{dx}(x)=1$

777. $\frac{d}{dx}(ax+b)=a$

778. $\frac{d}{dx}(ax^2+bx+c)=ax+b$

779. $\frac{d}{dx}(x^n)=nx^{n-1}$

780. $\frac{d}{dx}(x^{-n})=-\frac{n}{x^{n+1}}$

781. $\frac{d}{dx}\left(\frac{1}{x}\right)=-\frac{1}{x^2}$

782. $\frac{d}{dx}(\sqrt{x})=\frac{1}{2\sqrt{x}}$

783. $\frac{d}{dx}(\sqrt[n]{x})=\frac{1}{n\sqrt[n]{x^{n-1}}}$

784. $\frac{d}{dx}(\ln x)=\frac{1}{x}$

785. $\frac{d}{dx}(\log_a x)=\frac{1}{x\ln a}$, $a>0$, $a\neq 1$.

786. $\frac{d}{dx}\left(a^x\right)=a^x \ln a,\ a>0,\ a\neq 1.$

787. $\frac{d}{dx}\left(e^x\right)=e^x$

788. $\frac{d}{dx}(\sin x)=\cos x$

789. $\frac{d}{dx}(\cos x)=-\sin x$

790. $\frac{d}{dx}(\tan x)=\frac{1}{\cos^2 x}=\sec^2 x$

791. $\frac{d}{dx}(\cot x)=-\frac{1}{\sin^2 x}=-\csc^2 x$

792. $\frac{d}{dx}(\sec x)=\tan x\cdot \sec x$

793. $\frac{d}{dx}(\csc x)=-\cot x\cdot \csc x$

794. $\frac{d}{dx}(\arcsin x)=\frac{1}{\sqrt{1-x^2}}$

795. $\frac{d}{dx}(\arccos x)=-\frac{1}{\sqrt{1-x^2}}$

796. $\frac{d}{dx}(\arctan x)=\frac{1}{1+x^2}$

797. $\frac{d}{dx}(\operatorname{arccot} x) = -\frac{1}{1+x^2}$

798. $\frac{d}{dx}(\operatorname{arcsec} x) = \frac{1}{|x|\sqrt{x^2-1}}$

799. $\frac{d}{dx}(\operatorname{arccsc} x) = -\frac{1}{|x|\sqrt{x^2-1}}$

800. $\frac{d}{dx}(\sinh x) = \cosh x$

801. $\frac{d}{dx}(\cosh x) = \sinh x$

802. $\frac{d}{dx}(\tanh x) = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x$

803. $\frac{d}{dx}(\coth x) = -\frac{1}{\sinh^2 x} = -\operatorname{csch}^2 x$

804. $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \cdot \tanh x$

805. $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \cdot \coth x$

806. $\frac{d}{dx}(\operatorname{arcsinh} x) = \frac{1}{\sqrt{x^2+1}}$

807. $\frac{d}{dx}(\operatorname{arccosh} x) = \frac{1}{\sqrt{x^2-1}}$

808. $\frac{d}{dx}(\operatorname{arctanh} x) = \frac{1}{1-x^2}$, $|x| < 1$.

809. $\frac{d}{dx}(\operatorname{arccoth} x) = -\frac{1}{x^2-1}$, $|x| > 1$.

810. $\frac{d}{dx}(u^v) = vu^{v-1} \cdot \frac{du}{dx} + u^v \ln u \cdot \frac{dv}{dx}$

8.5 Higher Order Derivatives

Functions: f, y, u, v
Independent variable: x
Natural number: n

811. Second derivative

$$f'' = (f')' = \left(\frac{dy}{dx}\right)' = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

812. Higher-Order derivative

$$f^{(n)} = \frac{d^ny}{dx^n} = y^{(n)} = \left(f^{(n-1)}\right)'$$

813. $(u+v)^{(n)} = u^{(n)} + v^{(n)}$

814. $(u-v)^{(n)} = u^{(n)} - v^{(n)}$

815. Leibnitz's Formulas

$$(uv)'' = u''v + 2u'v' + uv''$$

$$(uv)''' = u'''v + 3u''v' + 3u'v'' + uv'''$$

$$(uv)^{(n)} = u^{(n)}v + nu^{(n-1)}v' + \frac{n(n-1)}{1 \cdot 2}u^{(n-2)}v'' + \ldots + uv^{(n)}$$

816. $$(x^m)^{(n)} = \frac{m!}{(m-n)!}x^{m-n}$$

817. $$(x^n)^{(n)} = n!$$

818. $$(\log_a x)^{(n)} = \frac{(-1)^{n-1}(n-1)!}{x^n \ln a}$$

819. $$(\ln x)^{(n)} = \frac{(-1)^{n-1}(n-1)!}{x^n}$$

820. $$(a^x)^{(n)} = a^x \ln^n a$$

821. $$(e^x)^{(n)} = e^x$$

822. $$(a^{mx})^{(n)} = m^n a^{mx} \ln^n a$$

823. $$(\sin x)^{(n)} = \sin\left(x + \frac{n\pi}{2}\right)$$

824. $$(\cos x)^{(n)} = \cos\left(x + \frac{n\pi}{2}\right)$$

8.6 Applications of Derivative

Functions: f, g, y
Position of an object: s
Velocity: v
Acceleration: w
Independent variable: x
Time: t
Natural number: n

825. Velocity and Acceleration
$s = f(t)$ is the position of an object relative to a fixed coordinate system at a time t,
$v = s' = f'(t)$ is the instantaneous velocity of the object,
$w = v' = s'' = f''(t)$ is the instantaneous acceleration of the object.

826. Tangent Line
$y - y_0 = f'(x_0)(x - x_0)$

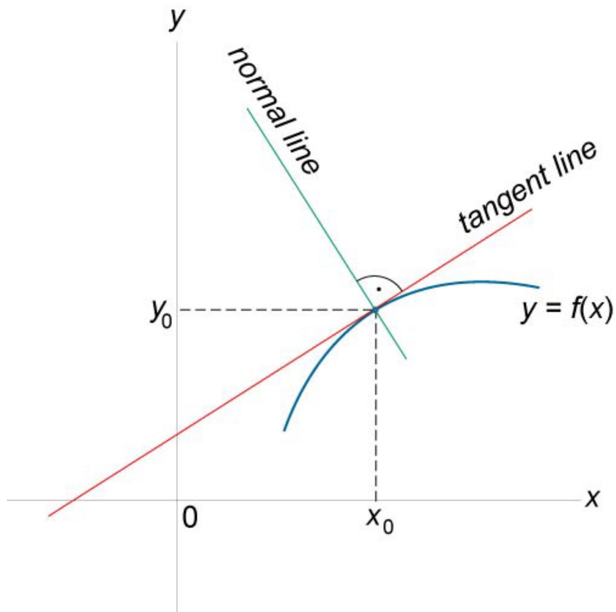


Figure 176.

827. Normal Line

$$y - y_0 = -\frac{1}{f'(x_0)}(x - x_0) \text{ (Fig 176)}$$

828. Increasing and Decreasing Functions.
If $f'(x_0) > 0$, then f(x) is increasing at x_0. (Fig 177, $x < x_1$, $x_2 < x$),
If $f'(x_0) < 0$, then f(x) is decreasing at x_0. (Fig 177, $x_1 < x < x_2$),
If $f'(x_0)$ does not exist or is zero, then the test fails.

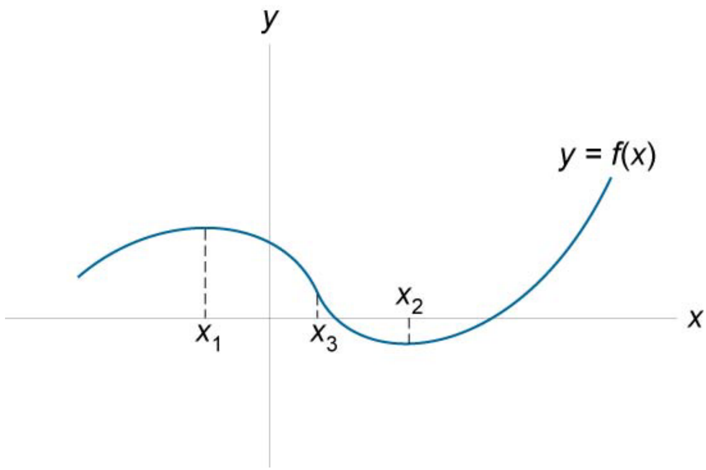


Figure 177.

829. Local extrema
A function f(x) has a local maximum at x_1 if and only if there exists some interval containing x_1 such that $f(x_1) \geq f(x)$ for all x in the interval (Fig.177).

A function f(x) has a local minimum at x_2 if and only if there exists some interval containing x_2 such that $f(x_2) \leq f(x)$ for all x in the interval (Fig.177).

830. Critical Points
A critical point on f(x) occurs at x_0 if and only if either $f'(x_0)$ is zero or the derivative doesn't exist.

831. First Derivative Test for Local Extrema.
If f(x) is increasing ($f'(x) > 0$) for all x in some interval $(a, x_1]$ and f(x) is decreasing ($f'(x) < 0$) for all x in some interval $[x_1, b)$, then f(x) has a local maximum at x_1 (Fig.177).

832. If f(x) is decreasing ($f'(x)<0$) for all x in some interval $(a, x_2]$ and f(x) is increasing ($f'(x)>0$) for all x in some interval $[x_2, b)$, then f(x) has a local minimum at x_2. (Fig.177).

833. Second Derivative Test for Local Extrema.
If $f'(x_1)=0$ and $f''(x_1)<0$, then f(x) has a local maximum at x_1.
If $f'(x_2)=0$ and $f''(x_2)>0$, then f(x) has a local minimum at x_2. (Fig.177)

834. Concavity.
f(x) is concave upward at x_0 if and only if $f'(x)$ is increasing at x_0 (Fig.177, $x_3<x$).
f(x) is concave downward at x_0 if and only if $f'(x)$ is decreasing at x_0. (Fig.177, $x<x_3$).

835. Second Derivative Test for Concavity.
If $f''(x_0)>0$, then f(x) is concave upward at x_0.
If $f''(x_0)<0$, then f(x) is concave downward at x_0.
If $f''(x)$ does not exist or is zero, then the test fails.

836. Inflection Points
If $f'(x_3)$ exists and $f''(x)$ changes sign at $x=x_3$, then the point $(x_3, f(x_3))$ is an inflection point of the graph of $f(x)$. If $f''(x_3)$ exists at the inflection point, then $f''(x_3)=0$ (Fig.177).

837. L'Hopital's Rule

$$\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{f'(x)}{g'(x)} \text{ if } \lim_{x\to c} f(x)=\lim_{x\to c} g(x)=\begin{cases}0\\ \infty\end{cases}.$$

8.7 Differential

Functions: f, u, v
Independent variable: x
Derivative of a function: $y'(x)$, $f'(x)$
Real constant: C
Differential of function $y = f(x)$: dy
Differential of x: dx
Small change in x: Δx
Small change in y: Δy

838. $dy = y'dx$

839. $f(x + \Delta x) = f(x) + f'(x)\Delta x$

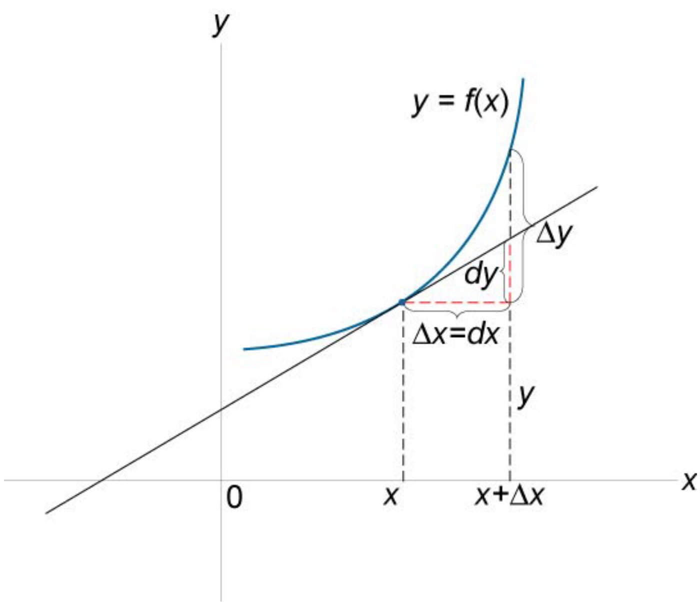


Figure 178.

840. Small Change in y
$\Delta y = f(x + \Delta x) - f(x)$

841. $d(u + v) = du + dv$

842. $d(u - v) = du - dv$

843. $d(Cu) = Cdu$

844. $d(uv) = vdu + udv$

845. $d\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2}$

8.8 Multivariable Functions

Functions of two variables: $z(x,y)$, $f(x,y)$, $g(x,y)$, $h(x,y)$
Arguments: x, y, t
Small changes in x, y, z, respectively: Δx, Δy, Δz.

846. First Order Partial Derivatives
The partial derivative with respect to x
$\frac{\partial f}{\partial x} = f_x$ (also $\frac{\partial z}{\partial x} = z_x$),
The partial derivative with respect to y
$\frac{\partial f}{\partial y} = f_y$ (also $\frac{\partial z}{\partial y} = z_y$).

847. Second Order Partial Derivatives

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial x^2}=f_{xx},$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial y^2}=f_{yy},$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial y\partial x}=f_{xy},$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial x\partial y}=f_{yx}.$$

If the derivatives are continuous, then

$$\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial^2 f}{\partial x\partial y}.$$

848. Chain Rules

If $f(x,y)=g(h(x,y))$ (g is a function of one variable h), then

$$\frac{\partial f}{\partial x}=g'(h(x,y))\frac{\partial h}{\partial x},\quad \frac{\partial f}{\partial y}=g'(h(x,y))\frac{\partial h}{\partial y}.$$

If $h(t)=f(x(t),y(t))$, then $h'(t)=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}$.

If $z=f(x(u,v),y(u,v))$, then

$$\frac{\partial z}{\partial u}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial u},\quad \frac{\partial z}{\partial v}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

849. Small Changes

$$\Delta z\approx\frac{\partial f}{\partial x}\Delta x+\frac{\partial f}{\partial y}\Delta y$$

850. Local Maxima and Minima
$f(x,y)$ has a local maximum at (x_0,y_0) if $f(x,y)\le f(x_0,y_0)$ for all (x,y) sufficiently close to (x_0,y_0).

$f(x,y)$ has a local minimum at (x_0,y_0) if $f(x,y)\ge f(x_0,y_0)$ for all (x,y) sufficiently close to (x_0,y_0).

851. Stationary Points
$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}=0.$$
Local maxima and local minima occur at stationary points.

852. Saddle Point
A stationary point which is neither a local maximum nor a local minimum

853. Second Derivative Test for Stationary Points
Let (x_0,y_0) be a stationary point ($\frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}=0$).
$$D=\begin{vmatrix} f_{xx}(x_0,y_0) & f_{xy}(x_0,y_0) \\ f_{yx}(x_0,y_0) & f_{yy}(x_0,y_0) \end{vmatrix}.$$

If $D>0$, $f_{xx}(x_0,y_0)>0$, (x_0,y_0) is a point of local minima.
If $D>0$, $f_{xx}(x_0,y_0)<0$, (x_0,y_0) is a point of local maxima.
If $D<0$, (x_0,y_0) is a saddle point.
If $D=0$, the test fails.

854. Tangent Plane
The equation of the tangent plane to the surface $z=f(x,y)$ at (x_0,y_0,z_0) is
$$z-z_0=f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0).$$

855. Normal to Surface
The equation of the normal to the surface $z = f(x, y)$ at (x_0, y_0, z_0) is

$$\frac{x - x_0}{f_x(x_0, y_0)} = \frac{y - y_0}{f_y(x_0, y_0)} = \frac{z - z_0}{-1}.$$

8.9 Differential Operators

Unit vectors along the coordinate axes: $\vec{i}$, $\vec{j}$, $\vec{k}$
Scalar functions (scalar fields): $f(x, y, z)$, $u(x_1, x_2, \ldots, x_n)$
Gradient of a scalar field: $\text{grad}\, u$, ∇u
Directional derivative: $\frac{\partial f}{\partial l}$
Vector function (vector field): $\vec{F}(P, Q, R)$
Divergence of a vector field: $\text{div}\, \vec{F}$, $\nabla \cdot \vec{F}$
Curl of a vector field: $\text{curl}\, \vec{F}$, $\nabla \times \vec{F}$
Laplacian operator: ∇^2

856. Gradient of a Scalar Function

$$\text{grad}\, f = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right),$$

$$\text{grad}\, u = \nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n} \right).$$

857. Directional Derivative

$$\frac{\partial f}{\partial l} = \frac{\partial f}{\partial x} \cos\alpha + \frac{\partial f}{\partial y} \cos\beta + \frac{\partial f}{\partial z} \cos\gamma,$$

where the direction is defined by the vector
$\vec{l}(\cos\alpha, \cos\beta, \cos\gamma)$, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

858. Divergence of a Vector Field

$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

859. Curl of a Vector Field

$$\operatorname{curl}\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial x} & \frac{\partial}{\partial x} \\ P & Q & R \end{vmatrix}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

860. Laplacian Operator

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

861. $\operatorname{div}\left(\operatorname{curl}\vec{F}\right) = \nabla\cdot\left(\nabla\times\vec{F}\right) \equiv 0$

862. $\operatorname{curl}\left(\operatorname{grad} f\right) = \nabla\times\left(\nabla f\right) \equiv 0$

863. $\operatorname{div}\left(\operatorname{grad} f\right) = \nabla\cdot\left(\nabla f\right) = \nabla^2 f$

864. $\operatorname{curl}\left(\operatorname{curl}\vec{F}\right) = \operatorname{grad}\left(\operatorname{div}\vec{F}\right) - \nabla^2\vec{F} = \nabla\left(\nabla\cdot\vec{F}\right) - \nabla^2\vec{F}$

Chapter 9

Integral Calculus

Functions: f, g, u, v
Independent variables: x, t, ξ
Indefinite integral of a function: $\int f(x)dx$, $\int g(x)dx$, ...
Derivative of a function: $y'(x)$, $f'(x)$, $F'(x)$, ...
Real constants: C, a, b, c, d, k
Natural numbers: m, n, i, j

9.1 Indefinite Integral

865. $\int f(x)dx = F(x) + C$ if $F'(x) = f(x)$.

866. $\left(\int f(x)dx\right)' = f(x)$

867. $\int kf(x)dx = k\int f(x)dx$

868. $\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$

869. $\int [f(x) - g(x)]dx = \int f(x)dx - \int g(x)dx$

870. $\int f(ax)dx = \frac{1}{a}F(ax) + C$

871. $\int f(ax+b)dx = \frac{1}{a}F(ax+b)+C$

872. $\int f(x)f'(x)dx = \frac{1}{2}f^2(x)+C$

873. $\int \frac{f'(x)}{f(x)}dx = \ln|f(x)|+C$

874. Method of Substitution
$\int f(x)dx = \int f(u(t))u'(t)dt$ if $x = u(t)$.

875. Integration by Parts
$\int udv = uv - \int vdu$,
where $u(x)$, $v(x)$ are differentiable functions.

9.2 Integrals of Rational Functions

876. $\int adx = ax + C$

877. $\int xdx = \frac{x^2}{2} + C$

878. $\int x^2dx = \frac{x^3}{3} + C$

879. $\int x^p dx = \frac{x^{p+1}}{p+1} + C$, $p \neq -1$.

880. $$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C,\ n \neq -1.$$

881. $$\int \frac{dx}{x} = \ln|x| + C$$

882. $$\int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b| + C$$

883. $$\int \frac{ax+b}{cx+d}dx = \frac{a}{c}x + \frac{bc-ad}{c^2}\ln|cx+d| + C$$

884. $$\int \frac{dx}{(x+a)(x+b)} = \frac{1}{a-b}\ln\left|\frac{x+b}{x+a}\right| + C,\ a \neq b.$$

885. $$\int \frac{xdx}{a+bx} = \frac{1}{b^2}\left(a+bx-a\ln|a+bx|\right) + C$$

886. $$\int \frac{x^2dx}{a+bx} = \frac{1}{b^3}\left[\frac{1}{2}(a+bx)^2 - 2a(a+bx) + a^2\ln|a+bx|\right] + C$$

887. $$\int \frac{dx}{x(a+bx)} = \frac{1}{a}\ln\left|\frac{a+bx}{x}\right| + C$$

888. $$\int \frac{dx}{x^2(a+bx)} = -\frac{1}{ax} + \frac{b}{a^2}\ln\left|\frac{a+bx}{x}\right| + C$$

889. $$\int \frac{xdx}{(a+bx)^2} = \frac{1}{b^2}\left(\ln|a+bx| + \frac{a}{a+bx}\right) + C$$

890. $$\int \frac{x^2 dx}{(a+bx)^2} = \frac{1}{b^3}\left(a + bx - 2a\ln|a+bx| - \frac{a^2}{a+bx} \right) + C$$

891. $$\int \frac{dx}{x(a+bx)^2} = \frac{1}{a(a+bx)} + \frac{1}{a^2}\ln\left|\frac{a+bx}{x}\right| + C$$

892. $$\int \frac{dx}{x^2-1} = \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C$$

893. $$\int \frac{dx}{1-x^2} = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + C$$

894. $$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$$

895. $$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$$

896. $$\int \frac{dx}{1+x^2} = \tan^{-1} x + C$$

897. $$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

898. $$\int \frac{xdx}{x^2+a^2} = \frac{1}{2}\ln\left(x^2+a^2\right) + C$$

899. $$\int \frac{dx}{a+bx^2} = \frac{1}{\sqrt{ab}}\arctan\left(x\sqrt{\frac{b}{a}} \right) + C,\ ab > 0.$$

900. $\int \frac{xdx}{a+bx^2} = \frac{1}{2b}\ln\left|x^2 + \frac{a}{b}\right| + C$

901. $\int \frac{dx}{x\left(a+bx^2\right)} = \frac{1}{2a}\ln\left|\frac{x^2}{a+bx^2}\right| + C$

902. $\int \frac{dx}{a^2 - b^2x^2} = \frac{1}{2ab}\ln\left|\frac{a+bx}{a-bx}\right| + C$

903. $\int \frac{dx}{ax^2+bx+c} = \frac{1}{\sqrt{b^2-4ac}}\ln\left|\frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right| + C$,

$b^2 - 4ac > 0$.

904. $\int \frac{dx}{ax^2+bx+c} = \frac{2}{\sqrt{4ac-b^2}}\arctan\frac{2ax+b}{\sqrt{4ac-b^2}} + C$,

$b^2 - 4ac < 0$.

9.3 Integrals of Irrational Functions

905. $\int \frac{dx}{\sqrt{ax+b}} = \frac{2}{a}\sqrt{ax+b} + C$

906. $\int \sqrt{ax+b}\, dx = \frac{2}{3a}(ax+b)^{3/2} + C$

907. $\int \frac{xdx}{\sqrt{ax+b}} = \frac{2(ax-2b)}{3a^2}\sqrt{ax+b} + C$

908. $$\int x\sqrt{ax+b}\,dx = \frac{2(3ax-2b)}{15a^2}(ax+b)^{3/2}+C$$

909. $$\int \frac{dx}{(x+c)\sqrt{ax+b}} = \frac{1}{\sqrt{b-ac}}\ln\left|\frac{\sqrt{ax+b}-\sqrt{b-ac}}{\sqrt{ax+b}+\sqrt{b-ac}}\right|+C,$$
$b-ac>0$.

910. $$\int \frac{dx}{(x+c)\sqrt{ax+b}} = \frac{1}{\sqrt{ac-b}}\arctan\sqrt{\frac{ax+b}{ac-b}}+C,$$
$b-ac<0$.

911. $$\int \sqrt{\frac{ax+b}{cx+d}}\,dx = \frac{1}{c}\sqrt{(ax+b)(cx+d)}-$$
$$-\frac{ad-bc}{c\sqrt{ac}}\ln\left|\sqrt{a(cx+d)}+\sqrt{c(ax+b)}\right|+C,\ a>0.$$

912. $$\int \sqrt{\frac{ax+b}{cx+d}}\,dx = \frac{1}{c}\sqrt{(ax+b)(cx+d)}-$$
$$-\frac{ad-bc}{c\sqrt{ac}}\arctan\sqrt{\frac{a(cx+d)}{c(ax+b)}}+C,\ (a<0,\ c>0).$$

913. $$\int x^2\sqrt{a+bx}\,dx = \frac{2(8a^2-12abx+15b^2x^2)}{105b^3}\sqrt{(a+bx)^3}+C$$

914. $$\int \frac{x^2dx}{\sqrt{a+bx}} = \frac{2(8a^2-4abx+3b^2x^2)}{15b^3}\sqrt{a+bx}+C$$

915. $$\int \frac{dx}{x\sqrt{a+bx}} = \frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{a+bx}-\sqrt{a}}{\sqrt{a+bx}+\sqrt{a}}\right|+C,\ a>0.$$

916. $\int \frac{dx}{x\sqrt{a+bx}} = \frac{2}{\sqrt{-a}} \arctan\left|\frac{a+bx}{-a}\right| + C$, $a<0$.

917. $\int \sqrt{\frac{a-x}{b+x}}\,dx = \sqrt{(a-x)(b+x)} + (a+b)\arcsin\sqrt{\frac{x+b}{a+b}} + C$

918. $\int \sqrt{\frac{a+x}{b-x}}\,dx = -\sqrt{(a+x)(b-x)} - (a+b)\arcsin\sqrt{\frac{b-x}{a+b}} + C$

919. $\int \sqrt{\frac{1+x}{1-x}}\,dx = -\sqrt{1-x^2} + \arcsin x + C$

920. $\int \frac{dx}{\sqrt{(x-a)(b-a)}} = 2\arcsin\sqrt{\frac{x-a}{b-a}} + C$

921. $\int \sqrt{a+bx-cx^2}\,dx = \frac{2cx-b}{4c}\sqrt{a+bx-cx^2} +$

$+ \frac{b^2-4ac}{8\sqrt{c^3}} \arcsin \frac{2cx-b}{\sqrt{b^2+4ac}} + C$

922. $\int \frac{dx}{\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{a}} \ln\left|2ax+b+2\sqrt{a\left(ax^2+bx+c\right)}\right| + C$,
$a>0$.

923. $\int \frac{dx}{\sqrt{ax^2+bx+c}} = -\frac{1}{\sqrt{a}} \arcsin \frac{2ax+b}{4a}\sqrt{b^2-4ac} + C$, $a<0$.

924. $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right| + C$

925. $$\int x\sqrt{x^2+a^2}\,dx = \frac{1}{3}\left(x^2+a^2\right)^{3/2} + C$$

926. $$\int x^2\sqrt{x^2+a^2}\,dx = \frac{x}{8}\left(2x^2+a^2\right)\sqrt{x^2+a^2} - $$
$$-\frac{a^4}{8}\ln\left|x+\sqrt{x^2+a^2}\right| + C$$

927. $$\int \frac{\sqrt{x^2+a^2}}{x^2}\,dx = -\frac{\sqrt{x^2+a^2}}{x} + \ln\left|x+\sqrt{x^2+a^2}\right| + C$$

928. $$\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left|x+\sqrt{x^2+a^2}\right| + C$$

929. $$\int \frac{\sqrt{x^2+a^2}}{x}\,dx = \sqrt{x^2+a^2} + a\ln\left|\frac{x}{a+\sqrt{x^2+a^2}}\right| + C$$

930. $$\int \frac{x\,dx}{\sqrt{x^2+a^2}} = \sqrt{x^2+a^2} + C$$

931. $$\int \frac{x^2\,dx}{\sqrt{x^2+a^2}} = \frac{x}{2}\sqrt{x^2+a^2} - \frac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right| + C$$

932. $$\int \frac{dx}{x\sqrt{x^2+a^2}} = \frac{1}{a}\ln\left|\frac{x}{a+\sqrt{x^2+a^2}}\right| + C$$

933. $$\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + C$$

934. $$\int x\sqrt{x^2-a^2}\,dx = \frac{1}{3}\left(x^2-a^2\right)^{3/2} + C$$

935. $$\int \frac{\sqrt{x^2-a^2}}{x}dx = \sqrt{x^2-a^2} + a\arcsin\frac{a}{x} + C$$

936. $$\int \frac{\sqrt{x^2-a^2}}{x^2}dx = -\frac{\sqrt{x^2-a^2}}{x} + \ln\left|x+\sqrt{x^2-a^2}\right| + C$$

937. $$\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|x+\sqrt{x^2-a^2}\right| + C$$

938. $$\int \frac{xdx}{\sqrt{x^2-a^2}} = \sqrt{x^2-a^2} + C$$

939. $$\int \frac{x^2dx}{\sqrt{x^2-a^2}} = \frac{x}{2}\sqrt{x^2-a^2} + \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + C$$

940. $$\int \frac{dx}{x\sqrt{x^2-a^2}} = -\frac{1}{a}\arcsin\frac{a}{x} + C$$

941. $$\int \frac{dx}{(x+a)\sqrt{x^2-a^2}} = \frac{1}{a}\sqrt{\frac{x-a}{x+a}} + C$$

942. $$\int \frac{dx}{(x-a)\sqrt{x^2-a^2}} = -\frac{1}{a}\sqrt{\frac{x+a}{x-a}} + C$$

943. $$\int \frac{dx}{x^2\sqrt{x^2-a^2}} = \frac{\sqrt{x^2-a^2}}{a^2x} + C$$

944. $$\int \frac{dx}{\left(x^2-a^2\right)^{3/2}} = -\frac{x}{a^2\sqrt{x^2-a^2}} + C$$

945. $$\int \left(x^2-a^2\right)^{3/2} dx = -\frac{x}{8}\left(2x^2-5a^2\right)\sqrt{x^2-a^2} + \frac{3a^4}{8}\ln\left|x+\sqrt{x^2-a^2}\right| + C$$

946. $$\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a} + C$$

947. $$\int x\sqrt{a^2-x^2}\,dx = -\frac{1}{3}\left(a^2-x^2\right)^{3/2} + C$$

948. $$\int x^2\sqrt{a^2-x^2}\,dx = \frac{x}{8}\left(2x^2-a^2\right)\sqrt{a^2-x^2} + \frac{a^4}{8}\arcsin\frac{x}{a} + C$$

949. $$\int \frac{\sqrt{a^2-x^2}}{x}dx = \sqrt{a^2-x^2} + a\ln\left|\frac{x}{a+\sqrt{a^2-x^2}}\right| + C$$

950. $$\int \frac{\sqrt{a^2-x^2}}{x^2}dx = -\frac{\sqrt{a^2-x^2}}{x} - \arcsin\frac{x}{a} + C$$

951. $$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$$

952. $$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin\frac{x}{a} + C$$

953. $$\int \frac{x\,dx}{\sqrt{a^2-x^2}} = -\sqrt{a^2-x^2} + C$$

954. $$\int \frac{x^2dx}{\sqrt{a^2-x^2}} = -\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a} + C$$

955. $\int \frac{dx}{(x+a)\sqrt{a^2-x^2}} = -\frac{1}{2}\sqrt{\frac{a-x}{a+x}} + C$

956. $\int \frac{dx}{(x-a)\sqrt{a^2-x^2}} = -\frac{1}{2}\sqrt{\frac{a+x}{a-x}} + C$

957. $\int \frac{dx}{(x+b)\sqrt{a^2-x^2}} = \frac{1}{\sqrt{b^2-a^2}} \arcsin \frac{bx+a^2}{a(x+b)} + C,\ b > a.$

958. $\int \frac{dx}{(x+b)\sqrt{a^2-x^2}} = \frac{1}{\sqrt{a^2-b^2}} \ln \left| \frac{x+b}{\sqrt{a^2-b^2}\sqrt{a^2-x^2}+a^2+bx} \right| + C,$ $b < a.$

959. $\int \frac{dx}{x^2\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{a^2 x} + C$

960. $\int (a^2-x^2)^{3/2} dx = \frac{x}{8}(5a^2-2x^2)\sqrt{a^2-x^2} + \frac{3a^4}{8} \arcsin \frac{x}{a} + C$

961. $\int \frac{dx}{(a^2-x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2-x^2}} + C$

9.4 Integrals of Trigonometric Functions

962. $\int \sin x dx = -\cos x + C$

963. $\int \cos x dx = \sin x + C$

964. $\int \sin^2 x\,dx = \frac{x}{2} - \frac{1}{4}\sin 2x + C$

965. $\int \cos^2 x\,dx = \frac{x}{2} + \frac{1}{4}\sin 2x + C$

966. $\int \sin^3 x\,dx = \frac{1}{3}\cos^3 x - \cos x + C = \frac{1}{12}\cos 3x - \frac{3}{4}\cos x + C$

967. $\int \cos^3 x\,dx = \sin x - \frac{1}{3}\sin^3 x + C = \frac{1}{12}\sin 3x + \frac{3}{4}\sin x + C$

968. $\int \frac{dx}{\sin x} = \int \csc x\,dx = \ln\left|\tan\frac{x}{2}\right| + C$

969. $\int \frac{dx}{\cos x} = \int \sec x\,dx = \ln\left|\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| + C$

970. $\int \frac{dx}{\sin^2 x} = \int \csc^2 x\,dx = -\cot x + C$

971. $\int \frac{dx}{\cos^2 x} = \int \sec^2 x\,dx = \tan x + C$

972. $\int \frac{dx}{\sin^3 x} = \int \csc^3 x\,dx = -\frac{\cos x}{2\sin^2 x} + \frac{1}{2}\ln\left|\tan\frac{x}{2}\right| + C$

973. $\int \frac{dx}{\cos^3 x} = \int \sec^3 x\,dx = \frac{\sin x}{2\cos^2 x} + \frac{1}{2}\ln\left|\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| + C$

974. $\int \sin x \cos x\,dx = -\frac{1}{4}\cos 2x + C$

975. $\int \sin^2 x \cos x \, dx = \frac{1}{3}\sin^3 x + C$

976. $\int \sin x \cos^2 x \, dx = -\frac{1}{3}\cos^3 x + C$

977. $\int \sin^2 x \cos^2 x \, dx = \frac{x}{8} - \frac{1}{32}\sin 4x + C$

978. $\int \tan x dx = -\ln|\cos x| + C$

979. $\int \frac{\sin x}{\cos^2 x} dx = \frac{1}{\cos x} + C = \sec x + C$

980. $\int \frac{\sin^2 x}{\cos x} dx = \ln\left|\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| - \sin x + C$

981. $\int \tan^2 x \, dx = \tan x - x + C$

982. $\int \cot x dx = \ln|\sin x| + C$

983. $\int \frac{\cos x}{\sin^2 x} dx = -\frac{1}{\sin x} + C = -\csc x + C$

984. $\int \frac{\cos^2 x}{\sin x} dx = \ln\left|\tan\frac{x}{2}\right| + \cos x + C$

985. $\int \cot^2 x \, dx = -\cot x - x + C$

986. $\int \frac{dx}{\cos x \sin x} = \ln|\tan x| + C$

987. $\int \frac{dx}{\sin^2 x \cos x} = -\frac{1}{\sin x} + \ln\left|\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| + C$

988. $\int \frac{dx}{\sin x \cos^2 x} = \frac{1}{\cos x} + \ln\left|\tan\frac{x}{2}\right| + C$

989. $\int \frac{dx}{\sin^2 x \cos^2 x} = \tan x - \cot x + C$

990. $\int \sin mx \sin nx \, dx = -\frac{\sin(m+n)x}{2(m+n)} + \frac{\sin(m-n)x}{2(m-n)} + C,$

$m^2 \neq n^2.$

991. $\int \sin mx \cos nx \, dx = -\frac{\cos(m+n)x}{2(m+n)} - \frac{\cos(m-n)x}{2(m-n)} + C,$

$m^2 \neq n^2.$

992. $\int \cos mx \cos nx \, dx = \frac{\sin(m+n)x}{2(m+n)} + \frac{\sin(m-n)x}{2(m-n)} + C,$

$m^2 \neq n^2.$

993. $\int \sec x \tan x dx = \sec x + C$

994. $\int \csc x \cot x dx = -\csc x + C$

995. $\int \sin x \cos^n x \, dx = -\frac{\cos^{n+1} x}{n+1} + C$

996. $\int \sin^n x \cos x \, dx = \frac{\sin^{n+1} x}{n+1} + C$

997. $\int \arcsin x\, dx = x \arcsin x + \sqrt{1-x^2} + C$

998. $\int \arccos x\, dx = x \arccos x - \sqrt{1-x^2} + C$

999. $\int \arctan x\, dx = x \arctan x - \frac{1}{2}\ln\left(x^2+1\right) + C$

1000. $\int \operatorname{arccot} x\, dx = x \operatorname{arccot} x + \frac{1}{2}\ln\left(x^2+1\right) + C$

9.5 Integrals of Hyperbolic Functions

1001. $\int \sinh x dx = \cosh x + C$

1002. $\int \cosh x dx = \sinh x + C$

1003. $\int \tanh x\, dx = \ln \cosh x + C$

1004. $\int \coth x\, dx = \ln\left|\sinh x\right| + C$

1005. $\int \operatorname{sech}^2 x dx = \tanh x + C$

1006. $\int \operatorname{csch}^2 x dx = -\coth x + C$

1007. $\int \operatorname{sech} x \tanh x dx = -\operatorname{sech} x + C$

1008. $\int \operatorname{csch} x \coth x dx = -\operatorname{csch} x + C$

9.6 Integrals of Exponential and Logarithmic Functions

1009. $\int e^x dx = e^x + C$

1010. $\int a^x dx = \frac{a^x}{\ln a} + C$

1011. $\int e^{ax} dx = \frac{e^{ax}}{a} + C$

1012. $\int xe^{ax} dx = \frac{e^{ax}}{a^2}(ax - 1) + C$

1013. $\int \ln x dx = x \ln x - x + C$

1014. $\int \frac{dx}{x \ln x} = \ln|\ln x| + C$

1015. $\int x^n \ln x dx = x^{n+1}\left[\frac{\ln x}{n+1} - \frac{1}{(n+1)^2}\right] + C$

1016. $\int e^{ax} \sin bx dx = \frac{a \sin bx - b \cos bx}{a^2 + b^2} e^{ax} + C$

1017. $\int e^{ax} \cos bx \, dx = \frac{a \cos bx + b \sin bx}{a^2 + b^2} e^{ax} + C$

9.7 Reduction Formulas

1018. $\int x^n e^{mx} dx = \frac{1}{m} x^n e^{mx} - \frac{n}{m} \int x^{n-1} e^{mx} dx$

1019. $\int \frac{e^{mx}}{x^n} dx = -\frac{e^{mx}}{(n-1)x^{n-1}} + \frac{m}{n-1} \int \frac{e^{mx}}{x^{n-1}} dx, \; n \neq 1.$

1020. $\int \sinh^n x dx = \frac{1}{n} \sinh^{n-1} x \cosh x - \frac{n-1}{n} \int \sinh^{n-2} x dx$

1021. $\int \frac{dx}{\sinh^n x} = -\frac{\cosh x}{(n-1)\sinh^{n-1} x} - \frac{n-2}{n-1} \int \frac{dx}{\sinh^{n-2} x}, \; n \neq 1.$

1022. $\int \cosh^n x dx = \frac{1}{n} \sinh x \cosh^{n-1} x \cosh x + \frac{n-1}{n} \int \cosh^{n-2} x dx$

1023. $\int \frac{dx}{\cosh^n x} = -\frac{\sinh x}{(n-1)\cosh^{n-1} x} + \frac{n-2}{n-1} \int \frac{dx}{\cosh^{n-2} x}, \; n \neq 1.$

1024. $\int \sinh^n x \cosh^m x dx = \frac{\sinh^{n+1} x \cosh^{m-1} x}{n+m}$

$$+ \frac{m-1}{n+m} \int \sinh^n x \cosh^{m-2} x dx$$

1025. $\int \sinh^n x \cosh^m x dx = \frac{\sinh^{n-1} x \cosh^{m+1} x}{n+m}$

$$-\frac{n-1}{n+m}\int \sinh^{n-2} x \cosh^m x dx$$

1026. $\int \tanh^n x dx = -\frac{1}{n-1}\tanh^{n-1} x + \int \tanh^{n-2} x dx \, , \; n \neq 1 \, .$

1027. $\int \coth^n x dx = -\frac{1}{n-1}\coth^{n-1} x + \int \coth^{n-2} x dx \, , \; n \neq 1 \, .$

1028. $\int \operatorname{sech}^n x dx = \frac{\operatorname{sech}^{n-2} x \tanh x}{n-1} + \frac{n-2}{n-1}\int \operatorname{sech}^{n-2} x dx \, , \; n \neq 1 \, .$

1029. $\int \sin^n x dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x dx$

1030. $\int \frac{dx}{\sin^n x} = -\frac{\cos x}{(n-1)\sin^{n-1} x} + \frac{n-2}{n-1}\int \frac{dx}{\sin^{n-2} x} \, , \; n \neq 1 \, .$

1031. $\int \cos^n x dx = \frac{1}{n}\sin x \cos^{n-1} x + \frac{n-1}{n}\int \cos^{n-2} x dx$

1032. $\int \frac{dx}{\cos^n x} = \frac{\sin x}{(n-1)\cos^{n-1} x} + \frac{n-2}{n-1}\int \frac{dx}{\cos^{n-2} x} \, , \; n \neq 1 \, .$

1033. $\int \sin^n x \cos^m x dx = \frac{\sin^{n+1} x \cos^{m-1} x}{n+m}$

$$+\frac{m-1}{n+m}\int \sin^n x \cos^{m-2} x dx$$

1034. $\int \sin^n x \cos^m x dx = -\frac{\sin^{n-1} x \cos^{m+1} x}{n+m}$

$$+\frac{n-1}{n+m}\int \sin^{n-2} x \cos^m x dx$$

1035. $\int \tan^n x dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x dx,\ n \neq 1.$

1036. $\int \cot^n x dx = -\frac{1}{n-1}\cot^{n-1} x - \int \cot^{n-2} x dx,\ n \neq 1.$

1037. $\int \sec^n x dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x dx,\ n \neq 1.$

1038. $\int \csc^n x dx = -\frac{\csc^{n-2} x \cot x}{n-1} + \frac{n-2}{n-1}\int \csc^{n-2} x dx,\ n \neq 1.$

1039. $\int x^n \ln^m x dx = \frac{x^{n+1}\ln^m x}{n+1} - \frac{m}{n+1}\int x^n \ln^{m-1} x dx$

1040. $\int \frac{\ln^m x}{x^n} dx = -\frac{\ln^m x}{(n-1)x^{n-1}} + \frac{m}{n-1}\int \frac{\ln^{m-1} x}{x^n} dx,\ n \neq 1.$

1041. $\int \ln^n x dx = x \ln^n x - n\int \ln^{n-1} x dx$

1042. $\int x^n \sinh x dx = x^n \cosh x - n\int x^{n-1} \cosh x dx$

1043. $\int x^n \cosh x dx = x^n \sinh x - n\int x^{n-1} \sinh x dx$

1044. $\int x^n \sin x dx = -x^n \cos x + n\int x^{n-1} \cos x dx$

1045. $\int x^n \cos x dx = x^n \sin x - n\int x^{n-1} \sin x dx$

1046. $\int x^n \sin^{-1} x dx = \frac{x^{n+1}}{n+1} \sin^{-1} x - \frac{1}{n+1} \int \frac{x^{n+1}}{\sqrt{1-x^2}} dx$

1047. $\int x^n \cos^{-1} x dx = \frac{x^{n+1}}{n+1} \cos^{-1} x + \frac{1}{n+1} \int \frac{x^{n+1}}{\sqrt{1-x^2}} dx$

1048. $\int x^n \tan^{-1} x dx = \frac{x^{n+1}}{n+1} \tan^{-1} x - \frac{1}{n+1} \int \frac{x^{n+1}}{1+x^2} dx$

1049. $\int \frac{x^n dx}{ax^n + b} = \frac{x}{a} - \frac{b}{a} \int \frac{dx}{ax^n + b}$

1050. $\int \frac{dx}{(ax^2 + bx + c)^n} = \frac{-2ax - b}{(n-1)(b^2 - 4ac)(ax^2 + bx + c)^{n-1}}$
$- \frac{2(2n-3)a}{(n-1)(b^2 - 4ac)} \int \frac{dx}{(ax^2 + bx + c)^{n-1}}, \; n \neq 1.$

1051. $\int \frac{dx}{(x^2 + a^2)^n} = \frac{x}{2(n-1)a^2 (x^2 + a^2)^{n-1}} + \frac{2n-3}{2(n-1)a^2} \int \frac{dx}{(x^2 + a^2)^{n-1}},$
$n \neq 1.$

1052. $\int \frac{dx}{(x^2 - a^2)^n} = - \frac{x}{2(n-1)a^2 (x^2 - a^2)^{n-1}}$
$- \frac{2n-3}{2(n-1)a^2} \int \frac{dx}{(x^2 - a^2)^{n-1}}, \; n \neq 1.$

9.8 Definite Integral

Definite integral of a function: $\int_a^b f(x)dx$, $\int_a^b g(x)dx$, ...

Riemann sum: $\sum_{i=1}^{n} f(\xi_i)\Delta x_i$

Small changes: Δx_i

Antiderivatives: $F(x)$, $G(x)$

Limits of integrations: a, b, c, d

1053. $\int_a^b f(x)dx = \lim_{\substack{n\to\infty \\ \max \Delta x_i \to 0}} \sum_{i=1}^{n} f(\xi_i)\Delta x_i$,

where $\Delta x_i = x_i - x_{i-1}$, $x_{i-1} \le \xi_i \le x_i$.

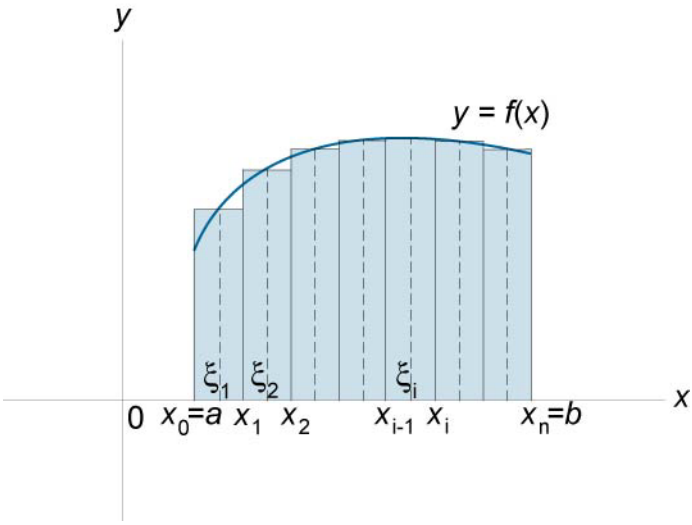


Figure 179.

1054. $\int_a^b 1dx = b - a$

1055. $\int_a^b kf(x)dx = k\int_a^b f(x)dx$

1056. $\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

1057. $\int_a^b [f(x) - g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$

1058. $\int_a^a f(x)dx = 0$

1059. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

1060. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ for $a < c < b$.

1061. $\int_a^b f(x)dx \geq 0$ if $f(x) \geq 0$ on $[a, b]$.

1062. $\int_a^b f(x)dx \leq 0$ if $f(x) \leq 0$ on $[a, b]$.

1063. Fundamental Theorem of Calculus

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a) \text{ if } F'(x) = f(x).$$

1064. Method of Substitution

If $x = g(t)$, then

$$\int_a^b f(x)dx = \int_c^d f(g(t))g'(t)dt,$$

where

$$c = g^{-1}(a),\ d = g^{-1}(b).$$

1065. Integration by Parts

$$\int_a^b u\,dv = (uv)\Big|_a^b - \int_a^b v\,du$$

1066. Trapezoidal Rule

$$\int_a^b f(x)dx = \frac{b-a}{2n}\left[f(x_0) + f(x_n) + 2\sum_{i=1}^{n-1} f(x_i)\right]$$

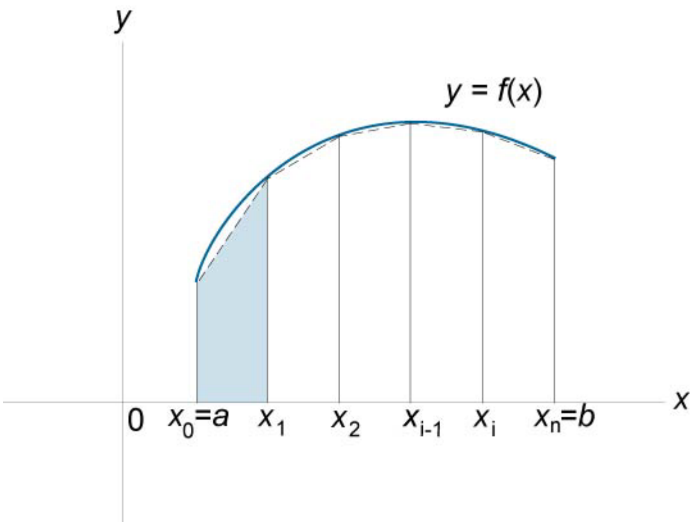


Figure 180.

1067. Simpson's Rule

$$\int_a^b f(x)dx = \frac{b-a}{3n}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \right.$$

$$\left. + 2f(x_4) + \ldots + 4f(x_{n-1}) + f(x_n)\right],$$

where

$$x_i = a + \frac{b-a}{n} i,\ i = 0, 1, 2, \ldots, n.$$

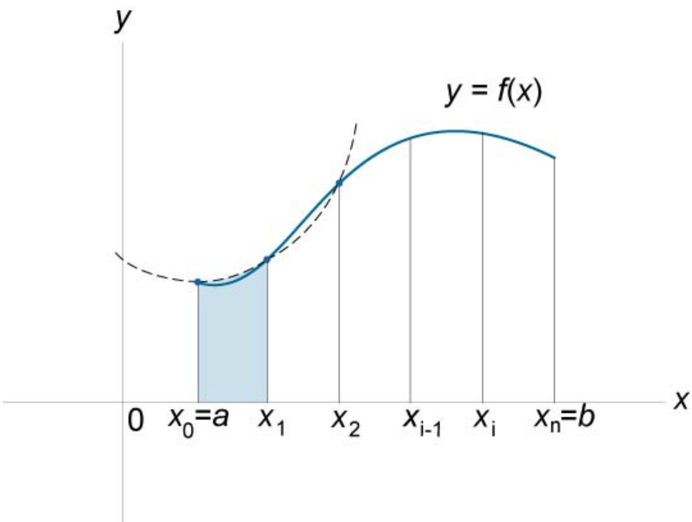


Figure 181.

1068. Area Under a Curve

$$S = \int_a^b f(x)dx = F(b) - F(a),$$

where $F'(x) = f(x)$.

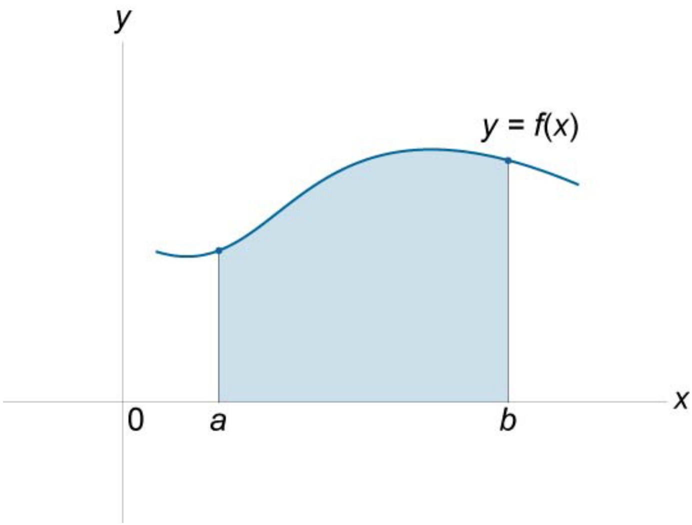


Figure 182.

1069. Area Between Two Curves

$$S = \int_a^b [f(x) - g(x)]dx = F(b) - G(b) - F(a) + G(a),$$

where $F'(x) = f(x)$, $G'(x) = g(x)$.

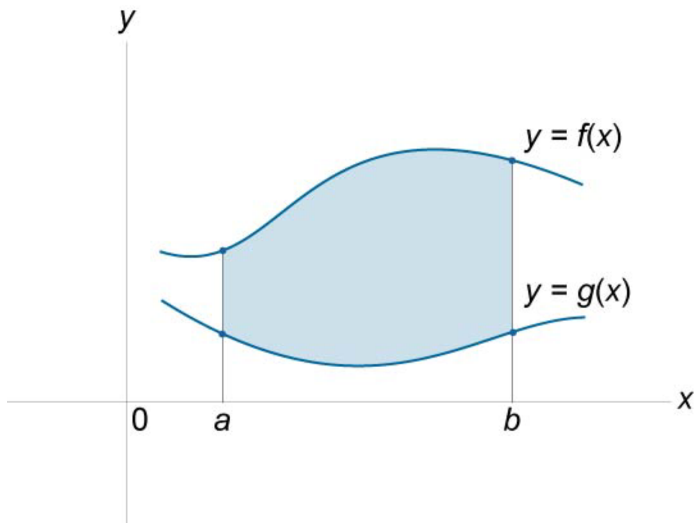


Figure 183.

9.9 Improper Integral

1070. The definite integral $\int_a^b f(x)dx$ is called an improper integral if

- a or b is infinite,
- $f(x)$ has one or more points of discontinuity in the interval $[a,b]$.

1071. If $f(x)$ is a continuous function on $[a,\infty)$, then

$$\int_a^\infty f(x)dx = \lim_{n\to\infty}\int_a^n f(x)dx.$$

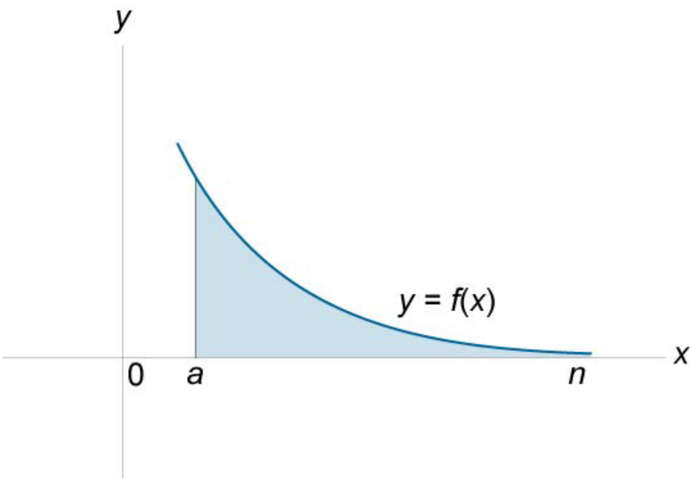


Figure 184.

1072. If $f(x)$ is a continuous function on $(-\infty, b]$, then

$$\int_{-\infty}^{b} f(x)dx = \lim_{n \to -\infty} \int_{n}^{b} f(x)dx.$$

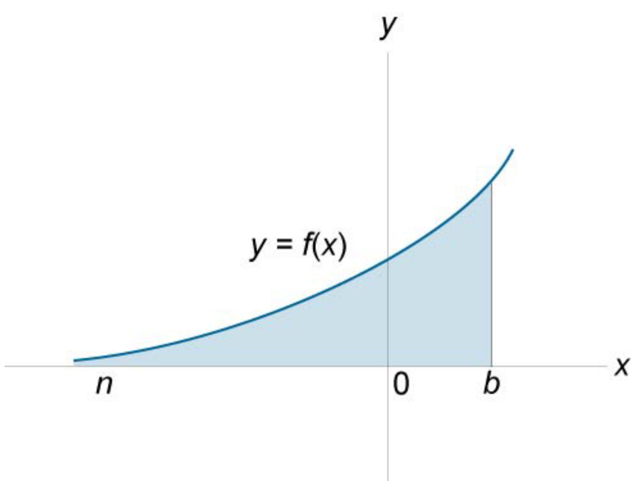


Figure 185.

Note : The improper integrals in 1071, 1072 are convergent if the limits exist and are finite; otherwise the integrals are divergent.

1073. $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{\infty} f(x)dx$

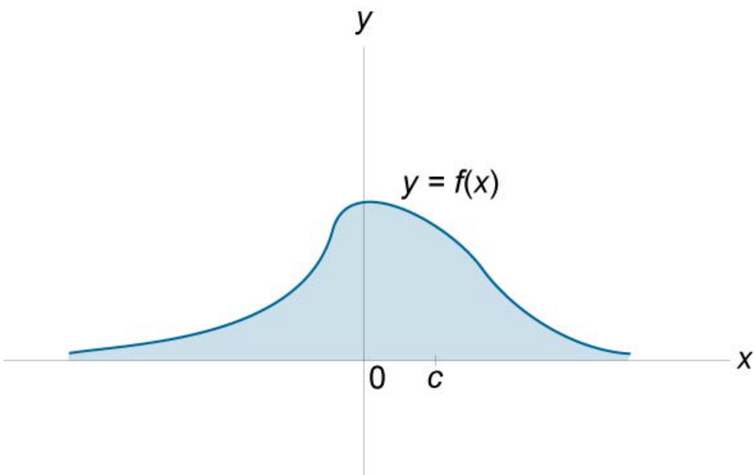


Figure 186.

If for some real number c, both of the integrals in the right side are convergent, then the integral $\int_{-\infty}^{\infty} f(x)dx$ is also convergent; otherwise it is divergent.

1074. Comparison Theorems
Let $f(x)$ and $g(x)$ be continuous functions on the closed interval $[a,\infty)$. Suppose that $0 \le g(x) \le f(x)$ for all x in $[a,\infty)$.

- If $\int_a^{\infty} f(x)dx$ is convergent, then $\int_a^{\infty} g(x)dx$ is also convergent,
- If $\int_a^{\infty} g(x)dx$ is divergent, then $\int_a^{\infty} f(x)dx$ is also divergent.

1075. Absolute Convergence

If $\int_a^{\infty} |f(x)|dx$ is convergent, then the integral $\int_a^{\infty} f(x)dx$ is absolutely convergent.

1076. Discontinuous Integrand

Let $f(x)$ be a function which is continuous on the interval $[a,b)$ but is discontinuous at $x=b$. Then

$$\int_a^b f(x)dx = \lim_{\varepsilon \to 0+} \int_a^{b-\varepsilon} f(x)dx$$

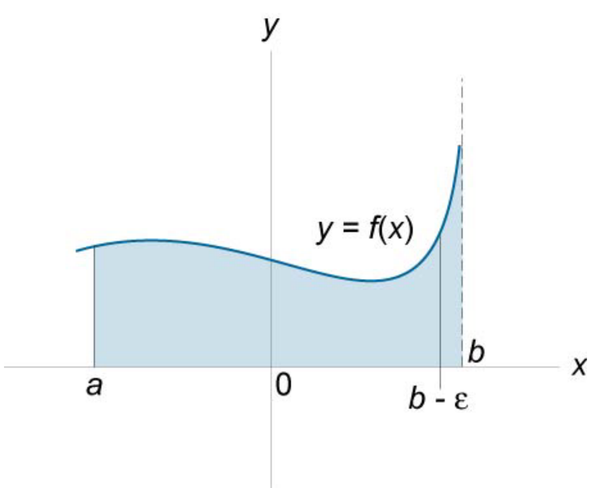


Figure 187.

1077. Let $f(x)$ be a continuous function for all real numbers x in the interval $[a,b]$ except for some point c in (a,b). Then

$$\int_a^b f(x)dx = \lim_{\varepsilon \to 0+} \int_a^{c-\varepsilon} f(x)dx + \lim_{\delta \to 0+} \int_{c+\delta}^b f(x)dx.$$

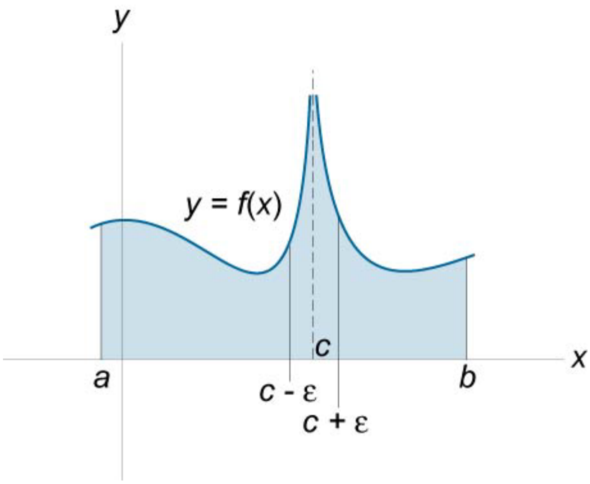


Figure 188.

9.10 Double Integral

Functions of two variables: $f(x,y)$, $f(u,v)$, ...

Double integrals: $\iint_R f(x,y)dxdy$, $\iint_R g(x,y)dxdy$, ...

Riemann sum: $\sum_{i=1}^{m}\sum_{j=1}^{n} f(u_i, v_j)\Delta x_i \Delta y_j$

Small changes: Δx_i, Δy_j

Regions of integration: R, S

Polar coordinates: r, θ

Area: A
Surface area: S
Volume of a solid: V
Mass of a lamina: m
Density: $\rho(x, y)$
First moments: M_x, M_y
Moments of inertia: I_x, I_y, I_0
Charge of a plate: Q
Charge density: $\sigma(x, y)$
Coordinates of center of mass: $\bar{x}$, $\bar{y}$
Average of a function: μ

1078. Definition of Double Integral
The double integral over a rectangle $[a, b] \times [c, d]$ is defined to be

$$\iint\limits_{[a,\, b]\times[c,\, d]} f(x, y)\, dA = \lim_{\substack{\max \Delta x_i \to 0 \\ \max \Delta y_j \to 0}} \sum_{i=1}^{m} \sum_{j=1}^{n} f(u_i, v_j) \Delta x_i \Delta y_j \,,$$

where (u_i, v_j) is some point in the rectangle $(x_{i-1}, x_i) \times (y_{j-1}, y_j)$, and $\Delta x_i = x_i - x_{i-1}$, $\Delta y_j = y_j - y_{j-1}$.

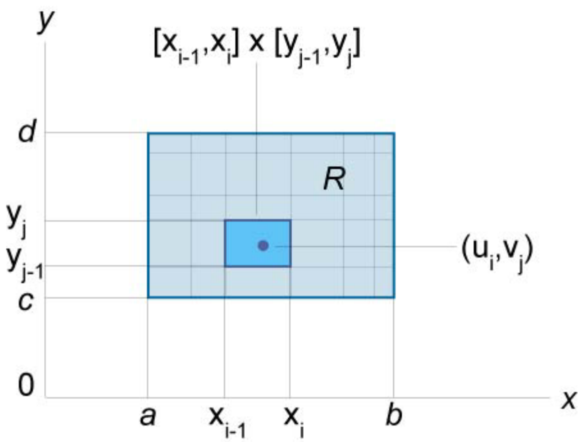


Figure 189.

The double integral over a general region R is

$$\iint\limits_R f(x,y)dA = \iint\limits_{[a,\,b]\times[c,\,d]} g(x,y)dA,$$

where rectangle $[a,\, b]\times[c,\, d]$ contains R,

$g(x,y)=f(x,y)$ if $f(x,y)$ is in R and $g(x,y)=0$ otherwise.

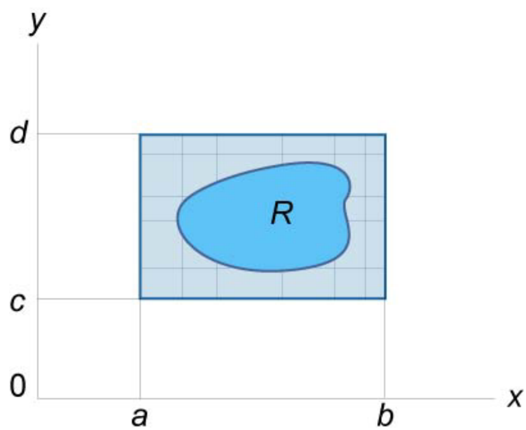


Figure 190.

1079. $\iint\limits_R [f(x,y)+g(x,y)]dA = \iint\limits_R f(x,y)dA + \iint\limits_R g(x,y)dA$

1080. $\iint\limits_R [f(x,y)-g(x,y)]dA = \iint\limits_R f(x,y)dA - \iint\limits_R g(x,y)dA$

1081. $\iint\limits_R kf(x,y)dA = k\iint\limits_R f(x,y)dA$,

where k is a constant.

1082. If $f(x,y)\le g(x,y)$ on R, then $\iint\limits_R f(x,y)dA \le \iint\limits_R g(x,y)dA$.

1083. If $f(x,y)\ge 0$ on R and $S\subset R$, then

$$\iint_S f(x,y)\,dA \le \iint_R f(x,y)\,dA .$$

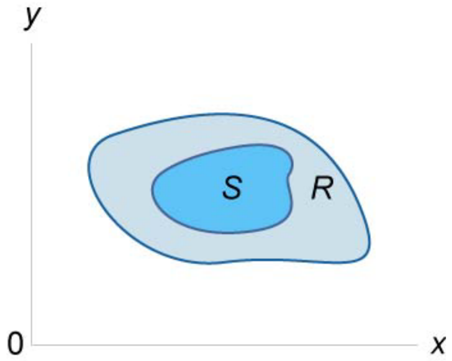


Figure 191.

1084. If $f(x,y) \ge 0$ on R and R and S are non-overlapping regions, then $\iint_{R\cup S} f(x,y)\,dA = \iint_R f(x,y)\,dA + \iint_S f(x,y)\,dA$.
Here $R \cup S$ is the union of the regions R and S.

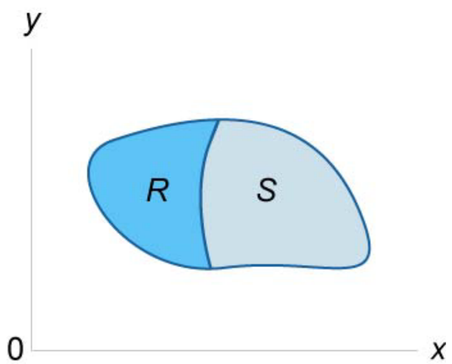


Figure 192.

1085. **Iterated Integrals and Fubini's Theorem**

$$\iint\limits_R f(x,y)\,dA = \int\limits_a^b \int\limits_{p(x)}^{q(x)} f(x,y)\,dy\,dx$$

for a region of type I,
$R = \{(x,y) \mid a \le x \le b,\ p(x) \le y \le q(x)\}$.

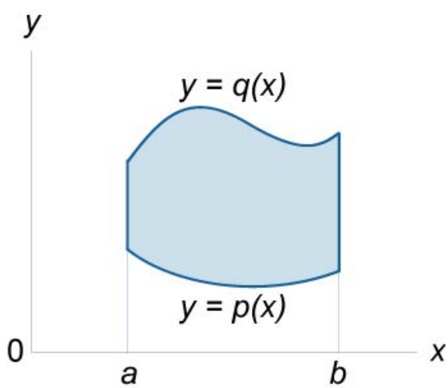


Figure 193.

$$\iint\limits_R f(x,y)\,dA = \int\limits_c^d \int\limits_{u(y)}^{v(y)} f(x,y)\,dx\,dy$$

for a region of type II,
$R = \{(x,y) \mid u(y) \le x \le v(y),\ c \le y \le d\}$.

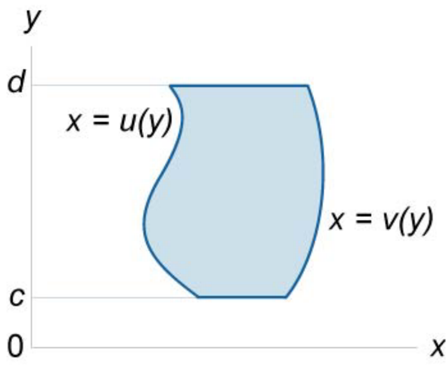


Figure 194.

1086. Double Integrals over Rectangular Regions

If R is the rectangular region $[a,b]\times[c,d]$, then

$$\iint_R f(x,y)dxdy = \int_a^b\left(\int_c^d f(x,y)dy\right)dx = \int_c^d\left(\int_a^b f(x,y)dx\right)dy .$$

In the special case where the integrand $f(x,y)$ can be written as $g(x)h(y)$ we have

$$\iint_R f(x,y)dxdy = \iint_R g(x)h(y)dxdy = \left(\int_a^b g(x)dx\right)\left(\int_c^d h(y)dy\right).$$

1087. Change of Variables

$$\iint_R f(x,y)dxdy = \iint_S f[x(u,v),y(u,v)]\left|\frac{\partial(x,y)}{\partial(u,v)}\right|dudv ,$$

where $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix} \neq 0$ is the jacobian of the transformations $(x,y)\to(u,v)$, and S is the pullback of R which can be computed by $x = x(u,v)$, $y = y(u,v)$ into the definition of R.

1088. Polar Coordinates
$x = r\cos\theta$, $y = r\sin\theta$.

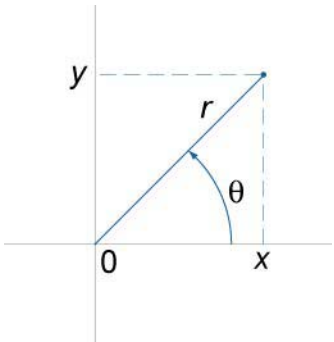


Figure 195.

1089. Double Integrals in Polar Coordinates

The Differential dxdy for Polar Coordinates is
$$dxdy = \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| drd\theta = rdrd\theta .$$

Let the region R is determined as follows:
$0 \le g(\theta) \le r \le h(\theta)$, $\alpha \le \theta \le \beta$, where $\beta - \alpha \le 2\pi$.
Then
$$\iint_R f(x,y)dxdy = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} f(r\cos\theta, r\sin\theta) rdrd\theta .$$

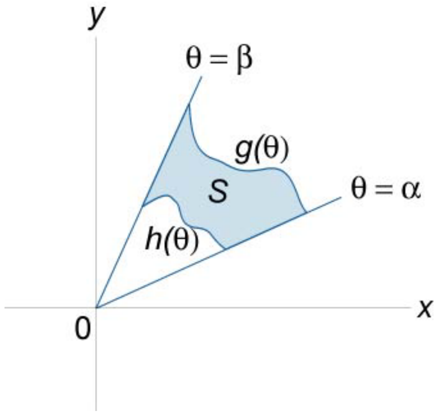


Figure 196.

If the region R is the polar rectangle given by $0 \le a \le r \le b$, $\alpha \le \theta \le \beta$, where $\beta - \alpha \le 2\pi$, then

$$\iint_R f(x,y)\,dx\,dy = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta .$$

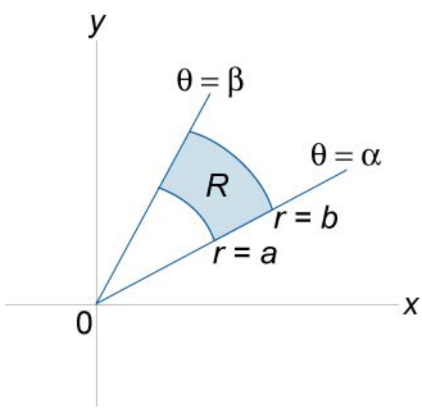


Figure 197.

1090. Area of a Region

$$A = \int_a^b \int_{g(x)}^{f(x)} dydx \ \text{(for a type I region)}.$$

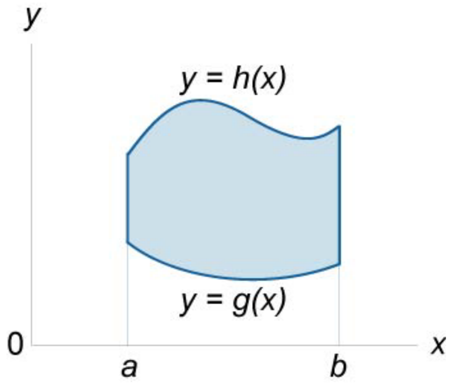


Figure 198.

$$A = \int_c^d \int_{p(y)}^{q(y)} dxdy \ \text{(for a type II region)}.$$

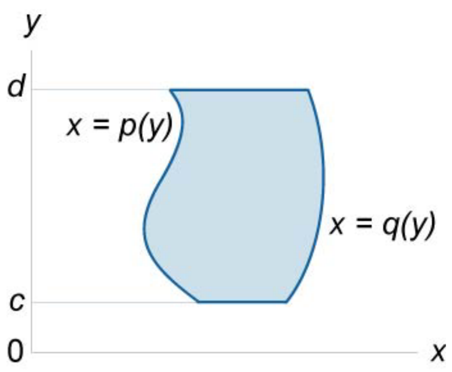


Figure 199.

1091. Volume of a Solid

$$V = \iint_R f(x,y)\,dA.$$

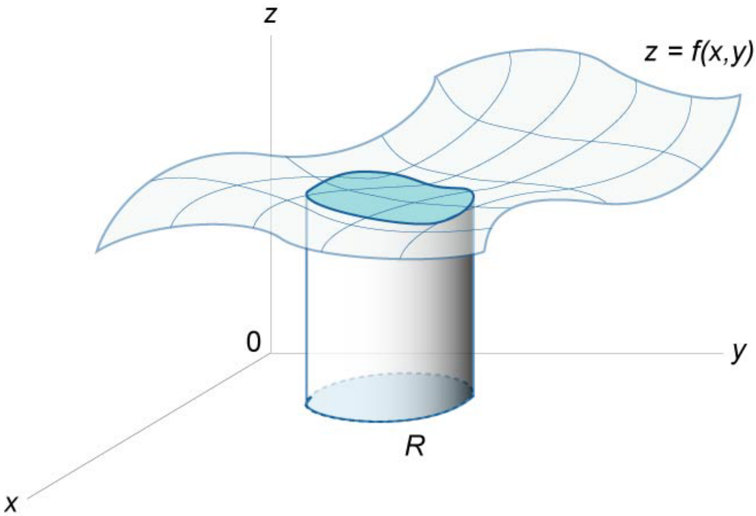


Figure 200.

If R is a type I region bounded by $x = a$, $x = b$, $y = h(x)$, $y = g(x)$, then

$$V = \iint_R f(x,y)\,dA = \int_a^b \int_{h(x)}^{g(x)} f(x,y)\,dy\,dx.$$

If R is a type II region bounded by $y = c$, $y = d$, $x = q(y)$, $x = p(y)$, then

$$V = \iint_R f(x,y)\,dA = \int_c^d \int_{p(y)}^{q(y)} f(x,y)\,dx\,dy.$$

If $f(x,y) \geq g(x,y)$ over a region R, then the volume of the solid between $z_1 = f(x,y)$ and $z_2 = g(x,y)$ over R is given by

$$V = \iint_R [f(x,y) - g(x,y)]dA .$$

1092. Area and Volume in Polar Coordinates

If S is a region in the xy-plane bounded by $\theta = \alpha$, $\theta = \beta$, $r = h(\theta)$, $r = g(\theta)$,

then

$$A = \iint_S dA = \int_{\alpha}^{\beta} \int_{h(\theta)}^{g(\theta)} r dr d\theta ,$$

$$V = \iint_S f(r,\theta) r dr d\theta .$$

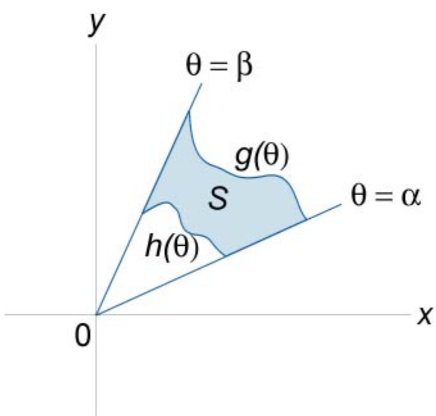


Figure 201.

1093. Surface Area

$$S = \iint_R \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dxdy$$

1094. Mass of a Lamina

$$m = \iint_R \rho(x,y)dA,$$

where the lamina occupies a region R and its density at a point (x,y) is $\rho(x,y)$.

1095. Moments

The moment of the lamina about the x-axis is given by formula

$$M_x = \iint_R y\rho(x,y)dA.$$

The moment of the lamina about the y-axis is

$$M_y = \iint_R x\rho(x,y)dA.$$

The moment of inertia about the x-axis is

$$I_x = \iint_R y^2\rho(x,y)dA.$$

The moment of inertia about the y-axis is

$$I_y = \iint_R x^2\rho(x,y)dA.$$

The polar moment of inertia is

$$I_0 = \iint_R (x^2 + y^2)\rho(x,y)dA.$$

1096. Center of Mass

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m}\iint_R x\rho(x,y)dA = \frac{\iint_R x\rho(x,y)dA}{\iint_R \rho(x,y)dA},$$

$$\bar{y} = \frac{M_x}{m} = \frac{1}{m}\iint_R y\rho(x,y)dA = \frac{\iint_R y\rho(x,y)dA}{\iint_R \rho(x,y)dA}.$$

1097. Charge of a Plate

$$Q = \iint_R \sigma(x,y)dA,$$

where electrical charge is distributed over a region R and its charge density at a point (x,y) is $\sigma(x,y)$.

1098. Average of a Function

$$\mu = \frac{1}{S}\iint_R f(x,y)dA,$$

where $S = \iint_R dA$.

9.11 Triple Integral

Functions of three variables: $f(x,y,z)$, $g(x,y,z)$, ...

Triple integrals: $\iiint_G f(x,y,z)dV$, $\iiint_G g(x,y,z)dV$, ...

Riemann sum: $\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} f(u_i, v_j, w_k)\Delta x_i \Delta y_j \Delta z_k$

Small changes: Δx_i, Δy_j, Δz_k

Limits of integration: a, b, c, d, r, s

Regions of integration: G, T, S

Cylindrical coordinates: r, θ, z

Spherical coordinates: r, θ, φ

Volume of a solid: V

Mass of a solid: m
Density: $\mu(x,y,z)$
Coordinates of center of mass: $\bar{x}$, $\bar{y}$, $\bar{z}$
First moments: M_{xy}, M_{yz}, M_{xz}
Moments of inertia: I_{xy}, I_{yz}, I_{xz}, I_x, I_y, I_z, I_0

1099. Definition of Triple Integral
The triple integral over a parallelepiped $[a, b]\times[c, d]\times[r, s]$ is defined to be

$$\iiint\limits_{[a,\, b]\times[c,\, d]\times[r,\, s]} f(x,y,z)dV = \lim_{\substack{\max \Delta x_i \to 0 \\ \max \Delta y_j \to 0 \\ \max \Delta z_k \to 0}} \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} f(u_i,v_j,w_k)\Delta x_i \Delta y_j \Delta z_k,$$

where (u_i,v_j,w_k) is some point in the parallelepiped $(x_{i-1},x_i)\times(y_{j-1},y_j)\times(z_{k-1},z_k)$, and $\Delta x_i = x_i - x_{i-1}$, $\Delta y_j = y_j - y_{j-1}$, $\Delta z_k = z_k - z_{k-1}$.

1100. $\iiint\limits_G [f(x,y,z)+g(x,y,z)]dV = \iiint\limits_G f(x,y,z)dV + \iiint\limits_G g(x,y,z)dV$

1101. $\iiint\limits_G [f(x,y,z)-g(x,y,z)]dV = \iiint\limits_G f(x,y,z)dV - \iiint\limits_G g(x,y,z)dV$

1102. $\iiint\limits_G kf(x,y,z)dV = k\iiint\limits_G f(x,y,z)dV$,
where k is a constant.

1103. If $f(x,y,z)\ge 0$ and G and T are nonoverlapping basic regions, then
$\iiint\limits_{G\cup T} f(x,y,z)dV = \iiint\limits_G f(x,y,z)dV + \iiint\limits_T f(x,y,z)dV$.
Here $G\cup T$ is the union of the regions G and T.

1104. Evaluation of Triple Integrals by Repeated Integrals

If the solid G is the set of points (x,y,z) such that $(x,y)\in R,\ \chi_1(x,y)\le z\le \chi_2(x,y)$, then

$$\iiint\limits_G f(x,y,z)\,dxdydz = \iint\limits_R \left[\int\limits_{\chi_1(x,y)}^{\chi_2(x,y)} f(x,y,z)\,dz\right] dxdy,$$

where R is projection of G onto the xy-plane.

If the solid G is the set of points (x,y,z) such that $a\le x\le b,\ \varphi_1(x)\le y\le \varphi_2(x),\ \chi_1(x,y)\le z\le \chi_2(x,y)$, then

$$\iiint\limits_G f(x,y,z)\,dxdydz = \int\limits_a^b \left[\int\limits_{\varphi_1(x)}^{\varphi_2(x)} \left(\int\limits_{\chi_1(x,y)}^{\chi_2(x,y)} f(x,y,z)\,dz\right) dy\right] dx$$

1105. Triple Integrals over Parallelepiped

If G is a parallelepiped $[a,\ b]\times[c,\ d]\times[r,\ s]$, then

$$\iiint\limits_G f(x,y,z)\,dxdydz = \int\limits_a^b \left[\int\limits_c^d \left(\int\limits_r^s f(x,y,z)\,dz\right) dy\right] dx.$$

In the special case where the integrand $f(x,y,z)$ can be written as $g(x)h(y)k(z)$ we have

$$\iiint\limits_G f(x,y,z)\,dxdydz = \left(\int\limits_a^b g(x)\,dx\right)\left(\int\limits_c^d h(y)\,dy\right)\left(\int\limits_r^s k(z)\,dz\right).$$

1106. Change of Variables

$$\iiint\limits_G f(x,y,z)\,dxdydz =$$

$$= \iiint\limits_S f[x(u,v,w),y(u,v,w),z(u,v,w)]\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dxdydz,$$

where $\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \neq 0$ is the jacobian of the transformations $(x,y,z) \to (u,v,w)$, and S is the pull-back of G which can be computed by $x = x(u,v,w)$,
$y = y(u,v,w)$
$z = z(u,v,w)$ into the definition of G.

1107. Triple Integrals in Cylindrical Coordinates
The differential dxdydz for cylindrical coordinates is

$$dxdydz = \left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| drd\theta dz = rdrd\theta dz.$$

Let the solid G is determined as follows:
$(x,y) \in R,\ \chi_1(x,y) \le z \le \chi_2(x,y)$,
where R is projection of G onto the xy-plane. Then

$$\iiint_G f(x,y,z)dxdydz = \iiint_S f(r\cos\theta, r\sin\theta, z)rdrd\theta dz$$

$$= \iint_{R(r,\theta)} \left[\int_{\chi_1(r\cos\theta, r\sin\theta)}^{\chi_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)dz \right] rdrd\theta.$$

Here S is the pullback of G in cylindrical coordinates.

1108. Triple Integrals in Spherical Coordinates
The Differential dxdydz for Spherical Coordinates is

$$dxdydz = \left|\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}\right| drd\theta d\varphi = r^2 \sin\theta drd\theta d\varphi$$

$$\iiint_G f(x,y,z)dxdydz =$$

$$= \iiint\limits_{S} f(r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta)\, r^2 \sin\theta\, dr\, d\theta\, d\varphi,$$

where the solid S is the pullback of G in spherical coordinates. The angle θ ranges from 0 to 2π, the angle φ ranges from 0 to π.

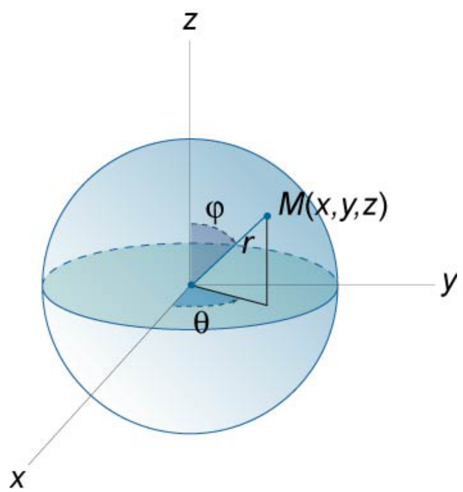


Figure 202.

1109. Volume of a Solid

$$V = \iiint\limits_{G} dx\,dy\,dz$$

1110. Volume in Cylindrical Coordinates

$$V = \iiint\limits_{S(r,\theta,z)} r\,dr\,d\theta\,dz$$

1111. Volume in Spherical Coordinates

$$V = \iiint\limits_{S(r,\theta,\varphi)} r^2 \sin\theta\, dr\, d\theta\, d\varphi$$

1112. Mass of a Solid

$$m = \iiint_G \mu(x,y,z)\,dV,$$

where the solid occupies a region G and its density at a point (x,y,z) is $\mu(x,y,z)$.

1113. Center of Mass of a Solid

$$\bar{x} = \frac{M_{yz}}{m}, \quad \bar{y} = \frac{M_{xz}}{m}, \quad \bar{z} = \frac{M_{xy}}{m},$$

where

$$M_{yz} = \iiint_G x\mu(x,y,z)\,dV,$$

$$M_{xz} = \iiint_G y\mu(x,y,z)\,dV,$$

$$M_{xy} = \iiint_G z\mu(x,y,z)\,dV$$

are the first moments about the coordinate planes $x = 0$, $y = 0$, $z = 0$, respectively, $\mu(x,y,z)$ is the density function.

1114. Moments of Inertia about the xy-plane (or $z = 0$), yz-plane ($x = 0$), and xz-plane ($y = 0$)

$$I_{xy} = \iiint_G z^2\mu(x,y,z)\,dV,$$

$$I_{yz} = \iiint_G x^2\mu(x,y,z)\,dV,$$

$$I_{xz} = \iiint_G y^2\mu(x,y,z)\,dV.$$

1115. Moments of Inertia about the x-axis, y-axis, and z-axis

$$I_x = I_{xy} + I_{xz} = \iiint_G \left(z^2 + y^2\right)\mu(x,y,z)\,dV,$$

$$I_y = I_{xy} + I_{yz} = \iiint_G \left(z^2 + x^2\right)\mu(x,y,z)\,dV,$$

$$I_z = I_{xz} + I_{yz} = \iiint\limits_G \left(y^2 + x^2\right)\mu(x,y,z)\,dV.$$

1116. Polar Moment of Inertia

$$I_0 = I_{xy} + I_{yz} + I_{xz} = \iiint\limits_G \left(x^2 + y^2 + z^2\right)\mu(x,y,z)\,dV$$

9.12 Line Integral

Scalar functions: $F(x,y,z)$, $F(x,y)$, $f(x)$
Scalar potential: $u(x,y,z)$
Curves: C, C_1, C_2
Limits of integrations: a, b, α, β
Parameters: t, s
Polar coordinates: r, θ
Vector field: $\vec{F}(P,Q,R)$
Position vector: $\vec{r}(s)$
Unit vectors: $\vec{i}$, $\vec{j}$, $\vec{k}$, $\vec{\tau}$
Area of region: S
Length of a curve: L
Mass of a wire: m
Density: $\rho(x,y,z)$, $\rho(x,y)$
Coordinates of center of mass: $\bar{x}$, $\bar{y}$, $\bar{z}$
First moments: M_{xy}, M_{yz}, M_{xz}
Moments of inertia: I_x, I_y, I_z
Volume of a solid: V
Work: W
Magnetic field: $\vec{B}$
Current: I
Electromotive force: ε
Magnetic flux: ψ

1117. Line Integral of a Scalar Function
Let a curve C be given by the vector function $\vec{r} = \vec{r}(s)$, $0 \le s \le S$, and a scalar function F is defined over the curve C. Then

$$\int_0^S F(\vec{r}(s))ds = \int_C F(x,y,z)ds = \int_C F ds,$$

where ds is the arc length differential.

1118. $$\int_{C_1 \cup C_2} F\,ds = \int_{C_1} F\,ds + \int_{C_2} F\,ds$$

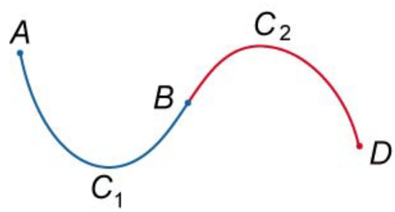


Figure 203.

1119. If the smooth curve C is parametrized by $\vec{r} = \vec{r}(t)$, $\alpha \le t \le \beta$, then

$$\int_C F(x,y,z)ds = \int_\alpha^\beta F(x(t),y(t),z(t))\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt.$$

1120. If C is a smooth curve in the xy-plane given by the equation $y = f(x)$, $a \le x \le b$, then

$$\int_C F(x,y)ds = \int_a^b F(x,f(x))\sqrt{1+(f'(x))^2}\,dx.$$

1121. Line Integral of Scalar Function in Polar Coordinates

$$\int_C F(x,y)ds = \int_\alpha^\beta F(r\cos\theta, r\sin\theta)\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta,$$

where the curve C is defined by the polar function $r(\theta)$.

1122. Line Integral of Vector Field

Let a curve C be defined by the vector function $\vec{r} = \vec{r}(s)$, $0 \le s \le S$. Then

$$\frac{d\vec{r}}{ds} = \vec{\tau} = (\cos\alpha, \cos\beta, \cos\gamma)$$

is the unit vector of the tangent line to this curve.

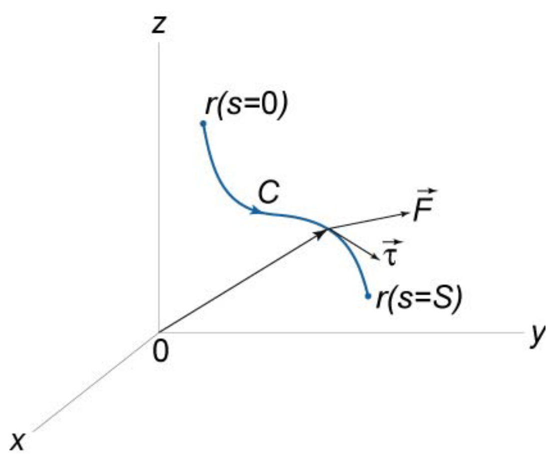


Figure 204.

Let a vector field $\vec{F}(P,Q,R)$ is defined over the curve C. Then the line integral of the vector field $\vec{F}$ along the curve C is

$$\int_C Pdx + Qdy + Rdz = \int_0^S (P\cos\alpha + Q\cos\beta + R\cos\gamma)ds.$$

1123. Properties of Line Integrals of Vector Fields

$$\int_{-C} \left(\vec{F} \cdot d\vec{r}\right) = -\int_{C} \left(\vec{F} \cdot d\vec{r}\right),$$

where -C denote the curve with the opposite orientation.

$$\int_{C} \left(\vec{F} \cdot d\vec{r}\right) = \int_{C_1 \cup C_2} \left(\vec{F} \cdot d\vec{r}\right) = \int_{C_1} \left(\vec{F} \cdot d\vec{r}\right) + \int_{C_2} \left(\vec{F} \cdot d\vec{r}\right),$$

where C is the union of the curves C_1 and C_2.

1124. If the curve C is parameterized by $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, $\alpha \le t \le \beta$, then

$$\int_{C} Pdx + Qdy + Rdz =$$

$$= \int_{\alpha}^{\beta} \left(P(x(t), y(t), z(t)) \frac{dx}{dt} + Q(x(t), y(t), z(t)) \frac{dy}{dt} + R(x(t), y(t), z(t)) \frac{dz}{dt} \right) dt$$

1125. If C lies in the xy-plane and given by the equation $y = f(x)$, then

$$\int_{C} Pdx + Qdy = \int_{a}^{b} \left(P(x, f(x)) + Q(x, f(x)) \frac{df}{dx} \right) dx .$$

1126. Green's Theorem

$$\iint_{R} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \oint_{C} Pdx + Qdy ,$$

where $\vec{F} = P(x,y)\vec{i} + Q(x,y)\vec{j}$ is a continuous vector function with continuous first partial derivatives $\frac{\partial P}{\partial y}$, $\frac{\partial Q}{\partial x}$ in a some domain R, which is bounded by a closed, piecewise smooth curve C.

1127. Area of a Region R Bounded by the Curve C

$$S = \iint_R dxdy = \frac{1}{2}\oint_C xdy - ydx$$

1128. Path Independence of Line Integrals

The line integral of a vector function $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is said to be path independent, if and only if P, Q, and R are continuous in a domain D, and if there exists some scalar function $u = u(x, y, z)$ (a scalar potential) in D such that

$\vec{F} = \operatorname{grad} u$, or $\frac{\partial u}{\partial x} = P$, $\frac{\partial u}{\partial y} = Q$, $\frac{\partial u}{\partial z} = R$.

Then

$$\int_C \vec{F}(\vec{r}) \cdot d\vec{r} = \int_C Pdx + Qdy + Rdz = u(B) - u(A).$$

1129. Test for a Conservative Field

A vector field of the form $\vec{F} = \operatorname{grad} u$ is called a conservative field. The line integral of a vector function $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is path independent if and only if

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \vec{0}.$$

If the line integral is taken in xy-plane so that

$$\int_C Pdx + Qdy = u(B) - u(A),$$

then the test for determining if a vector field is conservative can be written in the form

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

1130. Length of a Curve

$$L = \int_C ds = \int_\alpha^\beta \left|\frac{d\vec{r}}{dt}(t)\right| dt = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt,$$

where C ia a piecewise smooth curve described by the position vector $\vec{r}(t)$, $\alpha \le t \le \beta$.

If the curve C is two-dimensional, then

$$L = \int_C ds = \int_\alpha^\beta \left|\frac{d\vec{r}}{dt}(t)\right| dt = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

If the curve C is the graph of a function $y = f(x)$ in the xy-plane $(a \le x \le b)$, then

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

1131. Length of a Curve in Polar Coordinates

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta,$$

where the curve C is given by the equation $r = r(\theta)$, $\alpha \le \theta \le \beta$ in polar coordinates.

1132. Mass of a Wire

$$m = \int_C \rho(x, y, z)\, ds,$$

where $\rho(x, y, z)$ is the mass per unit length of the wire.

If C is a curve parametrized by the vector function $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$, then the mass can be computed by the formula

$$m = \int_{\alpha}^{\beta} \rho(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt .$$

If C is a curve in xy-plane, then the mass of the wire is given by

$$m = \int_C \rho(x, y)\, ds ,$$

or

$$m = \int_{\alpha}^{\beta} \rho(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \text{ (in parametric form).}$$

1133. Center of Mass of a Wire

$$\bar{x} = \frac{M_{yz}}{m}, \quad \bar{y} = \frac{M_{xz}}{m}, \quad \bar{z} = \frac{M_{xy}}{m},$$

where

$$M_{yz} = \int_C x\rho(x, y, z)\, ds ,$$

$$M_{xz} = \int_C y\rho(x, y, z)\, ds ,$$

$$M_{xy} = \int_C z\rho(x, y, z)\, ds .$$

1134. Moments of Inertia

The moments of inertia about the x-axis, y-axis, and z-axis are given by the formulas

$$I_x = \int_C (y^2 + z^2)\rho(x, y, z)\, ds ,$$

$$I_y = \int_C (x^2 + z^2)\rho(x, y, z)\, ds ,$$

$$I_z = \int_C (x^2 + y^2)\rho(x, y, z)\, ds .$$

1135. Area of a Region Bounded by a Closed Curve

$$S = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C x\,dy - y\,dx.$$

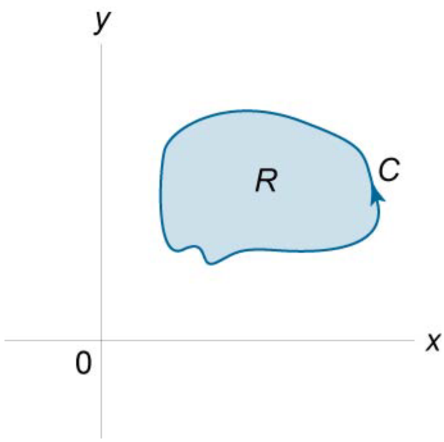


Figure 205.

If the closed curve C is given in parametric form $\vec{r}(t) = \langle x(t), y(t) \rangle$, then the area can be calculated by the formula

$$S = \int_\alpha^\beta x(t)\frac{dy}{dt}dt = -\int_\alpha^\beta y(t)\frac{dx}{dt}dt = \frac{1}{2}\int_\alpha^\beta \left(x(t)\frac{dy}{dt} - y(t)\frac{dx}{dt} \right) dt.$$

1136. Volume of a Solid Formed by Rotating a Closed Curve about the x-axis

$$V = -\pi\oint_C y^2 dx = -2\pi\oint_C xy\,dy = -\frac{\pi}{2}\oint_C 2xy\,dy + y^2 dx$$

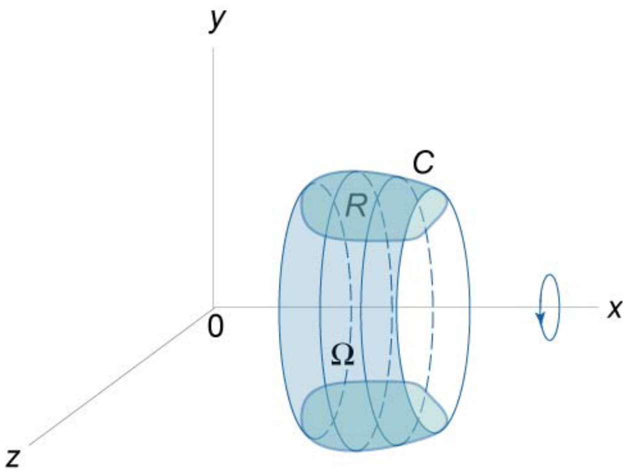


Figure 206.

1137. Work

Work done by a force $\vec{F}$ on an object moving along a curve C is given by the line integral

$$W = \int_C \vec{F} \cdot d\vec{r},$$

where $\vec{F}$ is the vector force field acting on the object, $d\vec{r}$ is the unit tangent vector.

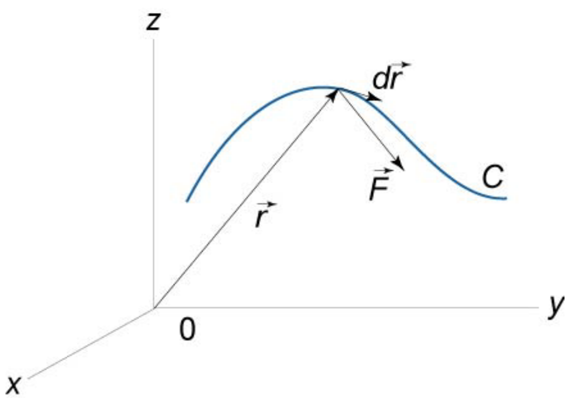


Figure 207.

If the object is moved along a curve C in the xy-plane, then
$$W = \int_C \vec{F} \cdot d\vec{r} = \int_C P dx + Q dy \, ,$$

If a path C is specified by a parameter t (t often means time), the formula for calculating work becomes
$$W = \int_\alpha^\beta \left[P(x(t), y(t), z(t)) \frac{dx}{dt} + Q(x(t), y(t), z(t)) \frac{dy}{dt} + R(x(t), y(t), z(t)) \frac{dz}{dt} \right] dt,$$
where t goes from α to β.

If a vector field $\vec{F}$ is conservative and $u(x, y, z)$ is a scalar potential of the field, then the work on an object moving from A to B can be found by the formula
$$W = u(B) - u(A).$$

1138. Ampere's Law
$$\oint_C \vec{B} \cdot d\vec{r} = \mu_0 I \, .$$
The line integral of a magnetic field $\vec{B}$ around a closed path C is equal to the total current I flowing through the area bounded by the path.

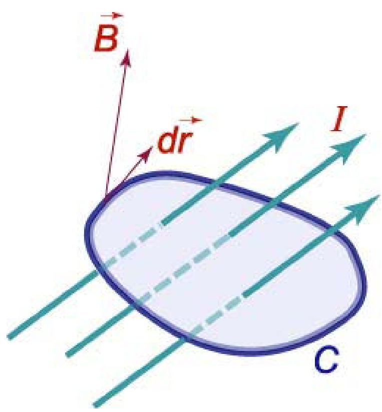


Figure 208.

1139. Faraday's Law

$$\varepsilon = \oint_C \vec{E} \cdot d\vec{r} = -\frac{d\psi}{dt}$$

The electromotive force (emf) ε induced around a closed loop C is equal to the rate of the change of magnetic flux ψ passing through the loop.

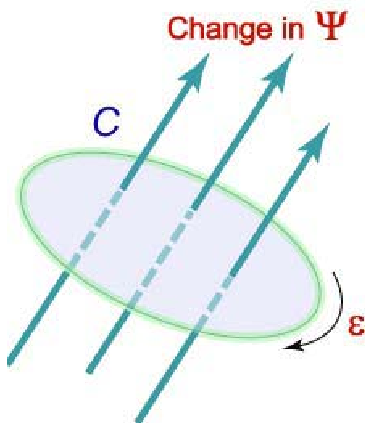


Figure 209.

9.13 Surface Integral

Scalar functions: $f(x,y,z)$, $z(x,y)$

Position vectors: $\vec{r}(u,v)$, $\vec{r}(x,y,z)$

Unit vectors: $\vec{i}$, $\vec{j}$, $\vec{k}$

Surface: S

Vector field: $\vec{F}(P,Q,R)$

Divergence of a vector field: $\text{div}\,\vec{F} = \nabla \cdot \vec{F}$

Curl of a vector field: $\text{curl}\,\vec{F} = \nabla \times \vec{F}$

Vector element of a surface: $d\vec{S}$

Normal to surface: $\vec{n}$

Surface area: A

Mass of a surface: m

Density: $\mu(x,y,z)$

Coordinates of center of mass: $\bar{x}$, $\bar{y}$, $\bar{z}$

First moments: M_{xy}, M_{yz}, M_{xz}

Moments of inertia: I_{xy}, I_{yz}, I_{xz}, I_x, I_y, I_z

Volume of a solid: V

Force: $\vec{F}$

Gravitational constant: G

Fluid velocity: $\vec{v}(\vec{r})$

Fluid density: ρ

Pressure: $p(\vec{r})$

Mass flux, electric flux: Φ

Surface charge: Q

Charge density: $\sigma(x,y)$

Magnitude of the electric field: $\vec{E}$

1140. Surface Integral of a Scalar Function

Let a surface S be given by the position vector

$\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$,

where (u,v) ranges over some domain $D(u,v)$ of the uv-plane.

The surface integral of a scalar function $f(x,y,z)$ over the surface S is defined as

$$\iint_S f(x,y,z)\,dS = \iint_{D(u,v)} f(x(u,v),y(u,v),z(u,v))\left|\frac{\partial\vec{r}}{\partial u} \times \frac{\partial\vec{r}}{\partial v}\right| du\,dv,$$

where the partial derivatives $\frac{\partial\vec{r}}{\partial u}$ and $\frac{\partial\vec{r}}{\partial v}$ are given by

$$\frac{\partial \vec{r}}{\partial u} = \frac{\partial x}{\partial u}(u,v)\vec{i} + \frac{\partial y}{\partial u}(u,v)\vec{j} + \frac{\partial z}{\partial u}(u,v)\vec{k},$$

$$\frac{\partial \vec{r}}{\partial v} = \frac{\partial x}{\partial v}(u,v)\vec{i} + \frac{\partial y}{\partial v}(u,v)\vec{j} + \frac{\partial z}{\partial v}(u,v)\vec{k}$$

and $\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}$ is the cross product.

1141. If the surface S is given by the equation $z = z(x,y)$ where $z(x,y)$ is a differentiable function in the domain $D(x,y)$, then

$$\iint_S f(x,y,z)dS = \iint_{D(x,y)} f(x,y,z(x,y))\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dxdy.$$

1142. Surface Integral of the Vector Field $\vec{F}$ over the Surface S

- If S is oriented outward, then

$$\iint_S \vec{F}(x,y,z)\cdot d\vec{S} = \iint_S \vec{F}(x,y,z)\cdot \vec{n}dS$$

$$= \iint_{D(u,v)} \vec{F}(x(u,v),y(u,v),z(u,v))\cdot\left[\frac{\partial \vec{r}}{\partial u}\times\frac{\partial \vec{r}}{\partial v}\right]dudv.$$

- If S is oriented inward, then

$$\iint_S \vec{F}(x,y,z)\cdot d\vec{S} = \iint_S \vec{F}(x,y,z)\cdot \vec{n}dS$$

$$= \iint_{D(u,v)} \vec{F}(x(u,v),y(u,v),z(u,v))\cdot\left[\frac{\partial \vec{r}}{\partial v}\times\frac{\partial \vec{r}}{\partial u}\right]dudv.$$

$d\vec{S} = \vec{n}dS$ is called the vector element of the surface. Dot means the scalar product of the appropriate vectors. The partial derivatives $\frac{\partial \vec{r}}{\partial u}$ and $\frac{\partial \vec{r}}{\partial v}$ are given by

$$\frac{\partial\vec{r}}{\partial u}=\frac{\partial x}{\partial u}(u,v)\cdot\vec{i}+\frac{\partial y}{\partial u}(u,v)\cdot\vec{j}+\frac{\partial z}{\partial u}(u,v)\cdot\vec{k},$$
$$\frac{\partial\vec{r}}{\partial v}=\frac{\partial x}{\partial v}(u,v)\cdot\vec{i}+\frac{\partial y}{\partial v}(u,v)\cdot\vec{j}+\frac{\partial z}{\partial v}(u,v)\cdot\vec{k}.$$

1143. If the surface S is given by the equation $z=z(x,y)$, where $z(x,y)$ is a differentiable function in the domain $D(x,y)$, then

- If S is oriented upward, i.e. the k-th component of the normal vector is positive, then
$$\iint_S \vec{F}(x,y,z)\cdot d\vec{S}=\iint_S \vec{F}(x,y,z)\cdot\vec{n}dS$$
$$=\iint_{D(x,y)}\vec{F}(x,y,z)\cdot\left(-\frac{\partial z}{\partial x}\vec{i}-\frac{\partial z}{\partial y}\vec{j}+\vec{k}\right)dxdy,$$

- If S is oriented downward, i.e. the k-th component of the normal vector is negative, then
$$\iint_S \vec{F}(x,y,z)\cdot d\vec{S}=\iint_S \vec{F}(x,y,z)\cdot\vec{n}dS$$
$$=\iint_{D(x,y)}\vec{F}(x,y,z)\cdot\left(\frac{\partial z}{\partial x}\vec{i}+\frac{\partial z}{\partial y}\vec{j}-\vec{k}\right)dxdy.$$

1144. $$\iint_S(\vec{F}\cdot\vec{n})dS=\iint_S Pdydz+Qdzdx+Rdxdy$$
$$=\iint_S(P\cos\alpha+Q\cos\beta+R\cos\gamma)dS,$$
where $P(x,y,z)$, $Q(x,y,z)$, $R(x,y,z)$ are the components of the vector field $\vec{F}$.
$\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the angles between the outer unit normal vector $\vec{n}$ and the x-axis, y-axis, and z-axis, respectively.

1145. If the surface S is given in parametric form by the vector $\vec{r}(x(u,v), y(u,v), z(u,v))$, then the latter formula can be written as

$$\iint_S (\vec{F}\cdot\vec{n})dS = \iint_S Pdydz + Qdzdx + Rdxdy = \iint_{D(u,v)} \begin{vmatrix} P & Q & R \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} dudv,$$

where (u,v) ranges over some domain $D(u,v)$ of the uv-plane.

1146. Divergence Theorem

$$\oint\!\!\oint_S \vec{F}\cdot d\vec{S} = \iiint_G (\nabla\cdot\vec{F})dV,$$

where

$$\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$$

is a vector field whose components P, Q, and R have continuous partial derivatives,

$$\nabla\cdot\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

is the divergence of $\vec{F}$, also denoted $\mathrm{div}\vec{F}$. The symbol $\oint\!\!\oint$ indicates that the surface integral is taken over a closed surface.

1147. Divergence Theorem in Coordinate Form

$$\oint\!\!\oint_S Pdydz + Qdxdz + Rdxdy = \iiint_G \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)dxdydz.$$

1148. Stoke's Theorem

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot d\vec{S},$$

where

$\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$

is a vector field whose components P, Q, and R have continuous partial derivatives,

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial x} & \frac{\partial}{\partial x} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

is the curl of $\vec{F}$, also denoted curl $\vec{F}$.

The symbol $\oint$ indicates that the line integral is taken over a closed curve.

1149. Stoke's Theorem in Coordinate Form

$$\oint_C P dx + Q dy + R dz$$

$$= \iint_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dydz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dzdx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

1150. Surface Area

$$A = \iint_S dS$$

1151. If the surface S is parameterized by the vector

$\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$,

then the surface area is

$$A = \iint_{D(u,v)} \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| dudv,$$

where $D(u,v)$ is the domain where the surface $\vec{r}(u,v)$ is defined.

1152. If S is given explicitly by the function $z(x,y)$, then the surface area is

$$A = \iint_{D(x,y)} \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dxdy,$$

where $D(x,y)$ is the projection of the surface S onto the xy-plane.

1153. Mass of a Surface

$$m = \iint_S \mu(x,y,z)dS,$$

where $\mu(x,y,z)$ is the mass per unit area (density function).

1154. Center of Mass of a Shell

$$\bar{x} = \frac{M_{yz}}{m},\ \bar{y} = \frac{M_{xz}}{m},\ \bar{z} = \frac{M_{xy}}{m},$$

where

$$M_{yz} = \iint_S x\mu(x,y,z)dS,$$

$$M_{xz} = \iint_S y\mu(x,y,z)dS,$$

$$M_{xy} = \iint_S z\mu(x,y,z)dS$$

are the first moments about the coordinate planes $x=0$, $y=0$, $z=0$, respectively. $\mu(x,y,z)$ is the density function.

1155. Moments of Inertia about the xy-plane (or $z=0$), yz-plane ($x=0$), and xz-plane ($y=0$)

$$I_{xy} = \iint_S z^2\mu(x,y,z)dS,$$

$$I_{yz} = \iint_S x^2\mu(x,y,z)dS,$$

$$I_{xz} = \iint_S y^2 \mu(x,y,z) dS.$$

1156. Moments of Inertia about the x-axis, y-axis, and z-axis

$$I_x = \iint_S (y^2 + z^2) \mu(x,y,z) dS,$$

$$I_y = \iint_S (x^2 + z^2) \mu(x,y,z) dS,$$

$$I_z = \iint_S (x^2 + y^2) \mu(x,y,z) dS.$$

1157. Volume of a Solid Bounded by a Closed Surface

$$V = \frac{1}{3} \left| \oiint_S x dydz + y dxdz + z dxdy \right|$$

1158. Gravitational Force

$$\vec{F} = Gm \iint_S \mu(x,y,z) \frac{\vec{r}}{r^3} dS,$$

where m is a mass at a point $\langle x_0, y_0, z_0 \rangle$ outside the surface,

$\vec{r} = \langle x - x_0, y - y_0, z - z_0 \rangle$,

$\mu(x,y,z)$ is the density function,

and G is gravitational constant.

1159. Pressure Force

$$\vec{F} = \iint_S p(\vec{r}) d\vec{S},$$

where the pressure $p(\vec{r})$ acts on the surface S given by the position vector $\vec{r}$.

1160. Fluid Flux (across the surface S)

$$\Phi = \oiint_S \vec{v}(\vec{r}) \cdot d\vec{S},$$

where $\vec{v}(\vec{r})$ is the fluid velocity.

1161. Mass Flux (across the surface S)

$$\Phi = \oiint_S \rho \vec{v}(\vec{r}) \cdot d\vec{S},$$

where $\vec{F} = \rho \vec{v}$ is the vector field, ρ is the fluid density.

1162. Surface Charge

$$Q = \iint_S \sigma(x, y) dS,$$

where $\sigma(x, y)$ is the surface charge density.

1163. Gauss' Law

The electric flux through any closed surface is proportional to the charge Q enclosed by the surface

$$\Phi = \oiint_S \vec{E} \cdot d\vec{S} = \frac{Q}{\varepsilon_0},$$

where

Φ is the electric flux,

$\vec{E}$ is the magnitude of the electric field strength,

$\varepsilon_0 = 8{,}85 \times 10^{-12} \frac{F}{m}$ is permittivity of free space.

Chapter 10

Differential Equations

Functions of one variable: y, p, q, u, g, h, G, H, r, z
Arguments (independent variables): x, y
Functions of two variables: $f(x,y)$, $M(x,y)$, $N(x,y)$

First order derivative: y', u', $\dot{y}$, $\frac{dy}{dt}$, ...

Second order derivatives: y'', $\ddot{y}$, $\frac{d^2I}{dt^2}$, ...

Partial derivatives: $\frac{\partial u}{\partial t}$, $\frac{\partial^2 u}{\partial x^2}$, ...

Natural number: n
Particular solutions: y_1, y_p
Real numbers: k, t, C, C_1, C_2, p, q, α, β
Roots of the characteristic equations: λ_1, λ_2
Time: t
Temperature: T, S
Population function: $P(t)$
Mass of an object: m
Stiffness of a spring: k
Displacement of the mass from equilibrium: y
Amplitude of the displacement: A
Frequency: ω
Damping coefficient: γ
Phase angle of the displacement: δ
Angular displacement: θ
Pendulum length: L

Acceleration of gravity: g
Current: I
Resistance: R
Inductance: L
Capacitance: C

10.1 First Order Ordinary Differential Equations

1164. Linear Equations

$$\frac{dy}{dx} + p(x)y = q(x).$$

The general solution is

$$y = \frac{\int u(x)q(x)dx + C}{u(x)},$$

where

$$u(x) = \exp\left(\int p(x)dx\right).$$

1165. Separable Equations

$$\frac{dy}{dx} = f(x,y) = g(x)h(y)$$

The general solution is given by

$$\int \frac{dy}{h(y)} = \int g(x)dx + C,$$

or

$$H(y) = G(x) + C.$$

1166. Homogeneous Equations

The differential equation $\frac{dy}{dx} = f(x, y)$ is homogeneous, if the function $f(x, y)$ is homogeneous, that is
$f(tx, ty) = f(x, y)$.

The substitution $z = \frac{y}{x}$ (then $y = zx$) leads to the separable equation
$$x\frac{dz}{dx} + z = f(1, z).$$

1167. Bernoulli Equation
$$\frac{dy}{dx} + p(x)y = q(x)y^n.$$

The substitution $z = y^{1-n}$ leads to the linear equation
$$\frac{dz}{dx} + (1-n)p(x)z = (1-n)q(x).$$

1168. Riccati Equation
$$\frac{dy}{dx} = p(x) + q(x)y + r(x)y^2$$

If a particular solution y_1 is known, then the general solution can be obtained with the help of substitution
$z = \frac{1}{y - y_1}$, which leads to the first order linear equation
$$\frac{dz}{dx} = -[q(x) + 2y_1 r(x)]z - r(x).$$

1169. Exact and Nonexact Equations

The equation

$M(x,y)dx + N(x,y)dy = 0$

is called exact if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},$$

and nonexact otherwise.

The general solution is

$\int M(x,y)dx + \int N(x,y)dy = C$.

1170. Radioactive Decay

$$\frac{dy}{dt} = -ky,$$

where $y(t)$ is the amount of radioactive element at time t, k is the rate of decay.

The solution is

$y(t) = y_0 e^{-kt}$, where $y_0 = y(0)$ is the initial amount.

1171. Newton's Law of Cooling

$$\frac{dT}{dt} = -k(T - S),$$

where $T(t)$ is the temperature of an object at time t, S is the temperature of the surrounding environment, k is a positive constant.

The solution is

$T(t) = S + (T_0 - S)e^{-kt}$,

where $T_0 = T(0)$ is the initial temperature of the object at time $t = 0$.

1172. Population Dynamics (Logistic Model)

$$\frac{dP}{dt}=kP\left(1-\frac{P}{M}\right),$$

where $P(t)$ is population at time t, k is a positive constant, M is a limiting size for the population.

The solution of the differential equation is

$P(t)=\dfrac{MP_0}{P_0+(M-P_0)e^{-kt}}$, where $P_0=P(0)$ is the initial population at time $t=0$.

10.2 Second Order Ordinary Differential Equations

1173. Homogeneous Linear Equations with Constant Coefficients

$$y''+py'+qy=0.$$

The characteristic equation is

$$\lambda^2+p\lambda+q=0.$$

If λ_1 and λ_2 are distinct real roots of the characteristic equation, then the general solution is

$y=C_1e^{\lambda_1 x}+C_2e^{\lambda_2 x}$, where

C_1 and C_2 are integration constants.

If $\lambda_1=\lambda_2=-\dfrac{p}{2}$, then the general solution is

$$y=(C_1+C_2x)e^{-\frac{p}{2}x}.$$

If λ_1 and λ_2 are complex numbers:

$\lambda_1 = \alpha + \beta i$, $\lambda_2 = \alpha - \beta i$, where

$\alpha = -\frac{p}{2}$, $\beta = \frac{\sqrt{4q - p^2}}{2}$,

then the general solution is

$y = e^{\alpha x}\left(C_1 \cos\beta x + C_2 \sin\beta x\right)$.

1174. Inhomogeneous Linear Equations with Constant Coefficients

$y'' + py' + qy = f(x)$.

The general solution is given by

$y = y_p + y_h$, where

y_p is a particular solution of the inhomogeneous equation and y_h is the general solution of the associated homogeneous equation (see the previous topic 1173).

If the right side has the form

$f(x) = e^{\alpha x}\left(P_1(x)\cos\beta x + P_1(x)\sin\beta x\right)$,

then the particular solution y_p is given by

$y_p = x^k e^{\alpha x}\left(R_1(x)\cos\beta x + R_2(x)\sin\beta x\right)$,

where the polynomials $R_1(x)$ and $R_2(x)$ have to be found by using the method of undetermined coefficients.

- If $\alpha + \beta i$ is not a root of the characteristic equation, then the power $k = 0$,
- If $\alpha + \beta i$ is a simple root, then $k = 1$,
- If $\alpha + \beta i$ is a double root, then $k = 2$.

1175. Differential Equations with y Missing

$y'' = f(x, y')$.

Set $u = y'$. Then the new equation satisfied by v is

$u' = f(x, u)$,

which is a first order differential equation.

1176. Differential Equations with x Missing

$y'' = f(y, y')$.

Set $u = y'$. Since

$$y'' = \frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx} = u\frac{du}{dy},$$

we have

$$u\frac{du}{dy} = f(y, u),$$

which is a first order differential equation.

1177. Free Undamped Vibrations

The motion of a Mass on a Spring is described by the equation

$m\ddot{y} + ky = 0$,

where

m is the mass of the object,

k is the stiffness of the spring,

y is displacement of the mass from equilibrium.

The general solution is

$y = A\cos(\omega_0 t - \delta)$,

where

A is the amplitude of the displacement,

ω_0 is the fundamental frequency, the period is $T = \frac{2\pi}{\omega_0}$,

δ is phase angle of the displacement.

This is an example of simple harmonic motion.

1178. Free Damped Vibrations

$m\ddot{y} + \gamma\dot{y} + ky = 0$, where

γ is the damping coefficient.

There are 3 cases for the general solution:

Case 1. $\gamma^2 > 4km$ (overdamped)

$y(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$,

where

$$\lambda_1 = \frac{-\gamma - \sqrt{\gamma^2 - 4km}}{2m},\ \lambda_2 = \frac{-\gamma + \sqrt{\gamma^2 - 4km}}{2m}.$$

Case 2. $\gamma^2 = 4km$ (critically damped)

$y(t) = (A + Bt)e^{\lambda t}$,

where

$$\lambda = -\frac{\gamma}{2m}.$$

Case 3. $\gamma^2 < 4km$ (underdamped)

$y(t) = e^{-\frac{\gamma}{2m}t} A\cos(\omega t - \delta)$, where

$\omega = \sqrt{4km - \gamma^2}$.

1179. Simple Pendulum

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0,$$

where θ is the angular displacement, L is the pendulum length, g is the acceleration of gravity.

The general solution for small angles θ is

$\theta(t) = \theta_{max} \sin\sqrt{\frac{g}{L}}t$, the period is $T = 2\pi\sqrt{\frac{L}{g}}$.

1180. RLC Circuit

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = V'(t) = \omega E_0 \cos(\omega t),$$

where I is the current in an RLC circuit with an ac voltage source $V(t)=E_0\sin(\omega t)$.

The general solution is
$I(t)=C_1e^{r_1t}+C_2e^{r_2t}+A\sin(\omega t-\varphi)$,
where

$$r_{1,2}=\frac{-R\pm\sqrt{R^2-\frac{4L}{C}}}{2L},$$

$$A=\frac{\omega E_0}{\sqrt{\left(L\omega^2-\frac{1}{C}\right)^2+R^2\omega^2}},$$

$$\varphi=\arctan\left(\frac{L\omega}{R}-\frac{1}{RC\omega}\right),$$

C_1, C_2 are constants depending on initial conditions.

10.3. Some Partial Differential Equations

1181. The Laplace Equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

applies to potential energy function $u(x,y)$ for a conservative force field in the xy-plane. Partial differential equations of this type are called elliptic.

1182. The Heat Equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial u}{\partial t}$$

applies to the temperature distribution $u(x,y)$ in the xy-plane when heat is allowed to flow from warm areas to cool ones. The equations of this type are called parabolic.

1183. The Wave Equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial t^2}$$

applies to the displacement $u(x,y)$ of vibrating membranes and other wave functions. The equations of this type are called hyperbolic.

Chapter 11

Series

11.1 Arithmetic Series

Initial term: a_1
Nth term: a_n
Difference between successive terms: d
Number of terms in the series: n
Sum of the first n terms: S_n

1184. $a_n = a_{n-1} + d = a_{n-2} + 2d = \ldots = a_1 + (n-1)d$

1185. $a_1 + a_n = a_2 + a_{n-1} = \ldots = a_i + a_{n+1-i}$

1186. $a_i = \dfrac{a_{i-1} + a_{i+1}}{2}$

1187. $S_n = \dfrac{a_1 + a_n}{2} \cdot n = \dfrac{2a_1 + (n-1)d}{2} \cdot n$

11.2 Geometric Series

Initial term: a_1
Nth term: a_n
Common ratio: q
Number of terms in the series: n
Sum of the first n terms: S_n
Sum to infinity: S

1188. $a_n = qa_{n-1} = a_1 q^{n-1}$

1189. $a_1 \cdot a_n = a_2 \cdot a_{n-1} = \ldots = a_i \cdot a_{n+1-i}$

1190. $a_i = \sqrt{a_{i-1} \cdot a_{i+1}}$

1191. $S_n = \dfrac{a_n q - a_1}{q-1} = \dfrac{a_1(q^n - 1)}{q-1}$

1192. $S = \lim_{n\to\infty} S_n = \dfrac{a_1}{1-q}$

For $|q| < 1$, the sum S converges as $n \to \infty$.

11.3 Some Finite Series

Number of terms in the series: n

1193. $1+2+3+\ldots+n=\frac{n(n+1)}{2}$

1194. $2+4+6+\ldots+2n=n(n+1)$

1195. $1+3+5+\ldots+(2n-1)=n^2$

1196. $k+(k+1)+(k+2)+\ldots+(k+n-1)=\frac{n(2k+n-1)}{2}$

1197. $1^2+2^2+3^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}$

1198. $1^3+2^3+3^3+\ldots+n^3=\left[\frac{n(n+1)}{2}\right]^2$

1199. $1^2+3^2+5^2+\ldots+(2n-1)^2=\frac{n(4n^2-1)}{3}$

1200. $1^3+3^3+5^3+\ldots+(2n-1)^3=n^2(2n^2-1)$

1201. $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^n}+\ldots=2$

1202. $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{n(n+1)}+\ldots=1$

1203. $1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{(n-1)!}+\ldots=e$

11.4 Infinite Series

Sequence: $\{a_n\}$
First term: a_1
Nth term: a_n

1204. Infinite Series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots + a_n + \ldots$$

1205. Nth Partial Sum

$$S_n = \sum_{n=1}^{n} a_n = a_1 + a_2 + \ldots + a_n$$

1206. Convergence of Infinite Series

$$\sum_{n=1}^{\infty} a_n = L, \text{ if } \lim_{n\to\infty} S_n = L$$

1207. Nth Term Test

- If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.
- If $\lim_{n\to\infty} a_n \neq 0$, then the series is divergent.

11.5 Properties of Convergent Series

Convergent Series: $\sum_{n=1}^{\infty} a_n = A$, $\sum_{n=1}^{\infty} b_n = B$

Real number: c

1208. $\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n = A + B$

1209. $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n = cA$.

11.6 Convergence Tests

1210. The Comparison Test

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series such that $0 < a_n \le b_n$ for all n.

- If $\sum_{n=1}^{\infty} b_n$ is convergent then $\sum_{n=1}^{\infty} a_n$ is also convergent.
- If $\sum_{n=1}^{\infty} a_n$ is divergent then $\sum_{n=1}^{\infty} b_n$ is also divergent.

1211. The Limit Comparison Test

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series such that a_n and b_n are positive for all n.

- If $0 < \lim_{n\to\infty} \frac{a_n}{b_n} < \infty$ then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are either both convergent or both divergent.
- If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ then $\sum_{n=1}^{\infty} b_n$ convergent implies that $\sum_{n=1}^{\infty} a_n$ is also convergent.

- If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ then $\sum_{n=1}^{\infty} b_n$ divergent implies that $\sum_{n=1}^{\infty} a_n$ is also divergent.

1212. p-series

p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$ and diverges for $0 < p \le 1$.

1213. The Integral Test

Let $f(x)$ be a function which is continuous, positive, and decreasing for all $x \ge 1$. The series

$$\sum_{n=1}^{\infty} f(n) = f(1) + f(2) + f(3) + \ldots + f(n) + \ldots$$

converges if $\int_1^{\infty} f(x)dx$ converges, and diverges if

$$\int_1^n f(x)dx \to \infty \text{ as } n \to \infty.$$

1214. The Ratio Test

Let $\sum_{n=1}^{\infty} a_n$ be a series with positive terms.

- If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$ then $\sum_{n=1}^{\infty} a_n$ is convergent.
- If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$ then $\sum_{n=1}^{\infty} a_n$ is divergent.
- If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$ then $\sum_{n=1}^{\infty} a_n$ may converge or diverge and the ratio test is inconclusive; some other tests must be used.

1215. The Root Test

Let $\sum_{n=1}^{\infty} a_n$ be a series with positive terms.

- If $\lim_{n\to\infty} \sqrt[n]{a_n} < 1$ then $\sum_{n=1}^{\infty} a_n$ is convergent.
- If $\lim_{n\to\infty} \sqrt[n]{a_n} > 1$ then $\sum_{n=1}^{\infty} a_n$ is divergent.
- If $\lim_{n\to\infty} \sqrt[n]{a_n} = 1$ then $\sum_{n=1}^{\infty} a_n$ may converge or diverge, but no conclusion can be drawn from this test.

11.7 Alternating Series

1216. The Alternating Series Test (Leibniz's Theorem)

Let $\{a_n\}$ be a sequence of positive numbers such that
$a_{n+1} < a_n$ for all n.
$\lim_{n\to\infty} a_n = 0$.

Then the alternating series $\sum_{n=1}^{\infty} (-1)^n a_n$ and $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ both converge.

1217. Absolute Convergence

- A series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if the series $\sum_{n=1}^{\infty} |a_n|$ is convergent.

- If the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent then it is convergent.

1218. Conditional Convergence

A series $\sum_{n=1}^{\infty} a_n$ is conditionally convergent if the series is convergent but is not absolutely convergent.

11.8 Power Series

Real numbers: $x,\ x_0$

Power series: $\sum_{n=0}^{\infty} a_n x^n,\ \sum_{n=0}^{\infty} a_n (x-x_0)^n$

Whole number: n

Radius of Convergence: R

1219. Power Series in x

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$$

1220. Power Series in $(x-x_0)$

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \ldots + a_n(x-x_0)^n + \ldots$$

1221. Interval of Convergence

The set of those values of x for which the function

$f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$ is convergent is called the interval of convergence.

1222. Radius of Convergence

If the interval of convergence is $(x_0 - R, x_0 + R)$ for some $R \geq 0$, the R is called the radius of convergence. It is given as

$$R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{a_n}} \text{ or } R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|.$$

11.9 Differentiation and Integration of Power Series

Continuous function: $f(x)$

Power series: $\sum_{n=0}^{\infty} a_n x^n$

Whole number: n

Radius of Convergence: R

1223. Differentiation of Power Series

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots$ for $|x| < R$.

Then, for $|x| < R$, $f(x)$ is continuous, the derivative $f'(x)$ exists and

$$f'(x) = \frac{d}{dx} a_0 + \frac{d}{dx} a_1 x + \frac{d}{dx} a_2 x^2 + \ldots$$

$$= a_1 + 2a_2 x + 3a_3 x^2 + \ldots = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

1224. Integration of Power Series

Let $f(x)=\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots$ for $|x|<R$.

Then, for $|x|<R$, the indefinite integral $\int f(x)dx$ exists and

$$\int f(x)dx = \int a_0 dx + \int a_1 x dx + \int a_2 x^2 dx + \ldots$$

$$= a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} + \ldots = \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n+1} + C.$$

11.10 Taylor and Maclaurin Series

Whole number: n
Differentiable function: $f(x)$
Remainder term: R_n

1225. Taylor Series

$$f(x)=\sum_{n=0}^{\infty} f^{(n)}(a)\frac{(x-a)^n}{n!} = f(a)+f'(a)(x-a)+\frac{f''(a)(x-a)^2}{2!}+\ldots$$

$$+\frac{f^{(n)}(a)(x-a)^n}{n!}+R_n.$$

1226. The Remainder After n+1 Terms is given by

$$R_n = \frac{f^{(n+1)}(\xi)(x-a)^{n+1}}{(n+1)!}, \quad a<\xi<x.$$

1227. Maclaurin Series

$$f(x)=\sum_{n=0}^{\infty} f^{(n)}(0)\frac{x^n}{n!} = f(0)+f'(0)x+\frac{f''(0)x^2}{2!}+\ldots+\frac{f^{(n)}(0)x^n}{n!}+R_n$$

11.11 Power Series Expansions for Some Functions

Whole number: n
Real number: x

1228. $e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^n}{n!}+\ldots$

1229. $a^x = 1+\frac{x\ln a}{1!}+\frac{(x\ln a)^2}{2!}+\frac{(x\ln a)^3}{3!}+\ldots+\frac{(x\ln a)^n}{n!}+\ldots$

1230. $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots+\frac{(-1)^n x^{n+1}}{n+1}\pm\ldots,\ -1<x\le 1.$

1231. $\ln\frac{1+x}{1-x}=2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}+\ldots\right),\ |x|<1.$

1232. $\ln x = 2\left[\frac{x-1}{x+1}+\frac{1}{3}\left(\frac{x-1}{x+1}\right)^3+\frac{1}{5}\left(\frac{x-1}{x+1}\right)^5\ldots\right],\ x>0.$

1233. $\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots+\frac{(-1)^n x^{2n}}{(2n)!}\pm\ldots$

1234. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} \pm \ldots$

1235. $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \ldots, \ |x| < \frac{\pi}{2}.$

1236. $\cot x = \frac{1}{x} - \left(\frac{x}{3} + \frac{x^3}{45} + \frac{2x^5}{945} + \frac{2x^7}{4725} + \ldots \right), \ |x| < \pi.$

1237. $\arcsin x = x + \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3x^5}{2 \cdot 4 \cdot 5} + \ldots + \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)x^{2n+1}}{2 \cdot 4 \cdot 6 \ldots (2n)(2n+1)} + \ldots,$

$|x| < 1.$

1238. $\arccos x = \frac{\pi}{2} - \left(x + \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3x^5}{2 \cdot 4 \cdot 5} + \ldots + \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)x^{2n+1}}{2 \cdot 4 \cdot 6 \ldots (2n)(2n+1)} + \ldots \right),$

$|x| < 1.$

1239. $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots + \frac{(-1)^n x^{2n+1}}{2n+1} \pm \ldots, \ |x| \le 1.$

1240. $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots + \frac{x^{2n}}{(2n)!} + \ldots$

1241. $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots + \frac{x^{2n+1}}{(2n+1)!} + \ldots$

11.12 Binomial Series

Whole numbers: n, m
Real number: x
Combinations: ${}^{n}C_{m}$

1242. $(1+x)^n = 1 + {}^{n}C_1 x + {}^{n}C_2 x^2 + \ldots + {}^{m}C_n x^m + \ldots + x^n$

1243. ${}^{n}C_m = \dfrac{n(n-1)\ldots[n-(m-1)]}{m!}, \ |x| < 1.$

1244. $\dfrac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots, \ |x| < 1.$

1245. $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots, \ |x| < 1.$

1246. $\sqrt{1+x} = 1 + \dfrac{x}{2} - \dfrac{x^2}{2\cdot 4} + \dfrac{1\cdot 3x^3}{2\cdot 4\cdot 6} - \dfrac{1\cdot 3\cdot 5x^4}{2\cdot 4\cdot 6\cdot 8} + \ldots, \ |x| \le 1.$

1247. $\sqrt[3]{1+x} = 1 + \dfrac{x}{3} - \dfrac{1\cdot 2x^2}{3\cdot 6} + \dfrac{1\cdot 2\cdot 5x^3}{3\cdot 6\cdot 9} - \dfrac{1\cdot 2\cdot 5\cdot 8x^4}{3\cdot 6\cdot 9\cdot 12} + \ldots, \ |x| \le 1.$

11.13 Fourier Series

Integrable function: $f(x)$
Fourier coefficients: a_0, a_n, b_n
Whole number: n

1248. $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

1249. $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$

1250. $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$

Chapter 12

Probability

12.1 Permutations and Combinations

Permutations: ${}^{n}P_{m}$
Combinations: ${}^{n}C_{m}$
Whole numbers: n, m

1251. Factorial

$$n! = 1 \cdot 2 \cdot 3 \ldots (n-2)(n-1)n$$

$$0! = 1$$

1252. ${}^{n}P_{n} = n!$

1253. ${}^{n}P_{m} = \frac{n!}{(n-m)!}$

1254. Binomial Coefficient

$${}^{n}C_{m} = \binom{n}{m} = \frac{n!}{m!(n-m)!}$$

1255. ${}^{n}C_{m} = {}^{n}C_{n-m}$

1256. ${}^{n}C_{m} + {}^{n}C_{m+1} = {}^{n+1}C_{m+1}$

1257. $^{n}C_0 + ^{n}C_1 + ^{n}C_2 + \ldots + ^{n}C_n = 2^n$

1258. Pascal's Triangle

Row 0							1						
Row 1						1		1					
Row 2					1		2		1				
Row 3				1		3		3		1			
Row 4			1		4		6		4		1		
Row 5		1		5		10		10		5		1	
Row 6	1		6		15		20		15		6		1

12.2 Probability Formulas

Events: A, B
Probability: P
Random variables: X, Y, Z
Values of random variables: x, y, z
Expected value of X: μ
Any positive real number: ε
Standard deviation: σ
Variance: σ^2
Density functions: $f(x)$, $f(t)$

1259. Probability of an Event

$$P(A) = \frac{m}{n},$$

where
m is the number of possible positive outcomes,
n is the total number of possible outcomes.

1260. Range of Probability Values
$0 \le P(A) \le 1$

1261. Certain Event
$P(A) = 1$

1262. Impossible Event
$P(A) = 0$

1263. Complement
$P(\overline{A}) = 1 - P(A)$

1264. Independent Events
$P(A/B) = P(A)$,
$P(B/A) = P(B)$

1265. Addition Rule for Independent Events
$P(A \cup B) = P(A) + P(B)$

1266. Multiplication Rule for Independent Events
$P(A \cap B) = P(A) \cdot P(B)$

1267. General Addition Rule
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$,
where
$A \cup B$ is the union of events A and B,
$A \cap B$ is the intersection of events A and B.

1268. Conditional Probability
$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

1269. $P(A \cap B) = P(B) \cdot P(A/B) = P(A) \cdot P(B/A)$

1270. Law of Total Probability

$$P(A)=\sum_{i=1}^{m} P(B_i)P(A/B_i),$$

where B_i is a sequence of mutually exclusive events.

1271. Bayes' Theorem

$$P(B/A)=\frac{P(A/B)\cdot P(B)}{P(A)}$$

1272. Bayes' Formula

$$P(B_i/A)=\frac{P(B_i)\cdot P(A/B_i)}{\sum_{k=1}^{m} P(B_i)\cdot P(A/B_i)},$$

where

B_i is a set of mutually exclusive events (hypotheses),

A is the final event,

$P(B_i)$ are the prior probabilities,

$P(B_i/A)$ are the posterior probabilities.

1273. Law of Large Numbers

$$P\left(\left|\frac{S_n}{n}-\mu\right|\geq\varepsilon\right)\rightarrow 0 \text{ as } n\rightarrow\infty,$$

$$P\left(\left|\frac{S_n}{n}-\mu\right|<\varepsilon\right)\rightarrow 1 \text{ as } n\rightarrow\infty,$$

where

S_n is the sum of random variables,

n is the number of possible outcomes.

1274. Chebyshev Inequality

$$P(|X-\mu|\geq\varepsilon)\leq\frac{V(X)}{\varepsilon^2},$$

where $V(X)$ is the variance of X.

1275. Normal Density Function

$$\varphi(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}},$$

where x is a particular outcome.

1276. Standard Normal Density Function

$$\varphi(z)=\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}$$

Average value $\mu=0$, deviation $\sigma=1$.

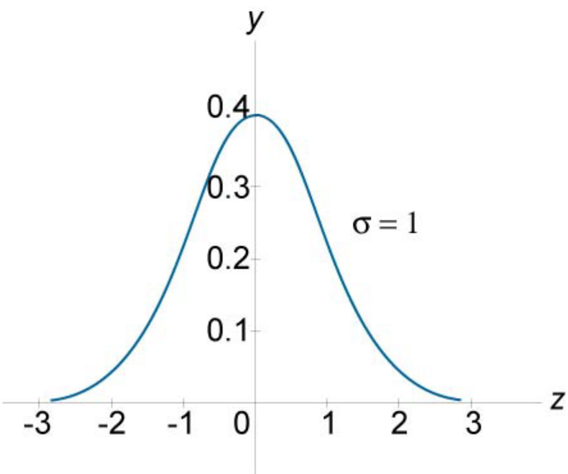


Figure 210.

1277. Standard Z Value

$$Z=\frac{X-\mu}{\sigma}$$

1278. Cumulative Normal Distribution Function

$$F(x)=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{(t-\mu)^2}{2\sigma^2}}dt,$$

where
x is a particular outcome,
t is a variable of integration.

1279. $P(\alpha < X < \beta) = F\left(\frac{\alpha - \mu}{\sigma}\right) - F\left(\frac{\beta - \mu}{\sigma}\right)$,

where
X is normally distributed random variable,
F is cumulative normal distribution function,
$P(\alpha < X < \beta)$ is interval probability.

1280. $P(|X - \mu| < \varepsilon) = 2F\left(\frac{\varepsilon}{\sigma}\right)$,

where
X is normally distributed random variable,
F is cumulative normal distribution function.

1281. Cumulative Distribution Function

$$F(x) = P(X < x) = \int_{-\infty}^{x} f(t)dt,$$

where t is a variable of integration.

1282. Bernoulli Trials Process

$\mu = np$, $\sigma^2 = npq$,

where
n is a sequence of experiments,
p is the probability of success of each experiments,
q is the probability of failure, $q = 1 - p$.

1283. Binomial Distribution Function

$$b(n,p,q) = \binom{n}{k} p^k q^{n-k},$$

$\mu = np$, $\sigma^2 = npq$,

$f(x) = (q + pe^x)^n$,

where

n is the number of trials of selections,

p is the probability of success,

q is the probability of failure, $q = 1 - p$.

1284. Geometric Distribution

$P(T = j) = q^{j-1}p$,

$\mu = \frac{1}{p}$, $\sigma^2 = \frac{q}{p^2}$,

where

T is the first successful event is the series,

j is the event number,

p is the probability that any one event is successful,

q is the probability of failure, $q = 1 - p$.

1285. Poisson Distribution

$P(X = k) \approx \frac{\lambda^k}{k!}e^{-\lambda}$, $\lambda = np$,

$\mu = \lambda$, $\sigma^2 = \lambda$,

where

λ is the rate of occurrence,

k is the number of positive outcomes.

1286. Density Function

$$P(a \le X \le b) = \int_a^b f(x)dx$$

1287. Continuous Uniform Density

$f = \frac{1}{b - a}$, $\mu = \frac{a + b}{2}$, where f is the density function.

1288. Exponential Density Function

$f(t) = \lambda e^{-\lambda t}$, $\mu = \lambda$, $\sigma^2 = \lambda^2$

where t is time, λ is the failure rate.

1289. Exponential Distribution Function

$F(t) = 1 - e^{-\lambda t}$,

where t is time, λ is the failure rate.

1290. Expected Value of Discrete Random Variables

$$\mu = E(X) = \sum_{i=1}^{n} x_i p_i ,$$

where x_i is a particular outcome, p_i is its probability.

1291. Expected Value of Continuous Random Variables

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

1292. Properties of Expectations

$E(X + Y) = E(X) + E(Y)$,

$E(X - Y) = E(X) - E(Y)$,

$E(cX) = cE(X)$,

$E(XY) = E(X) \cdot E(Y)$,

where c is a constant.

1293. $E(X^2) = V(X) + \mu^2$,

where

$\mu = E(X)$ is the expected value,

$V(X)$ is the variance.

1294. Markov Inequality

$$P(X > k) \le \frac{E(X)}{k},$$

where k is some constant.

1295. Variance of Discrete Random Variables

$$\sigma^2 = V(X) = E\left[(X-\mu)^2\right] = \sum_{i=1}^{n} (x_i - \mu)^2 p_i ,$$

where

x_i is a particular outcome,

p_i is its probability.

1296. Variance of Continuous Random Variables

$$\sigma^2 = V(X) = E\left[(X-\mu)^2\right] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$$

1297. Properties of Variance

$V(X+Y) = V(X) + V(Y)$,

$V(X-Y) = V(X) + V(Y)$,

$V(X+c) = V(X)$,

$V(cX) = c^2 V(X)$,

where c is a constant.

1298. Standard Deviation

$$D(X) = \sqrt{V(X)} = \sqrt{E\left[(X-\mu)^2\right]}$$

1299. Covariance

$$\operatorname{cov}(X,Y) = E\left[(X-\mu(X))(Y-\mu(Y))\right] = E(XY) - \mu(X)\mu(Y),$$

where

X is random variable,

$V(X)$ is the variance of X,

μ is the expected value of X or Y.

1300. Correlation

$$\rho(X,Y)=\frac{\operatorname{cov}(X,Y)}{\sqrt{V(X)V(Y)}},$$

where

$V(X)$ is the variance of X,

$V(Y)$ is the variance of Y.

Look for other handbooks and solved problem guides at
www.math-ebooks.com.