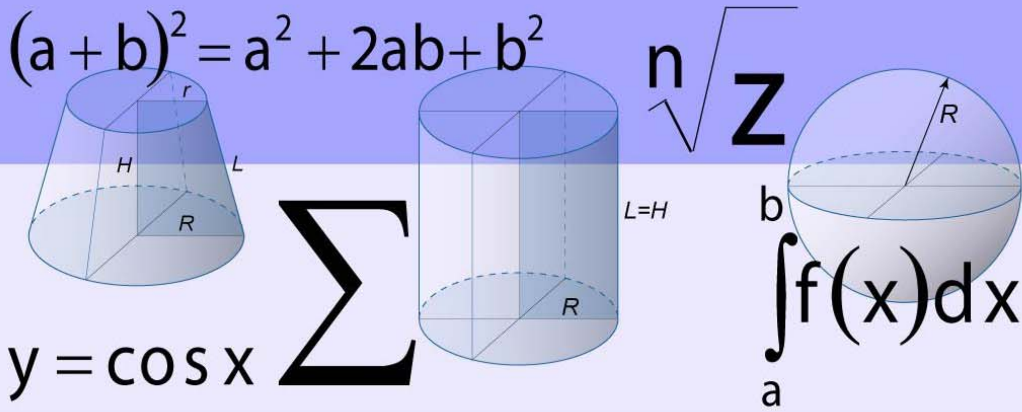

1300 Math Formulas

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Preface

This handbook is a complete desktop reference for students and engineers. It has everything from high school math to math for advanced undergraduates in engineering, economics, physical sciences, and mathematics. The ebook contains hundreds of formulas, tables, and figures from Number Sets, Algebra, Geometry, Trigonometry, Matrices and Determinants, Vectors, Analytic Geometry, Calculus, Differential Equations, Series, and Probability Theory.

The structured table of contents, links, and layout make finding the relevant information quick and painless, so it can be used as an everyday online reference guide.

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Chapter 1

Number Sets

1.1 Set Identities

Sets: A, B, C
Universal set: I
Complement : A'
Proper subset: $A \subset B$
Empty set: $\varnothing$
Union of sets: $A \cup B$
Intersection of sets: $A \cap B$
Difference of sets: $A \setminus B$

1. $A \subset I$

2. $A \subset A$

3. $A = B$ if $A \subset B$ and $B \subset A$.

4. Empty Set
$\varnothing \subset A$

5. Union of Sets
$C = A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

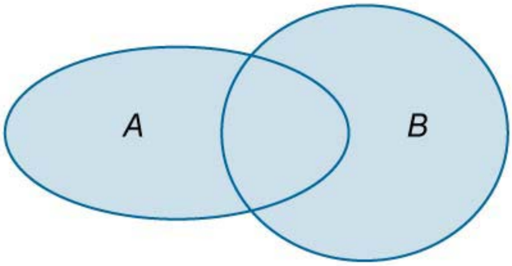


Figure 1.

6. Commutativity
$A \cup B = B \cup A$

7. Associativity
$A \cup (B \cup C) = (A \cup B) \cup C$

8. Intersection of Sets
$C = A \cup B = \{x \mid x \in A \text{ and } x \in B\}$

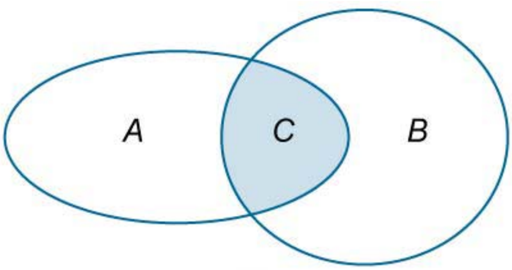


Figure 2.

9. Commutativity
$A \cap B = B \cap A$

10. Associativity
$A \cap (B \cap C) = (A \cap B) \cap C$

11. Distributivity
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$,
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

12. Idempotency
$A \cap A = A$,
$A \cup A = A$

13. Domination
$A \cap \varnothing = \varnothing$,
$A \cup I = I$

14. Identity
$A \cup \varnothing = A$,
$A \cap I = A$

15. Complement
$A' = \{x \in I \mid x \notin A\}$

16. Complement of Intersection and Union
$A \cup A' = I$,
$A \cap A' = \varnothing$

17. De Morgan's Laws
$(A \cup B)' = A' \cap B'$,
$(A \cap B)' = A' \cup B'$

18. Difference of Sets
$C = B \setminus A = \{x \mid x \in B \text{ and } x \notin A\}$

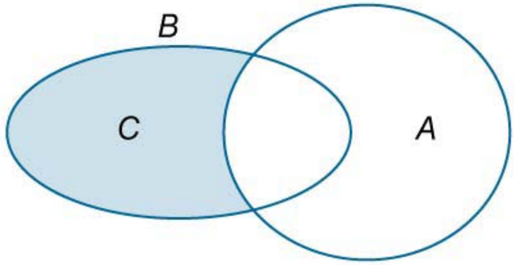


Figure 3.

19. $B \setminus A = B \setminus (A \cap B)$

20. $B \setminus A = B \cap A'$

21. $A \setminus A = \varnothing$

22. $A \setminus B = A$ if $A \cap B = \varnothing$.

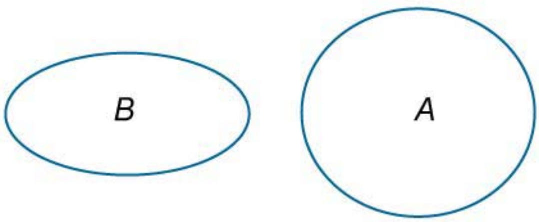


Figure 4.

23. $(A \setminus B) \cap C = (A \cap C) \setminus (B \cap C)$

24. $A' = I \setminus A$

25. Cartesian Product
$C = A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}$

1.2 Sets of Numbers

Natural numbers: N
Whole numbers: N_0
Integers: Z
Positive integers: Z^+
Negative integers: Z^-
Rational numbers: Q
Real numbers: R
Complex numbers: C

26. Natural Numbers
Counting numbers: $N = \{1, 2, 3, \ldots\}$.

27. Whole Numbers
Counting numbers and zero: $N_0 = \{0, 1, 2, 3, \ldots\}$.

28. Integers
Whole numbers and their opposites and zero:
$Z^+ = N = \{1, 2, 3, \ldots\}$,
$Z^- = \{\ldots, -3, -2, -1\}$,
$Z = Z^- \cup \{0\} \cup Z^+ = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

29. Rational Numbers
Repeating or terminating decimals:

$$Q = \left\{ x \mid x = \frac{a}{b} \text{ and } a \in Z \text{ and } b \in Z \text{ and } b \neq 0 \right\}.$$

30. Irrational Numbers
Nonrepeating and nonterminating decimals.

31. Real Numbers
Union of rational and irrational numbers: R.

32. Complex Numbers
$C = \{x + iy \mid x \in R \text{ and } y \in R\}$,
where i is the imaginary unit.

33. $N \subset Z \subset Q \subset R \subset C$

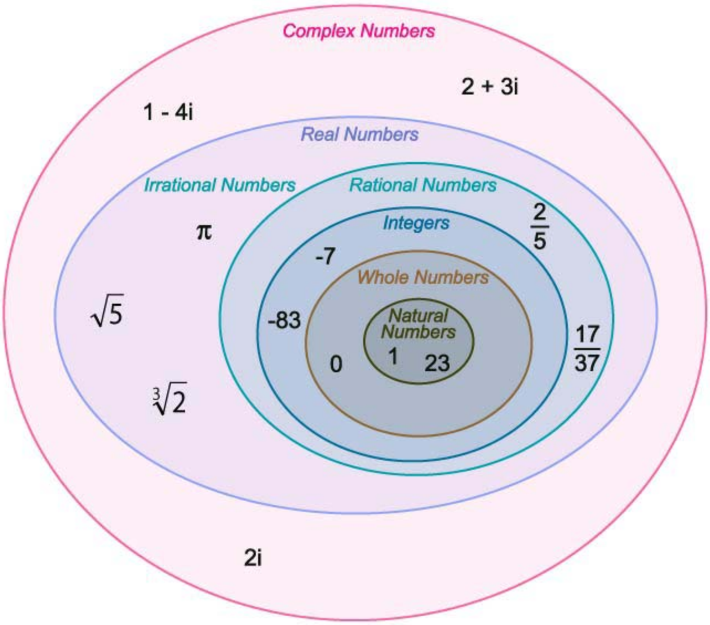


Figure 5.

1.3 Basic Identities

Real numbers: a, b, c

34. Additive Identity

$a + 0 = a$

35. Additive Inverse

$a + (-a) = 0$

36. Commutative of Addition

$a + b = b + a$

37. Associative of Addition

$(a + b) + c = a + (b + c)$

38. Definition of Subtraction

$a - b = a + (-b)$

39. Multiplicative Identity

$a \cdot 1 = a$

40. Multiplicative Inverse

$a \cdot \frac{1}{a} = 1, \; a \neq 0$

41. Multiplication Times 0

$a \cdot 0 = 0$

42. Commutative of Multiplication

$a \cdot b = b \cdot a$

43. Associative of Multiplication
$(a \cdot b) \cdot c = a \cdot (b \cdot c)$

44. Distributive Law
$a(b + c) = ab + ac$

45. Definition of Division
$$\frac{a}{b} = a \cdot \frac{1}{b}$$

1.4 Complex Numbers

Natural number: n
Imaginary unit: i
Complex number: z
Real part: a, c
Imaginary part: bi, di
Modulus of a complex number: r, r_1, r_2
Argument of a complex number: φ, φ_1, φ_2

46.

$i^1 = i$	$i^5 = i$	$i^{4n+1} = i$
$i^2 = -1$	$i^6 = -1$	$i^{4n+2} = -1$
$i^3 = -i$	$i^7 = -i$	$i^{4n+3} = -i$
$i^4 = 1$	$i^8 = 1$	$i^{4n} = 1$

47. $z = a + bi$

48. Complex Plane

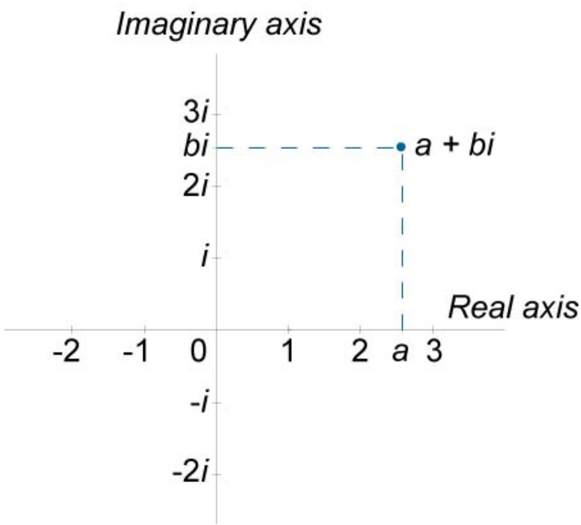


Figure 6.

49. $(a+bi)+(c+di)=(a+c)+(b+d)i$

50. $(a+bi)-(c+di)=(a-c)+(b-d)i$

51. $(a+bi)(c+di)=(ac-bd)+(ad+bc)i$

52. $\dfrac{a+bi}{c+di}=\dfrac{ac+bd}{c^2+d^2}+\dfrac{bc-ad}{c^2+d^2}\cdot i$

53. Conjugate Complex Numbers

$\overline{a+bi}=a-bi$

54. $a=r\cos\varphi,\ b=r\sin\varphi$

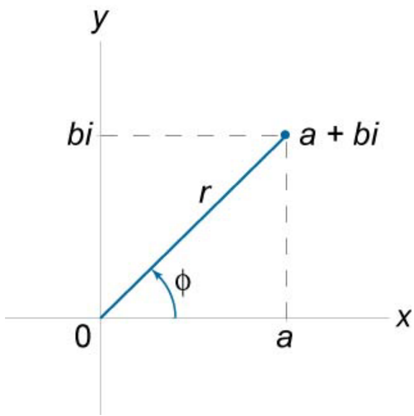


Figure 7.

55. Polar Presentation of Complex Numbers

$a + bi = r(\cos\varphi + i\sin\varphi)$

56. Modulus and Argument of a Complex Number

If $a + bi$ is a complex number, then

$r = \sqrt{a^2 + b^2}$ (modulus),

$\varphi = \arctan\frac{b}{a}$ (argument).

57. Product in Polar Representation

$$z_1 \cdot z_2 = r_1(\cos\varphi_1 + i\sin\varphi_1) \cdot r_2(\cos\varphi_2 + i\sin\varphi_2)$$
$$= r_1 r_2[\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2)]$$

58. Conjugate Numbers in Polar Representation

$$\overline{r(\cos\varphi + i\sin\varphi)} = r[\cos(-\varphi) + i\sin(-\varphi)]$$

59. Inverse of a Complex Number in Polar Representation

$$\frac{1}{r(\cos\varphi + i\sin\varphi)} = \frac{1}{r}[\cos(-\varphi) + i\sin(-\varphi)]$$

60. Quotient in Polar Representation

$$\frac{z_1}{z_2} = \frac{r_1(\cos\varphi_1 + i\sin\varphi_1)}{r_2(\cos\varphi_2 + i\sin\varphi_2)} = \frac{r_1}{r_2}[\cos(\varphi_1 - \varphi_2) + i\sin(\varphi_1 - \varphi_2)]$$

61. Power of a Complex Number

$$z^n = [r(\cos\varphi + i\sin\varphi)]^n = r^n[\cos(n\varphi) + i\sin(n\varphi)]$$

62. Formula “De Moivre”

$$(\cos\varphi + i\sin\varphi)^n = \cos(n\varphi) + i\sin(n\varphi)$$

63. Nth Root of a Complex Number

$$\sqrt[n]{z} = \sqrt[n]{r(\cos\varphi + i\sin\varphi)} = \sqrt[n]{r}\left(\cos\frac{\varphi + 2\pi k}{n} + i\sin\frac{\varphi + 2\pi k}{n}\right),$$

where

$k = 0, 1, 2, \ldots, n-1$.

64. Euler’s Formula

$$e^{ix} = \cos x + i\sin x$$

Chapter 2

Algebra

2.1 Factoring Formulas

Real numbers: a, b, c
Natural number: n

65. $a^2 - b^2 = (a+b)(a-b)$

66. $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

67. $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

68. $a^4 - b^4 = (a^2 - b^2)(a^2 + b^2) = (a-b)(a+b)(a^2 + b^2)$

69. $a^5 - b^5 = (a-b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4)$

70. $a^5 + b^5 = (a+b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4)$

71. If n is odd, then
$a^n + b^n = (a+b)(a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \ldots - ab^{n-2} + b^{n-1})$.

72. If n is even, then
$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + ab^{n-2} + b^{n-1})$,

$$a^n + b^n = (a+b)(a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \ldots + ab^{n-2} - b^{n-1}).$$

2.2 Product Formulas

Real numbers: a, b, c
Whole numbers: n, k

73. $(a-b)^2 = a^2 - 2ab + b^2$

74. $(a+b)^2 = a^2 + 2ab + b^2$

75. $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

76. $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

77. $(a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$

78. $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

79. Binomial Formula

$$(a+b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + \ldots + {}^nC_{n-1}ab^{n-1} + {}^nC_nb^n,$$

where ${}^nC_k = \dfrac{n!}{k!(n-k)!}$ are the binomial coefficients.

80. $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$

81. $(a+b+c+\ldots+u+v)^2 = a^2 + b^2 + c^2 + \ldots + u^2 + v^2 +$
$+ 2(ab + ac + \ldots + au + av + bc + \ldots + bu + bv + \ldots + uv)$

2.3 Powers

Bases (positive real numbers): a, b
Powers (rational numbers): n, m

82. $a^m a^n = a^{m+n}$

83. $\frac{a^m}{a^n} = a^{m-n}$

84. $(ab)^m = a^m b^m$

85. $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

86. $(a^m)^n = a^{mn}$

87. $a^0 = 1, \ a \neq 0$

88. $a^1 = 1$

89. $a^{-m} = \frac{1}{a^m}$

90. $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

2.4 Roots

Bases: a, b
Powers (rational numbers): n, m
$a, b \geq 0$ for even roots ($n = 2k$, $k \in N$)

91. $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$

92. $\sqrt[n]{a}\,\sqrt[m]{b} = \sqrt[nm]{a^m b^n}$

93. $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$, $b \neq 0$

94. $\frac{\sqrt[n]{a}}{\sqrt[m]{b}} = \frac{\sqrt[nm]{a^m}}{\sqrt[nm]{b^n}} = \sqrt[nm]{\frac{a^m}{b^n}}$, $b \neq 0$.

95. $\left(\sqrt[n]{a^m}\right)^p = \sqrt[n]{a^{mp}}$

96. $\left(\sqrt[n]{a}\right)^n = a$

97. $\sqrt[n]{a^m} = \sqrt[np]{a^{mp}}$

98. $\sqrt[n]{a^m} = a^{\frac{m}{n}}$

99. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$

100. $\left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$

101. $\frac{1}{\sqrt[n]{a}} = \frac{\sqrt[n]{a^{n-1}}}{a}, \ a \neq 0.$

102. $\sqrt{a \pm \sqrt{b}} = \sqrt{\frac{a + \sqrt{a^2 - b}}{2}} \pm \sqrt{\frac{a - \sqrt{a^2 - b}}{2}}$

103. $\frac{1}{\sqrt{a} \pm \sqrt{b}} = \frac{\sqrt{a} \mp \sqrt{b}}{a - b}$

2.5 Logarithms

Positive real numbers: x, y, a, c, k
Natural number: n

104. Definition of Logarithm
$y = \log_a x$ if and only if $x = a^y$, $a > 0$, $a \neq 1$.

105. $\log_a 1 = 0$

106. $\log_a a = 1$

107. $\log_a 0 = \begin{cases} -\infty & \text{if } a > 1 \\ +\infty & \text{if } a < 1 \end{cases}$

108. $\log_a (xy) = \log_a x + \log_a y$

109. $\log_a \frac{x}{y} = \log_a x - \log_a y$

110. $\log_a(x^n) = n\log_a x$

111. $\log_a \sqrt[n]{x} = \frac{1}{n}\log_a x$

112. $\log_a x = \frac{\log_c x}{\log_c a} = \log_c x \cdot \log_a c,\ c > 0,\ c \neq 1.$

113. $\log_a c = \frac{1}{\log_c a}$

114. $x = a^{\log_a x}$

115. Logarithm to Base 10
$\log_{10} x = \log x$

116. Natural Logarithm
$\log_e x = \ln x$,

where $e = \lim\limits_{k\to\infty}\left(1+\frac{1}{k}\right)^k = 2.718281828\ldots$

117. $\log x = \frac{1}{\ln 10}\ln x = 0.434294 \ln x$

118. $\ln x = \frac{1}{\log e}\log x = 2.302585 \log x$

2.6 Equations

Real numbers: a, b, c, p, q, u, v
Solutions: x_1, x_2, y_1, y_2, y_3

119. Linear Equation in One Variable

$$ax + b = 0,\ x = -\frac{b}{a}.$$

120. Quadratic Equation

$$ax^2 + bx + c = 0,\ x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

121. Discriminant

$$D = b^2 - 4ac$$

122. Viete's Formulas

If $x^2 + px + q = 0$, then

$$\begin{cases} x_1 + x_2 = -p \\ x_1 x_2 = q \end{cases}.$$

123. $ax^2 + bx = 0,\ x_1 = 0,\ x_2 = -\frac{b}{a}.$

124. $ax^2 + c = 0,\ x_{1,2} = \pm\sqrt{-\frac{c}{a}}.$

125. Cubic Equation. Cardano's Formula.

$$y^3 + py + q = 0,$$

$$y_1 = u + v\,,\; y_{2,3} = -\frac{1}{2}(u+v) \pm \frac{\sqrt{3}}{2}(u+v)i\,,$$

where

$$u = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^2}}\,,\; v = \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^2}}\,.$$

2.7 Inequalities

Variables: x, y, z

Real numbers: $\begin{cases} a, b, c, d \\ a_1, a_2, a_3, \ldots, a_n \end{cases}$, m, n

Determinants: $D,\ D_x,\ D_y,\ D_z$

126. Inequalities, Interval Notations and Graphs

Inequality	Interval Notation	Graph
$a \le x \le b$	$[a, b]$	
$a < x \le b$	$(a, b]$	
$a \le x < b$	$[a, b)$	
$a < x < b$	(a, b)	
$-\infty < x \le b$, $x \le b$	$(-\infty, b]$	
$-\infty < x < b$, $x < b$	$(-\infty, b)$	
$a \le x < \infty$, $x \ge a$	$[a, \infty)$	
$a < x < \infty$, $x > a$	(a, ∞)	

127. If $a > b$, then $b < a$.

128. If $a > b$, then $a - b > 0$ or $b - a < 0$.

129. If $a > b$, then $a + c > b + c$.

130. If $a > b$, then $a - c > b - c$.

131. If $a > b$ and $c > d$, then $a + c > b + d$.

132. If $a > b$ and $c > d$, then $a - d > b - c$.

133. If $a > b$ and $m > 0$, then $ma > mb$.

134. If $a > b$ and $m > 0$, then $\frac{a}{m} > \frac{b}{m}$.

135. If $a > b$ and $m < 0$, then $ma < mb$.

136. If $a > b$ and $m < 0$, then $\frac{a}{m} < \frac{b}{m}$.

137. If $0 < a < b$ and $n > 0$, then $a^n < b^n$.

138. If $0 < a < b$ and $n < 0$, then $a^n > b^n$.

139. If $0 < a < b$, then $\sqrt[n]{a} < \sqrt[n]{b}$.

140. $\sqrt{ab} \le \frac{a+b}{2}$,
where $a > 0$, $b > 0$; an equality is valid only if $a = b$.

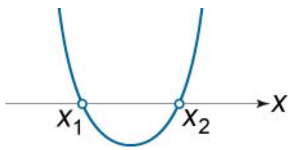
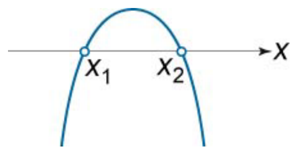
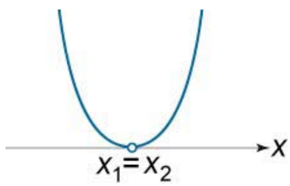
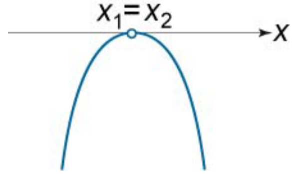
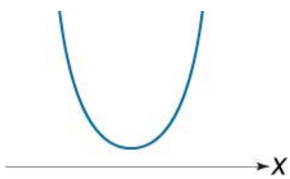
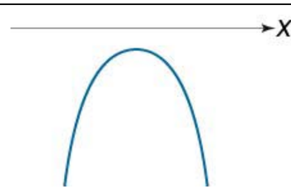
141. $a + \frac{1}{a} \ge 2$, where $a > 0$; an equality takes place only at $a = 1$.

142. $\sqrt[n]{a_1 a_2 \ldots a_n} \le \dfrac{a_1 + a_2 + \ldots + a_n}{n}$, where $a_1, a_2, \ldots, a_n > 0$.

143. If $ax + b > 0$ and $a > 0$, then $x > -\dfrac{b}{a}$.

144. If $ax + b > 0$ and $a < 0$, then $x < -\dfrac{b}{a}$.

145. $ax^2 + bx + c > 0$

	$a > 0$	$a < 0$
$D > 0$	 $x < x_1,\ x > x_2$	 $x_1 < x < x_2$
$D = 0$	 $x_1 < x,\ x > x_1$	 $x \in \varnothing$
$D < 0$	 $-\infty < x < \infty$	 $x \in \varnothing$

146. $|a+b| \le |a| + |b|$

147. If $|x| < a$, then $-a < x < a$, where $a > 0$.

148. If $|x| > a$, then $x < -a$ and $x > a$, where $a > 0$.

149. If $x^2 < a$, then $|x| < \sqrt{a}$, where $a > 0$.

150. If $x^2 > a$, then $|x| > \sqrt{a}$, where $a > 0$.

151. If $\frac{f(x)}{g(x)} > 0$, then $\begin{cases} f(x) \cdot g(x) > 0 \\ g(x) \neq 0 \end{cases}$.

152. $\frac{f(x)}{g(x)} < 0$, then $\begin{cases} f(x) \cdot g(x) < 0 \\ g(x) \neq 0 \end{cases}$.

2.8 Compound Interest Formulas

Future value: A
Initial deposit: C
Annual rate of interest: r
Number of years invested: t
Number of times compounded per year: n

153. General Compound Interest Formula

$$A = C\left(1 + \frac{r}{n}\right)^{nt}$$

154. **Simplified Compound Interest Formula**
If interest is compounded once per year, then the previous formula simplifies to:
$A = C(1 + r)^t$.

155. **Continuous Compound Interest**
If interest is compounded continually ($n \to \infty$), then
$A = Ce^{rt}$.

Chapter 3

Geometry

3.1 Right Triangle

Legs of a right triangle: a, b
Hypotenuse: c
Altitude: h
Medians: m_a, m_b, m_c
Angles: α, β
Radius of circumscribed circle: R
Radius of inscribed circle: r
Area: S

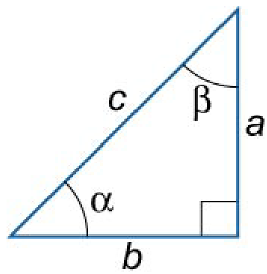


Figure 8.

156. $\alpha+\beta=90^\circ$

157. $\sin \alpha = \frac{a}{c} = \cos \beta$

158. $\cos \alpha = \frac{b}{c} = \sin \beta$

159. $\tan \alpha = \frac{a}{b} = \cot \beta$

160. $\cot \alpha = \frac{b}{a} = \tan \beta$

161. $\sec \alpha = \frac{c}{b} = \operatorname{cosec} \beta$

162. $\operatorname{cosec} \alpha = \frac{c}{a} = \sec \beta$

163. Pythagorean Theorem
$a^2 + b^2 = c^2$

164. $a^2 = fc$, $b^2 = gc$,
where f and c are projections of the legs a and b, respectively, onto the hypotenuse c.

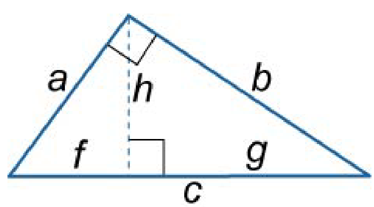


Figure 9.

165. $h^2 = fg$,
where h is the altitude from the right angle.

166. $m_a^2 = b^2 - \frac{a^2}{4}$, $m_b^2 = a^2 - \frac{b^2}{4}$,
where m_a and m_b are the medians to the legs a and b.

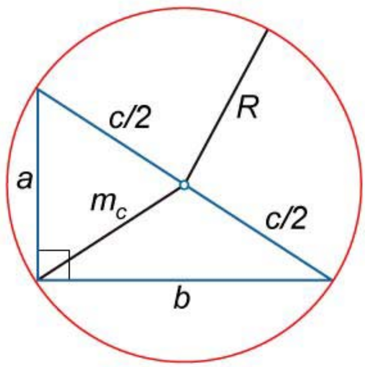


Figure 10.

167. $m_c = \frac{c}{2}$,
where m_c is the median to the hypotenuse c.

168. $R = \frac{c}{2} = m_c$

169. $r = \frac{a+b-c}{2} = \frac{ab}{a+b+c}$

170. $ab = ch$

171. $S = \frac{ab}{2} = \frac{ch}{2}$

3.2 Isosceles Triangle

Base: a
Legs: b
Base angle: β
Vertex angle: α
Altitude to the base: h
Perimeter: L
Area: S

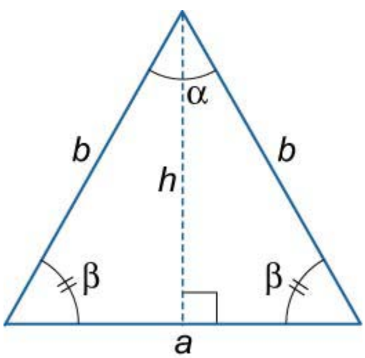


Figure 11.

172. $\beta = 90° - \frac{\alpha}{2}$

173. $h^2 = b^2 - \frac{a^2}{4}$

174. $L = a + 2b$

175. $S = \frac{ah}{2} = \frac{b^2}{2}\sin\alpha$

3.3 Equilateral Triangle

Side of a equilateral triangle: a
Altitude: h
Radius of circumscribed circle: R
Radius of inscribed circle: r
Perimeter: L
Area: S

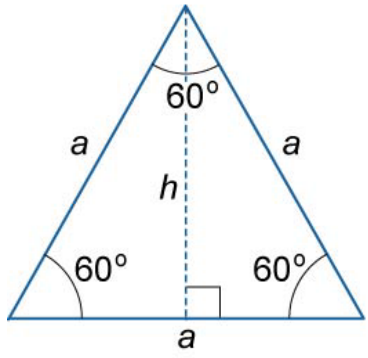


Figure 12.

176. $h = \frac{a\sqrt{3}}{2}$

177. $R = \frac{2}{3}h = \frac{a\sqrt{3}}{3}$

178. $r = \frac{1}{3}h = \frac{a\sqrt{3}}{6} = \frac{R}{2}$

179. $L = 3a$

180. $S = \frac{ah}{2} = \frac{a^2\sqrt{3}}{4}$

3.4 Scalene Triangle

(A triangle with no two sides equal)

Sides of a triangle: a, b, c
Semiperimeter: $p = \frac{a+b+c}{2}$
Angles of a triangle: α, β, γ
Altitudes to the sides a, b, c: h_a, h_b, h_c
Medians to the sides a, b, c: m_a, m_b, m_c
Bisectors of the angles α, β, γ: t_a, t_b, t_c
Radius of circumscribed circle: R
Radius of inscribed circle: r
Area: S

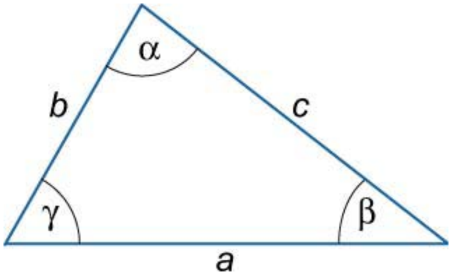


Figure 13.

181. $\alpha + \beta + \gamma = 180°$

182. $a + b > c$,
$b + c > a$,
$a + c > b$.

183. $|a - b| < c$,
$|b - c| < a$,
$|a - c| < b$.

184. Midline
$q = \frac{a}{2}$, $q \parallel a$.

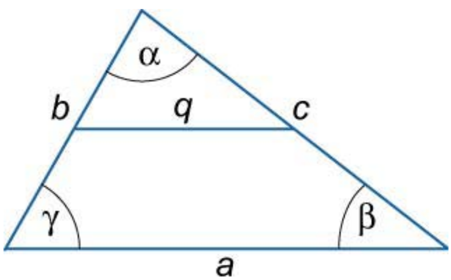


Figure 14.

185. Law of Cosines

$a^2 = b^2 + c^2 - 2bc\cos\alpha$,

$b^2 = a^2 + c^2 - 2ac\cos\beta$,

$c^2 = a^2 + b^2 - 2ab\cos\gamma$.

186. Law of Sines

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2R,$$

where R is the radius of the circumscribed circle.

187. $$R = \frac{a}{2\sin\alpha} = \frac{b}{2\sin\beta} = \frac{c}{2\sin\gamma} = \frac{bc}{2h_a} = \frac{ac}{2h_b} = \frac{ab}{2h_c} = \frac{abc}{4S}$$

188. $$r^2 = \frac{(p-a)(p-b)(p-c)}{p},$$

$$\frac{1}{r} = \frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}.$$

189. $$\sin\frac{\alpha}{2} = \sqrt{\frac{(p-b)(p-c)}{bc}},$$

$$\cos\frac{\alpha}{2} = \sqrt{\frac{p(p-a)}{bc}},$$

$$\tan\frac{\alpha}{2} = \sqrt{\frac{(p-b)(p-c)}{p(p-a)}}.$$

190. $$h_a = \frac{2}{a}\sqrt{p(p-a)(p-b)(p-c)},$$

$$h_b = \frac{2}{b}\sqrt{p(p-a)(p-b)(p-c)},$$

$$h_c = \frac{2}{c}\sqrt{p(p-a)(p-b)(p-c)}.$$

191. $h_a = b\sin\gamma = c\sin\beta$,
$h_b = a\sin\gamma = c\sin\alpha$,
$h_c = a\sin\beta = b\sin\alpha$.

192. $m_a^2 = \frac{b^2 + c^2}{2} - \frac{a^2}{4}$,
$m_b^2 = \frac{a^2 + c^2}{2} - \frac{b^2}{4}$,
$m_c^2 = \frac{a^2 + b^2}{2} - \frac{c^2}{4}$.

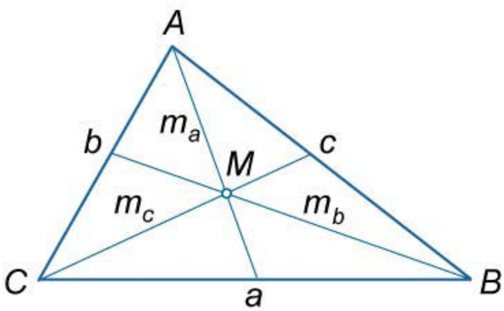


Figure 15.

193. $AM = \frac{2}{3}m_a$, $BM = \frac{2}{3}m_b$, $CM = \frac{2}{3}m_c$ (Fig.15).

194. $t_a^2 = \frac{4bcp(p-a)}{(b+c)^2}$,
$t_b^2 = \frac{4acp(p-b)}{(a+c)^2}$,
$t_c^2 = \frac{4abp(p-c)}{(a+b)^2}$.

195. $S = \frac{ah_a}{2} = \frac{bh_b}{2} = \frac{ch_c}{2}$,

$S = \frac{ab\sin\gamma}{2} = \frac{ac\sin\beta}{2} = \frac{bc\sin\alpha}{2}$,

$S = \sqrt{p(p-a)(p-b)(p-c)}$ (Heron's Formula),

$S = pr$,

$S = \frac{abc}{4R}$,

$S = 2R^2 \sin\alpha \sin\beta \sin\gamma$,

$S = p^2 \tan\frac{\alpha}{2} \tan\frac{\beta}{2} \tan\frac{\gamma}{2}$.

3.5 Square

Side of a square: a
Diagonal: d
Radius of circumscribed circle: R
Radius of inscribed circle: r
Perimeter: L
Area: S

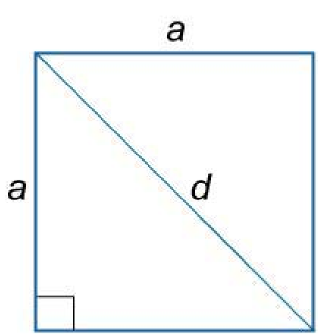


Figure 16.

196. $d = a\sqrt{2}$

197. $R = \frac{d}{2} = \frac{a\sqrt{2}}{2}$

198. $r = \frac{a}{2}$

199. $L = 4a$

200. $S = a^2$

3.6 Rectangle

Sides of a rectangle: a, b
Diagonal: d
Radius of circumscribed circle: R
Perimeter: L
Area: S

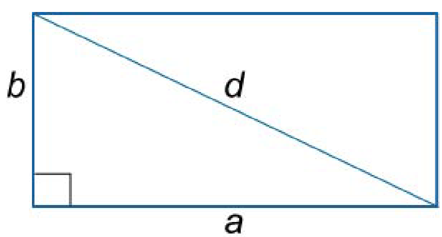


Figure 17.

201. $d = \sqrt{a^2 + b^2}$

202. $R = \frac{d}{2}$

203. $L = 2(a + b)$

204. $S = ab$

3.7 Parallelogram

Sides of a parallelogram: a, b
Diagonals: d_1, d_2
Consecutive angles: α, β
Angle between the diagonals: φ
Altitude: h
Perimeter: L
Area: S

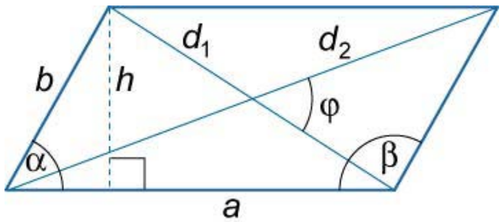


Figure 18.

205. $\alpha + \beta = 180°$

206. $d_1^2 + d_2^2 = 2(a^2 + b^2)$

207. $h = b \sin \alpha = b \sin \beta$

208. $L = 2(a + b)$

209. $S = ah = ab \sin \alpha$,

$S = \frac{1}{2} d_1 d_2 \sin \varphi$.

3.8 Rhombus

Side of a rhombus: a
Diagonals: d_1, d_2
Consecutive angles: α, β
Altitude: H
Radius of inscribed circle: r
Perimeter: L
Area: S

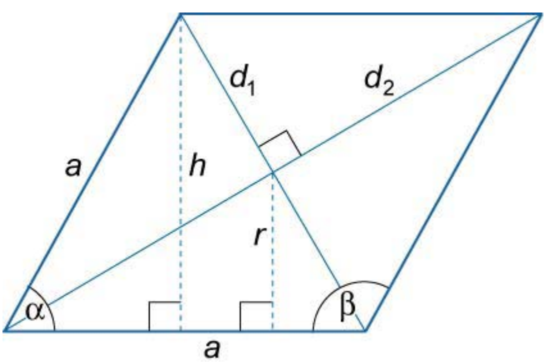


Figure 19.

210. $\alpha + \beta = 180°$

211. $d_1^2 + d_2^2 = 4a^2$

212. $h = a \sin\alpha = \dfrac{d_1 d_2}{2a}$

213. $r = \dfrac{h}{2} = \dfrac{d_1 d_2}{4a} = \dfrac{a \sin\alpha}{2}$

214. $L = 4a$

215. $S = ah = a^2 \sin\alpha$,

$S = \dfrac{1}{2} d_1 d_2$.

3.9 Trapezoid

Bases of a trapezoid: a, b
Midline: q
Altitude: h
Area: S

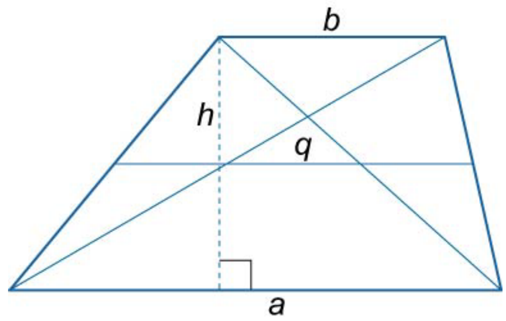


Figure 20.

216. $q = \frac{a+b}{2}$

217. $S = \frac{a+b}{2} \cdot h = qh$

3.10 Isosceles Trapezoid

Bases of a trapezoid: a, b
Leg: c
Midline: q
Altitude: h
Diagonal: d
Radius of circumscribed circle: R
Area: S

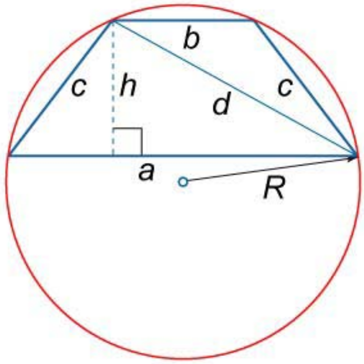


Figure 21.

218. $q = \frac{a+b}{2}$

219. $d = \sqrt{ab + c^2}$

220. $h = \sqrt{c^2 - \frac{1}{4}(b-a)^2}$

221. $R = \frac{c\sqrt{ab + c^2}}{\sqrt{(2c - a + b)(2c + a - b)}}$

222. $S = \frac{a+b}{2} \cdot h = qh$

3.11 Isosceles Trapezoid with Inscribed Circle

Bases of a trapezoid: a, b
Leg: c
Midline: q
Altitude: h
Diagonal: d
Radius of inscribed circle: R
Radius of circumscribed circle: r
Perimeter: L
Area: S

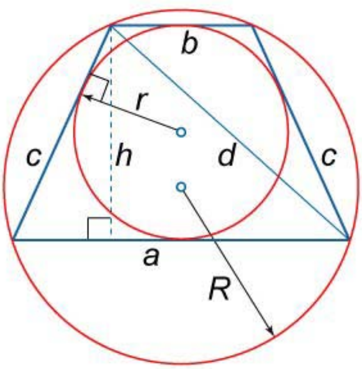


Figure 22.

223. $a + b = 2c$

224. $q = \dfrac{a+b}{2} = c$

225. $d^2 = h^2 + c^2$

226. $r = \frac{h}{2} = \frac{\sqrt{ab}}{2}$

227. $R = \frac{cd}{2h} = \frac{cd}{4r} = \frac{c}{2}\sqrt{1 + \frac{c^2}{ab}} = \frac{c}{2h}\sqrt{h^2 + c^2} = \frac{a+b}{8}\sqrt{\frac{a}{b} + 6 + \frac{b}{a}}$

228. $L = 2(a+b) = 4c$

229. $S = \frac{a+b}{2} \cdot h = \frac{(a+b)\sqrt{ab}}{2} = qh = ch = \frac{Lr}{2}$

3.12 Trapezoid with Inscribed Circle

Bases of a trapezoid: a, b
Lateral sides: c, d
Midline: q
Altitude: h
Diagonals: d_1, d_2
Angle between the diagonals: φ
Radius of inscribed circle: r
Radius of circumscribed circle: R
Perimeter: L
Area: S

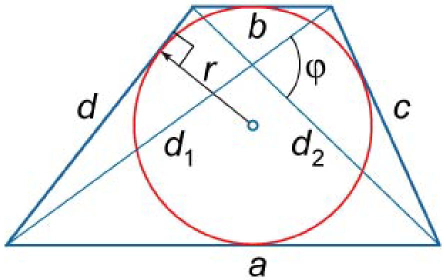


Figure 23.

230. $a + b = c + d$

231. $q = \frac{a+b}{2} = \frac{c+d}{2}$

232. $L = 2(a+b) = 2(c+d)$

233. $S = \frac{a+b}{2} \cdot h = \frac{c+d}{2} \cdot h = qh,$

$S = \frac{1}{2} d_1 d_2 \sin\varphi.$

3.13 Kite

Sides of a kite: a, b
Diagonals: d_1, d_2
Angles: α, β, γ
Perimeter: L
Area: S

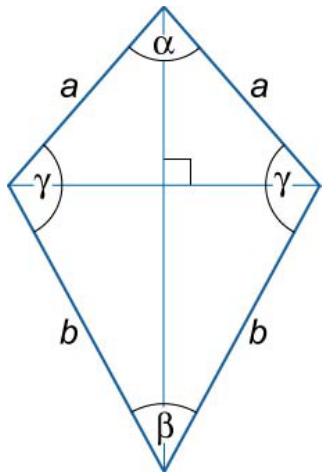


Figure 24.

234. $\alpha + \beta + 2\gamma = 360^\circ$

235. $L = 2(a + b)$

236. $S = \frac{d_1 d_2}{2}$

3.14 Cyclic Quadrilateral

Sides of a quadrilateral: a, b, c, d
Diagonals: d_1, d_2
Angle between the diagonals: φ
Internal angles: $\alpha, \beta, \gamma, \delta$
Radius of circumscribed circle: R
Perimeter: L
Semiperimeter: p
Area: S

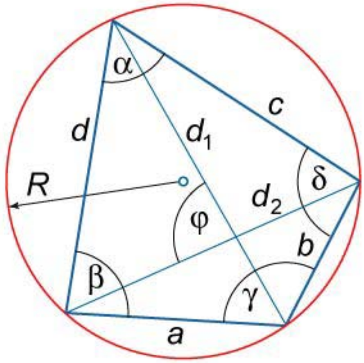


Figure 25.

237. $\alpha + \gamma = \beta + \delta = 180°$

238. Ptolemy's Theorem
$ac + bd = d_1 d_2$

239. $L = a + b + c + d$

240. $$R = \frac{1}{4}\sqrt{\frac{(ac+bd)(ad+bc)(ab+cd)}{(p-a)(p-b)(p-c)(p-d)}},$$
where $p = \frac{L}{2}$.

241. $$S = \frac{1}{2} d_1 d_2 \sin\varphi,$$
$$S = \sqrt{(p-a)(p-b)(p-c)(p-d)},$$
where $p = \frac{L}{2}$.

3.15 Tangential Quadrilateral

Sides of a quadrilateral: a, b, c, d
Diagonals: d_1, d_2
Angle between the diagonals: φ
Radius of inscribed circle: r
Perimeter: L
Semiperimeter: p
Area: S

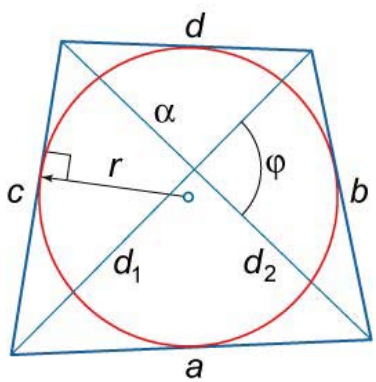


Figure 26.

242. $a + c = b + d$

243. $L = a + b + c + d = 2(a + c) = 2(b + d)$

244. $r = \dfrac{\sqrt{d_1^2 d_2^2 - (a - b)^2 (a + b - p)^2}}{2p}$,

where $p = \dfrac{L}{2}$.

245. $S = pr = \frac{1}{2} d_1 d_2 \sin \varphi$

3.16 General Quadrilateral

Sides of a quadrilateral: a, b, c, d
Diagonals: d_1, d_2
Angle between the diagonals: φ
Internal angles: $\alpha, \beta, \gamma, \delta$
Perimeter: L
Area: S

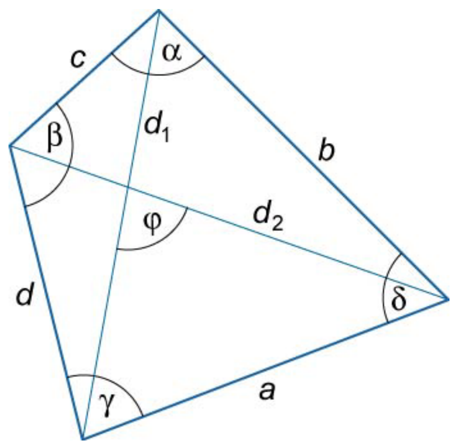


Figure 27.

246. $\alpha + \beta + \gamma + \delta = 360°$

247. $L = a + b + c + d$

248. $S = \frac{1}{2} d_1 d_2 \sin \varphi$

3.17 Regular Hexagon

Side: a
Internal angle: α
Slant height: m
Radius of inscribed circle: r
Radius of circumscribed circle: R
Perimeter: L
Semiperimeter: p
Area: S

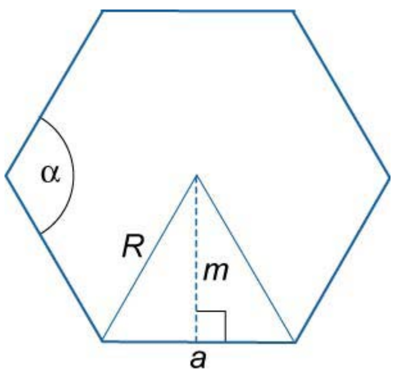


Figure 28.

249. $\alpha = 120°$

250. $r = m = \frac{a\sqrt{3}}{2}$

251. $R = a$

252. $L = 6a$

253. $S = pr = \frac{a^2 3\sqrt{3}}{2}$,

where $p = \frac{L}{2}$.

3.18 Regular Polygon

Side: a
Number of sides: n
Internal angle: α
Slant height: m
Radius of inscribed circle: r
Radius of circumscribed circle: R
Perimeter: L
Semiperimeter: p
Area: S

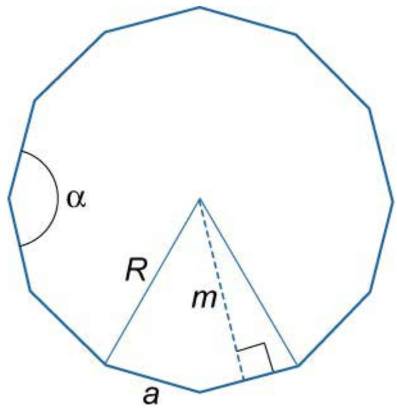


Figure 29.

254. $\alpha = \frac{n-2}{2} \cdot 180°$

255. $\alpha = \frac{n-2}{2} \cdot 180°$

256. $R = \frac{a}{2\sin\frac{\pi}{n}}$

257. $r = m = \frac{a}{2\tan\frac{\pi}{n}} = \sqrt{R^2 - \frac{a^2}{4}}$

258. $L = na$

259. $S = \frac{nR^2}{2}\sin\frac{2\pi}{n}$,

$S = pr = p\sqrt{R^2 - \frac{a^2}{4}}$,

where $p = \frac{L}{2}$.

3.19 Circle

Radius: R
Diameter: d
Chord: a
Secant segments: e, f
Tangent segment: g
Central angle: α
Inscribed angle: β
Perimeter: L
Area: S

260. $a = 2R\sin\frac{\alpha}{2}$

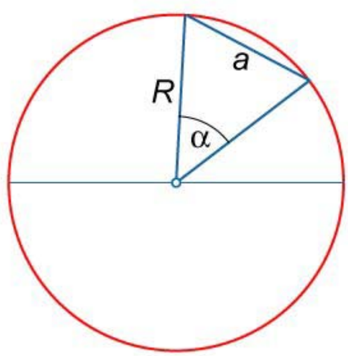


Figure 30.

261. $a_1a_2 = b_1b_2$

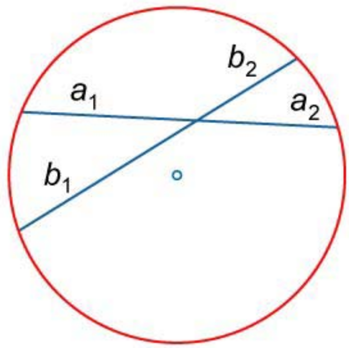


Figure 31.

262. $ee_1 = ff_1$

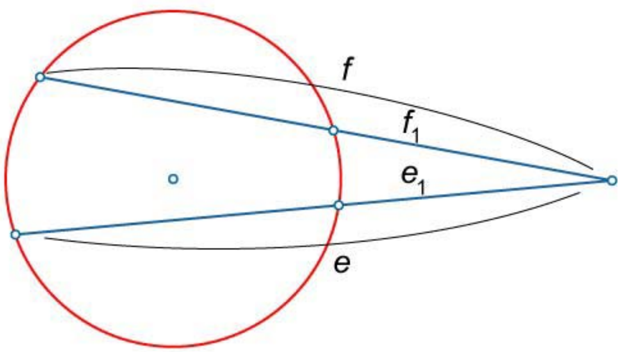


Figure 32.

263. $g^2 = ff_1$

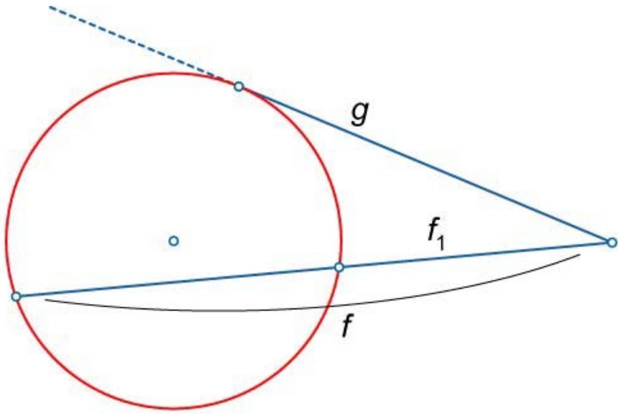


Figure 33.

264. $\beta = \dfrac{\alpha}{2}$

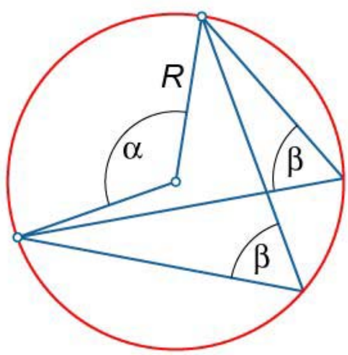


Figure 34.

265. $L = 2\pi R = \pi d$

266. $S = \pi R^2 = \dfrac{\pi d^2}{4} = \dfrac{LR}{2}$

3.20 Sector of a Circle

Radius of a circle: R
Arc length: s
Central angle (in radians): x
Central angle (in degrees): α
Perimeter: L
Area: S

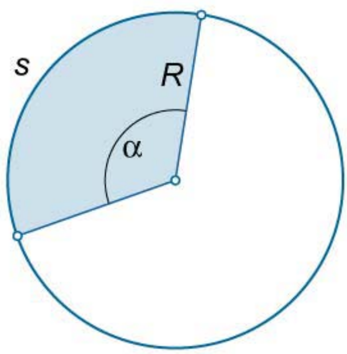


Figure 35.

267. $s = Rx$

268. $s = \dfrac{\pi R \alpha}{180^\circ}$

269. $L = s + 2R$

270. $S = \dfrac{Rs}{2} = \dfrac{R^2 x}{2} = \dfrac{\pi R^2 \alpha}{360^\circ}$

3.21 Segment of a Circle

Radius of a circle: R
Arc length: s
Chord: a
Central angle (in radians): x
Central angle (in degrees): α
Height of the segment: h
Perimeter: L
Area: S

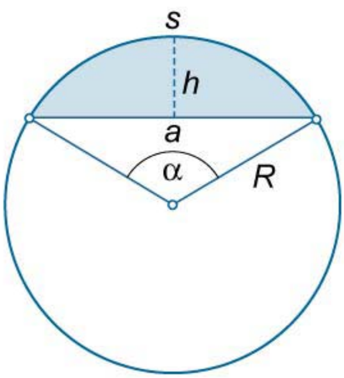


Figure 36.

271. $a = 2\sqrt{2hR - h^2}$

272. $h = R - \frac{1}{2}\sqrt{4R^2 - a^2}$, $h < R$

273. $L = s + a$

274. $S = \frac{1}{2}[sR - a(R - h)] = \frac{R^2}{2}\left(\frac{\alpha\pi}{180°} - \sin\alpha\right) = \frac{R^2}{2}(x - \sin x)$,

$S \approx \frac{2}{3}ha$.

3.22 Cube

Edge: a
Diagonal: d
Radius of inscribed sphere: r
Radius of circumscribed sphere: r
Surface area: S
Volume: V

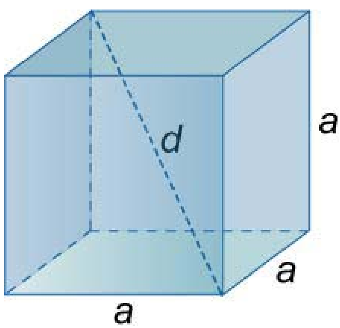


Figure 37.

275. $d = a\sqrt{3}$

276. $r = \frac{a}{2}$

277. $R = \frac{a\sqrt{3}}{2}$

278. $S = 6a^2$

279. $V = a^3$

3.23 Rectangular Parallelepiped

Edges: a, b, c
Diagonal: d
Surface area: S
Volume: V

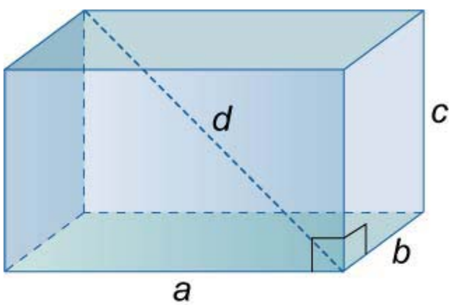


Figure 38.

280. $d = \sqrt{a^2 + b^2 + c^2}$

281. $S = 2(ab + ac + bc)$

282. $V = abc$

3.24 Prism

Lateral edge: l
Height: h
Lateral area: S_L
Area of base: S_B
Total surface area: S
Volume: V

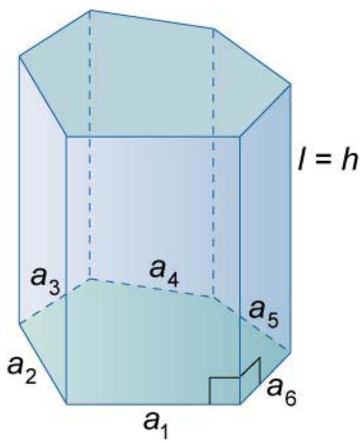


Figure 39.

283. $S = S_L + 2S_B$.

284. Lateral Area of a Right Prism
$S_L = (a_1 + a_2 + a_3 + \ldots + a_n)l$

285. Lateral Area of an Oblique Prism
$S_L = pl$,
where p is the perimeter of the cross section.

286. $V = S_B h$

287. Cavalieri's Principle
Given two solids included between parallel planes. If every plane cross section parallel to the given planes has the same area in both solids, then the volumes of the solids are equal.

3.25 Regular Tetrahedron

Triangle side length: a
Height: h
Area of base: S_B
Surface area: S
Volume: V

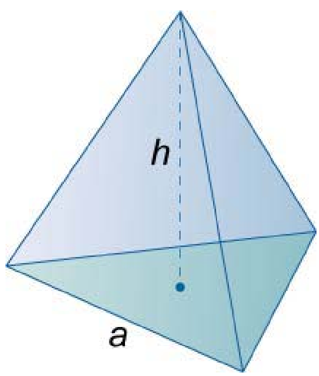


Figure 40.

288. $h = \sqrt{\frac{2}{3}}\, a$

289. $S_B = \frac{\sqrt{3}a^2}{4}$

290. $S = \sqrt{3}a^2$

291. $V = \frac{1}{3}S_B h = \frac{a^3}{6\sqrt{2}}$.

3.26 Regular Pyramid

Side of base: a
Lateral edge: b
Height: h
Slant height: m
Number of sides: n
Semiperimeter of base: p
Radius of inscribed sphere of base: r
Area of base: S_B
Lateral surface area: S_L
Total surface area: S
Volume: V

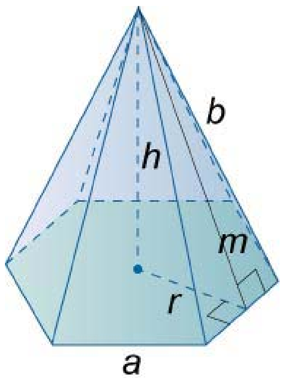


Figure 41.

292. $m = \sqrt{b^2 - \frac{a^2}{4}}$

293. $h = \frac{\sqrt{4b^2 \sin^2 \frac{\pi}{n} - a^2}}{2 \sin \frac{\pi}{n}}$

294. $S_L = \frac{1}{2} nam = \frac{1}{4} na\sqrt{4b^2 - a^2} = pm$

295. $S_B = pr$

296. $S = S_B + S_L$

297. $V = \frac{1}{3} S_B h = \frac{1}{3} prh$

3.27 Frustum of a Regular Pyramid

Base and top side lengths: $\begin{cases} a_1, a_2, a_3, \ldots, a_n \\ b_1, b_2, b_3, \ldots, b_n \end{cases}$

Height: h
Slant height: m
Area of bases: S_1, S_2
Lateral surface area: S_L
Perimeter of bases: P_1, P_2
Scale factor: k
Total surface area: S
Volume: V

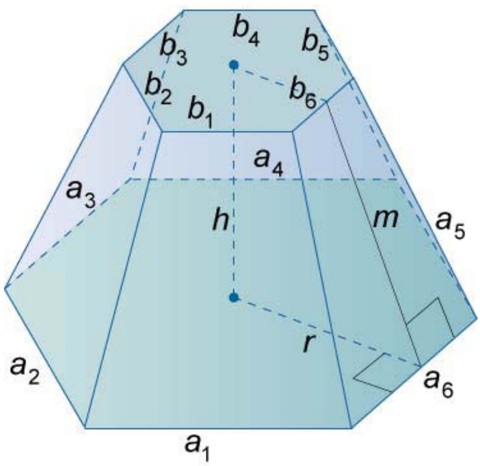


Figure 42.

298. $\frac{b_1}{a_1} = \frac{b_2}{a_2} = \frac{b_3}{a_3} = \ldots = \frac{b_n}{a_n} = \frac{b}{a} = k$

299. $\frac{S_2}{S_1} = k^2$

300. $S_L = \frac{m(P_1 + P_2)}{2}$

301. $S = S_L + S_1 + S_2$

302. $V = \frac{h}{3}\left(S_1 + \sqrt{S_1 S_2} + S_2\right)$

303. $V = \frac{hS_1}{3}\left[1 + \frac{b}{a} + \left(\frac{b}{a}\right)^2\right] = \frac{hS_1}{3}\left[1 + k + k^2\right]$

3.28 Rectangular Right Wedge

Sides of base: a, b
Top edge: c
Height: h
Lateral surface area: S_L
Area of base: S_B
Total surface area: S
Volume: V

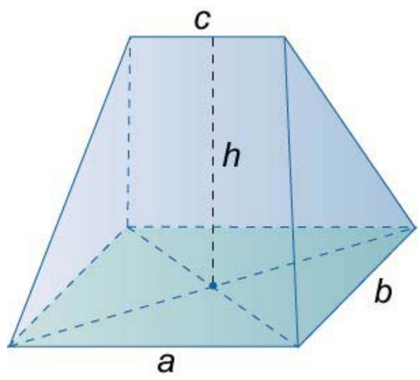


Figure 43.

304. $S_L = \frac{1}{2}(a+c)\sqrt{4h^2+b^2} + b\sqrt{h^2+(a-c)^2}$

305. $S_B = ab$

306. $S = S_B + S_L$

307. $V = \frac{bh}{6}(2a+c)$

3.29 Platonic Solids

Edge: a
Radius of inscribed circle: r
Radius of circumscribed circle: R
Surface area: S
Volume: V

308. Five Platonic Solids
The platonic solids are convex polyhedra with equivalent faces composed of congruent convex regular polygons.

Solid	Number of Vertices	Number of Edges	Number of Faces	Section
Tetrahedron	4	6	4	3.25
Cube	8	12	6	3.22
Octahedron	6	12	8	3.27
Icosahedron	12	30	20	3.27
Dodecahedron	20	30	12	3.27

Octahedron

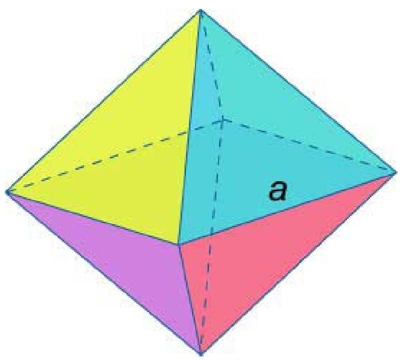


Figure 44.

309. $r = \frac{a\sqrt{6}}{6}$

310. $R = \frac{a\sqrt{2}}{2}$

311. $S = 2a^2\sqrt{3}$

312. $V = \frac{a^3\sqrt{2}}{3}$

Icosahedron

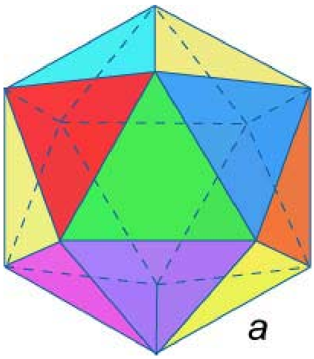


Figure 45.

313. $r = \frac{a\sqrt{3}\left(3+\sqrt{5}\right)}{12}$

314. $R = \frac{a}{4}\sqrt{2\left(5+\sqrt{5}\right)}$

315. $S = 5a^2\sqrt{3}$

316. $V = \frac{5a^3\left(3+\sqrt{5}\right)}{12}$

Dodecahedron

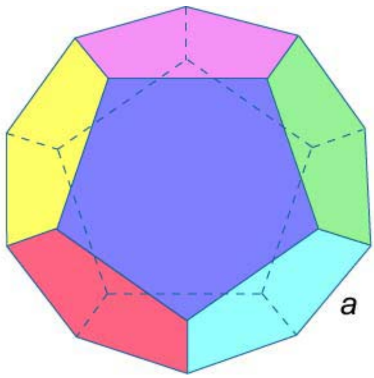


Figure 46.

317. $r = \frac{a\sqrt{10\left(25 + 11\sqrt{5}\right)}}{2}$

318. $R = \frac{a\sqrt{3}\left(1 + \sqrt{5}\right)}{4}$

319. $S = 3a^2\sqrt{5\left(5 + 2\sqrt{5}\right)}$

320. $V = \frac{a^3\left(15 + 7\sqrt{5}\right)}{4}$

3.30 Right Circular Cylinder

Radius of base: R
Diameter of base: d

Height: H
Lateral surface area: S_L
Area of base: S_B
Total surface area: S
Volume: V

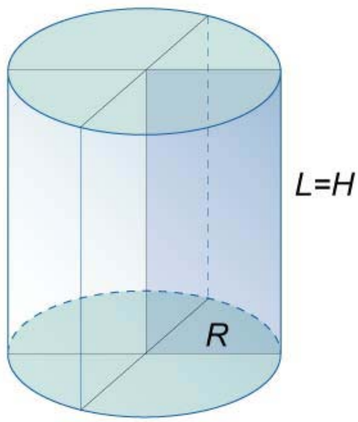


Figure 47.

321. $S_L = 2\pi RH$

322. $S = S_L + 2S_B = 2\pi R(H+R) = \pi d\left(H + \frac{d}{2}\right)$

323. $V = S_B H = \pi R^2 H$

3.31 Right Circular Cylinder with an Oblique Plane Face

Radius of base: R
The greatest height of a side: h_1
The shortest height of a side: h_2
Lateral surface area: S_L
Area of plane end faces: S_B
Total surface area: S
Volume: V

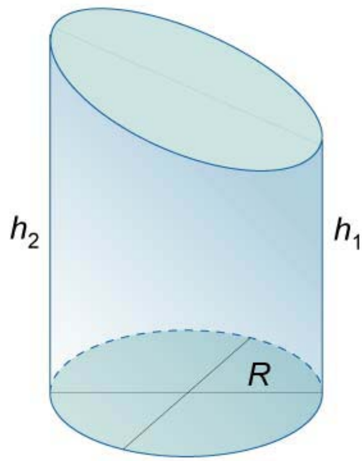


Figure 48.

324. $S_L = \pi R(h_1 + h_2)$

325. $S_B = \pi R^2 + \pi R\sqrt{R^2 + \left(\frac{h_1 - h_2}{2}\right)^2}$

326. $S = S_L + S_B = \pi R\left[h_1 + h_2 + R + \sqrt{R^2 + \left(\frac{h_1 - h_2}{2}\right)^2}\right]$

327. $V = \frac{\pi R^2}{2}(h_1 + h_2)$

3.32 Right Circular Cone

Radius of base: R
Diameter of base: d
Height: H
Slant height: m
Lateral surface area: S_L
Area of base: S_B
Total surface area: S
Volume: V

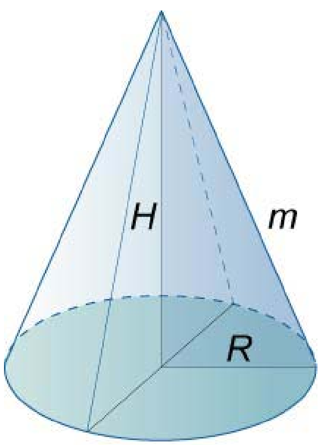


Figure 49.

328. $H=\sqrt{m^2-R^2}$

329. $S_L=\pi Rm=\dfrac{\pi md}{2}$

330. $S_B=\pi R^2$

331. $S=S_L+S_B=\pi R(m+R)=\dfrac{1}{2}\pi d\left(m+\dfrac{d}{2}\right)$

332. $V=\dfrac{1}{3}S_BH=\dfrac{1}{3}\pi R^2H$

3.33 Frustum of a Right Circular Cone

Radius of bases: R, r
Height: H
Slant height: m
Scale factor: k
Area of bases: S_1, S_2
Lateral surface area: S_L
Total surface area: S
Volume: V

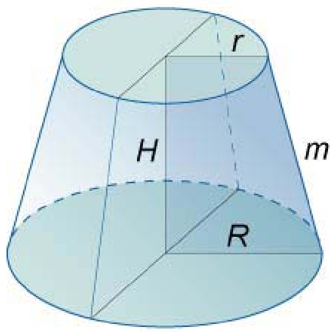


Figure 50.

333. $H=\sqrt{m^2-(R-r)^2}$

334. $\frac{R}{r}=k$

335. $\frac{S_2}{S_1}=\frac{R^2}{r^2}=k^2$

336. $S_L=\pi m(R+r)$

337. $S=S_1+S_2+S_L=\pi\left[R^2+r^2+m(R+r)\right]$

338. $V=\frac{h}{3}\left(S_1+\sqrt{S_1S_2}+S_2\right)$

339. $V=\frac{hS_1}{3}\left[1+\frac{R}{r}+\left(\frac{R}{r}\right)^2\right]=\frac{hS_1}{3}\left[1+k+k^2\right]$

3.34 Sphere

Radius: R
Diameter: d
Surface area: S
Volume: V

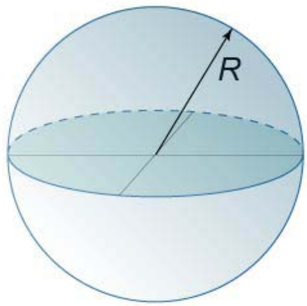


Figure 51.

340. $S = 4\pi R^2$

341. $V = \frac{4}{3}\pi R^3 H = \frac{1}{6}\pi d^3 = \frac{1}{3}SR$

3.35 Spherical Cap

Radius of sphere: R
Radius of base: r
Height: h
Area of plane face: S_B
Area of spherical cap: S_C
Total surface area: S
Volume: V

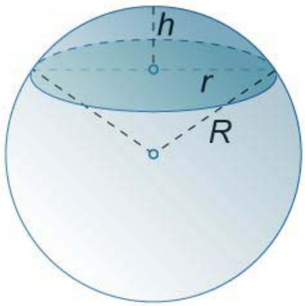


Figure 52.

342. $R = \dfrac{r^2 + h^2}{2h}$

343. $S_B = \pi r^2$

344. $S_C = \pi\left(h^2 + r^2\right)$

345. $S = S_B + S_C = \pi\left(h^2 + 2r^2\right) = \pi\left(2Rh + r^2\right)$

346. $V = \dfrac{\pi}{6} h^2 \left(3R - h\right) = \dfrac{\pi}{6} h\left(3r^2 + h^2\right)$

3.36 Spherical Sector

Radius of sphere: R
Radius of base of spherical cap: r
Height: h
Total surface area: S
Volume: V

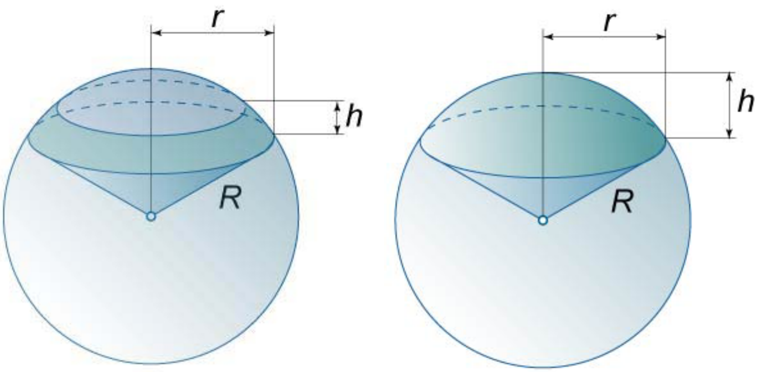


Figure 53.

347. $S = \pi R(2h + r)$

348. $V = \frac{2}{3}\pi R^2 h$

Note: The given formulas are correct both for "open" and "closed" spherical sector.

3.37 Spherical Segment

Radius of sphere: R
Radius of bases: r_1, r_2
Height: h
Area of spherical surface: S_S
Area of plane end faces: S_1, S_2
Total surface area: S
Volume: V

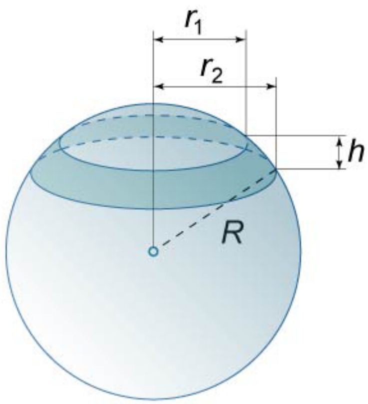


Figure 54.

349. $S_S = 2\pi Rh$

350. $S = S_S + S_1 + S_2 = \pi\left(2Rh + r_1^2 + r_2^2\right)$

351. $V = \frac{1}{6}\pi h\left(3r_1^2 + 3r_2^2 + h^2\right)$

3.38 Spherical Wedge

Radius: R
Dihedral angle in degrees: x
Dihedral angle in radians: α
Area of spherical lune: S_L
Total surface area: S
Volume: V

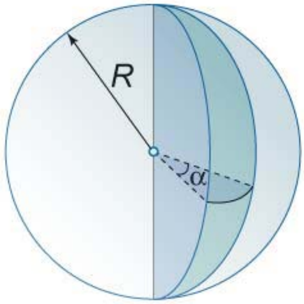


Figure 55.

352. $S_L = \frac{\pi R^2}{90}\alpha = 2R^2 x$

353. $S = \pi R^2 + \frac{\pi R^2}{90}\alpha = \pi R^2 + 2R^2 x$

354. $V = \frac{\pi R^3}{270}\alpha = \frac{2}{3}R^3 x$

3.39 Ellipsoid

Semi-axes: a, b, c
Volume: V

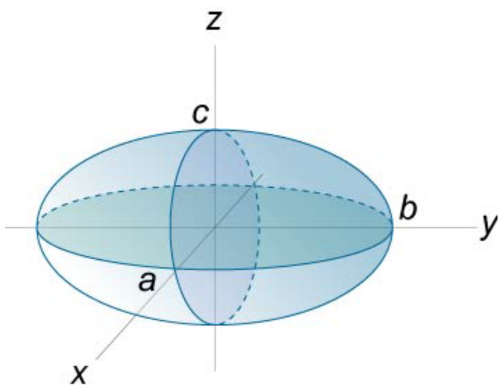


Figure 56.

355. $V = \frac{4}{3}\pi abc$

Prolate Spheroid

Semi-axes: a, b, b ($a > b$)
Surface area: S
Volume: V

356. $S = 2\pi b\left(b + \frac{a \arcsin e}{e}\right),$

where $e = \frac{\sqrt{a^2 - b^2}}{a}.$

357. $V = \frac{4}{3}\pi b^2 a$

Oblate Spheroid

Semi-axes: a, b, b ($a < b$)
Surface area: S
Volume: V

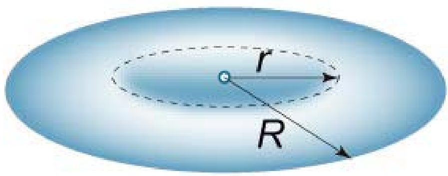
358. $$S = 2\pi b\left(b + \frac{a \operatorname{arcsinh}\left(\frac{be}{a}\right)}{be/a}\right),$$

where $e = \frac{\sqrt{b^2 - a^2}}{b}$.

359. $$V = \frac{4}{3}\pi b^2 a$$

3.40 Circular Torus

Major radius: R
Minor radius: r
Surface area: S
Volume: V



Picture 57.

360. $S = 4\pi^2 Rr$

361. $V = 2\pi^2 Rr^2$

Chapter 4

Trigonometry

Angles: α, β
Real numbers (coordinates of a point): x, y
Whole number: k

4.1 Radian and Degree Measures of Angles

362. $1 \text{ rad} = \dfrac{180^\circ}{\pi} \approx 57^\circ 17' 45''$

363. $1^\circ = \dfrac{\pi}{180} \text{ rad} \approx 0.017453 \text{ rad}$

364. $1' = \dfrac{\pi}{180 \cdot 60} \text{ rad} \approx 0.000291 \text{ rad}$

365. $1'' = \dfrac{\pi}{180 \cdot 3600} \text{ rad} \approx 0.000005 \text{ rad}$

366.

Angle (degrees)	0	30	45	60	90	180	270	360
Angle (radians)	0	$\dfrac{\pi}{6}$	$\dfrac{\pi}{4}$	$\dfrac{\pi}{3}$	$\dfrac{\pi}{2}$	π	$\dfrac{3\pi}{2}$	2π

4.2 Definitions and Graphs of Trigonometric Functions

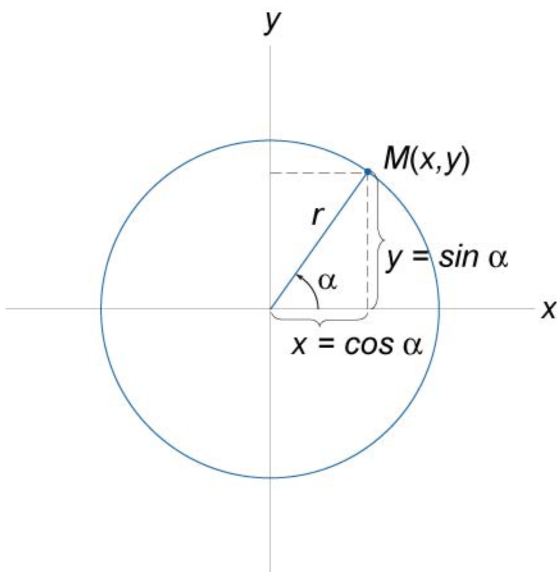


Figure 58.

367. $\sin\alpha = \frac{y}{r}$

368. $\cos\alpha = \frac{x}{r}$

369. $\tan\alpha = \frac{y}{x}$

370. $\cot\alpha = \frac{x}{y}$

371. $\sec\alpha = \dfrac{r}{x}$

372. $\operatorname{cosec}\alpha = \dfrac{r}{y}$

373. Sine Function
$y = \sin x, \ -1 \le \sin x \le 1.$

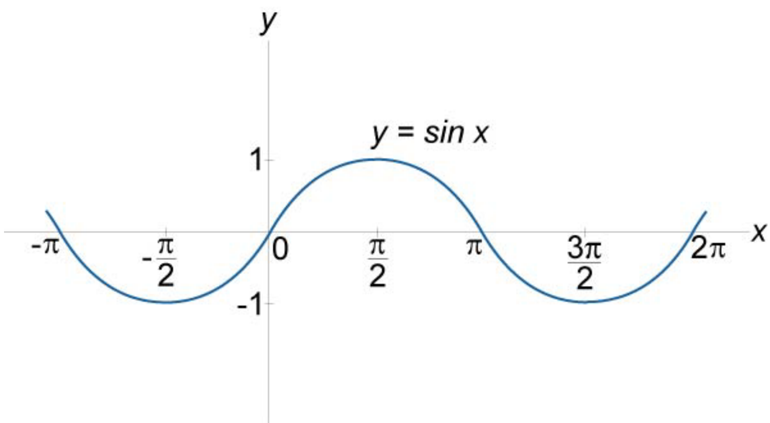


Figure 59.

374. Cosine Function
$y = \cos x, \ -1 \le \cos x \le 1.$

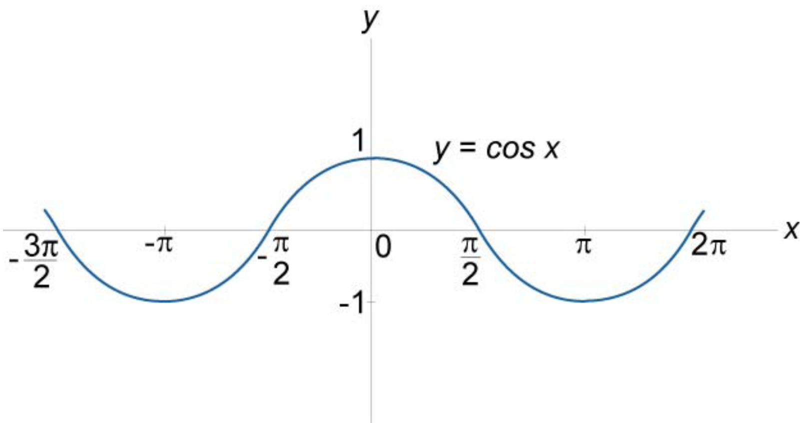


Figure 60.

375. Tangent Function

$$y = \tan x,\ x \neq (2k+1)\frac{\pi}{2},\ -\infty \leq \tan x \leq \infty.$$

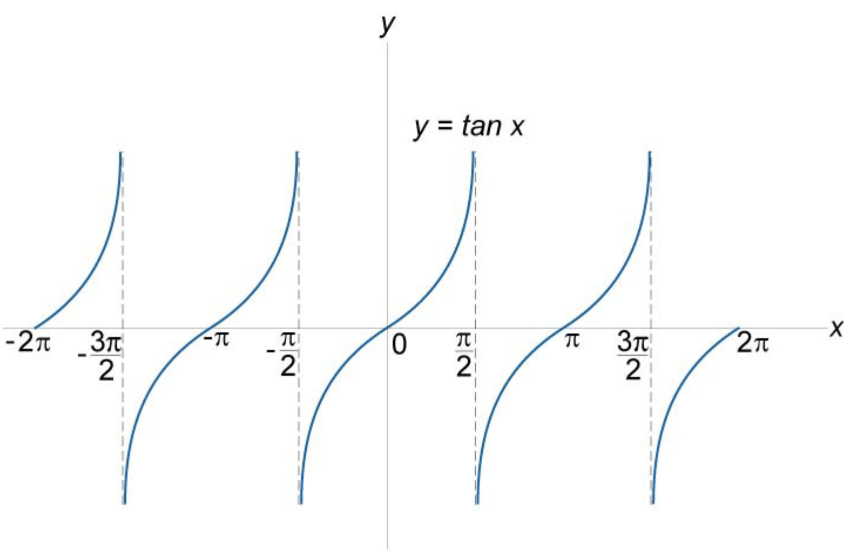


Figure 61.

376. Cotangent Function

$y = \cot x, \; x \neq k\pi, \; -\infty \leq \cot x \leq \infty.$

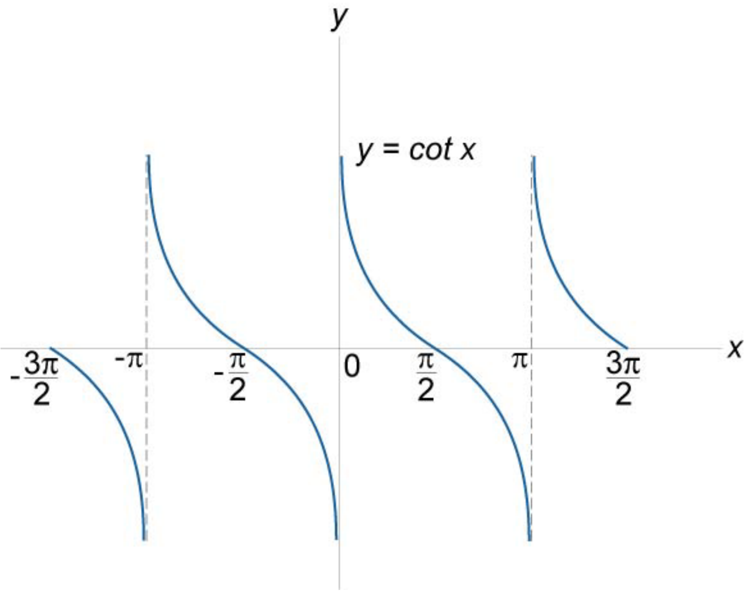


Figure 62.

377. Secant Function

$y = \sec x, \; x \neq (2k+1)\frac{\pi}{2}.$

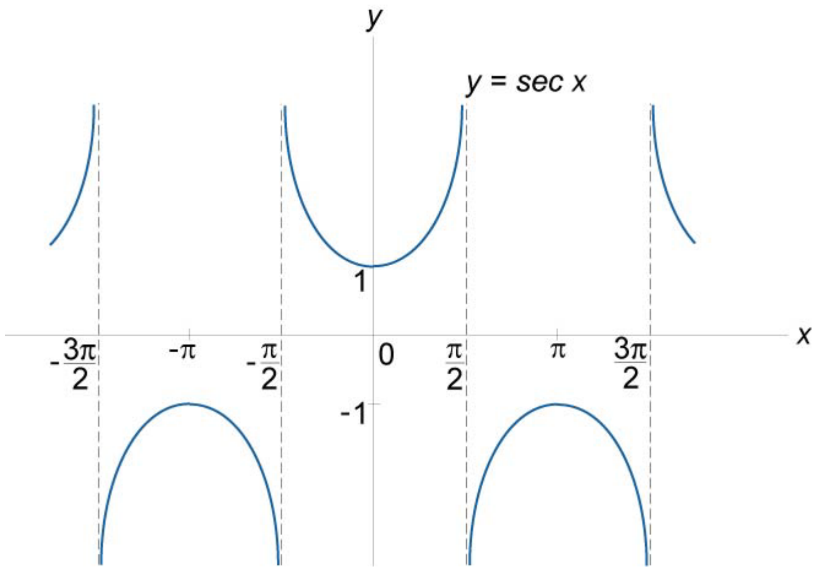


Figure 63.

378. Cosecant Function

$y = \operatorname{cosec} x, \ x \neq k\pi.$

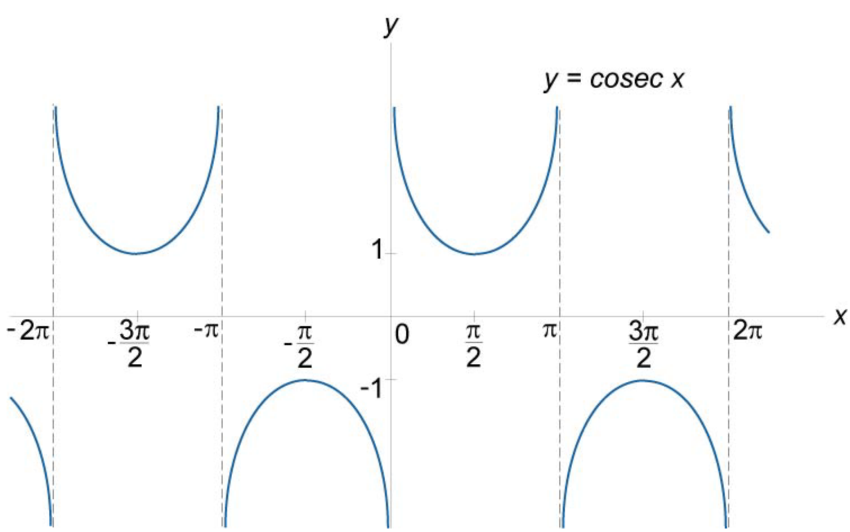


Figure 64.

4.3. Signs of Trigonometric Functions

379.

Quadrant	Sin α	Cos α	Tan α	Cot α	Sec α	Cosec α
I	+	+	+	+	+	+
II	+					+
III			+	+		
IV		+			+	

380.

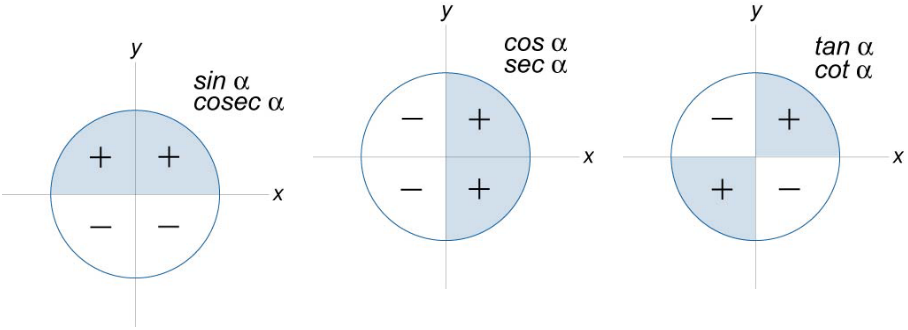


Figure 65.

4.4 Trigonometric Functions of Common Angles

381.

$\alpha°$	α rad	$\sin\alpha$	$\cos\alpha$	$\tan\alpha$	$\cot\alpha$	$\sec\alpha$	$\operatorname{cosec}\alpha$
0	0	0	1	0	∞	1	∞
30	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}}$	2
45	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$	2	$\frac{2}{\sqrt{3}}$
90	$\frac{\pi}{2}$	1	0	∞	0	∞	1
120	$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$	$-\frac{1}{\sqrt{3}}$	-2	$\frac{2}{\sqrt{3}}$
180	π	0	-1	0	∞	-1	∞
270	$\frac{3\pi}{2}$	-1	0	∞	0	∞	-1
360	2π	0	1	0	∞	1	∞

382.

$\alpha°$	α rad	$\sin\alpha$	$\cos\alpha$	$\tan\alpha$	$\cot\alpha$
15	$\frac{\pi}{12}$	$\frac{\sqrt{6}-\sqrt{2}}{4}$	$\frac{\sqrt{6}+\sqrt{2}}{4}$	$2-\sqrt{3}$	$2+\sqrt{3}$
18	$\frac{\pi}{10}$	$\frac{\sqrt{5}-1}{4}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\sqrt{\frac{5-2\sqrt{5}}{5}}$	$\sqrt{5+2\sqrt{5}}$
36	$\frac{\pi}{5}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$	$\frac{\sqrt{5}+1}{4}$	$\frac{\sqrt{10-2\sqrt{5}}}{\sqrt{5}+1}$	$\frac{\sqrt{5}+1}{\sqrt{10-2\sqrt{5}}}$
54	$\frac{3\pi}{10}$	$\frac{\sqrt{5}+1}{4}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$	$\frac{\sqrt{5}+1}{\sqrt{10-2\sqrt{5}}}$	$\frac{\sqrt{10-2\sqrt{5}}}{\sqrt{5}+1}$
72	$\frac{2\pi}{5}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{\sqrt{5}-1}{4}$	$\sqrt{5+2\sqrt{5}}$	$\sqrt{\frac{5-2\sqrt{5}}{5}}$
75	$\frac{5\pi}{12}$	$\frac{\sqrt{6}+\sqrt{2}}{4}$	$\frac{\sqrt{6}-\sqrt{2}}{4}$	$2+\sqrt{3}$	$2-\sqrt{3}$

4.5 Most Important Formulas

383. $\sin^2\alpha + \cos^2\alpha = 1$

384. $\sec^2\alpha - \tan^2\alpha = 1$

385. $\csc^2\alpha - \cot^2\alpha = 1$

386. $\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha}$

387. $\cot\alpha = \dfrac{\cos\alpha}{\sin\alpha}$

388. $\tan\alpha \cdot \cot\alpha = 1$

389. $\sec\alpha = \dfrac{1}{\cos\alpha}$

390. $\operatorname{cosec}\alpha = \dfrac{1}{\sin\alpha}$

4.6 Reduction Formulas

391.

β	$\sin\beta$	$\cos\beta$	$\tan\beta$	$\cot\beta$
$-\alpha$	$-\sin\alpha$	$+\cos\alpha$	$-\tan\alpha$	$-\cot\alpha$
$90^\circ-\alpha$	$+\cos\alpha$	$+\sin\alpha$	$+\cot\alpha$	$+\tan\alpha$
$90^\circ+\alpha$	$+\cos\alpha$	$-\sin\alpha$	$-\cot\alpha$	$-\tan\alpha$
$180^\circ-\alpha$	$+\sin\alpha$	$-\cos\alpha$	$-\tan\alpha$	$-\cot\alpha$
$180^\circ+\alpha$	$-\sin\alpha$	$-\cos\alpha$	$+\tan\alpha$	$+\cot\alpha$
$270^\circ-\alpha$	$-\cos\alpha$	$-\sin\alpha$	$+\cot\alpha$	$+\tan\alpha$
$270^\circ+\alpha$	$-\cos\alpha$	$+\sin\alpha$	$-\cot\alpha$	$-\tan\alpha$
$360^\circ-\alpha$	$-\sin\alpha$	$+\cos\alpha$	$-\tan\alpha$	$-\cot\alpha$
$360^\circ+\alpha$	$+\sin\alpha$	$+\cos\alpha$	$+\tan\alpha$	$+\cot\alpha$

4.7 Periodicity of Trigonometric Functions

392. $\sin(\alpha \pm 2\pi n) = \sin\alpha$, period 2π or $360°$.

393. $\cos(\alpha \pm 2\pi n) = \cos\alpha$, period 2π or $360°$.

394. $\tan(\alpha \pm \pi n) = \tan\alpha$, period π or $180°$.

395. $\cot(\alpha \pm \pi n) = \cot\alpha$, period π or $180°$.

4.8 Relations between Trigonometric Functions

396. $$\sin\alpha = \pm\sqrt{1-\cos^2\alpha} = \pm\sqrt{\frac{1}{2}(1-\cos 2\alpha)} = 2\cos^2\left(\frac{\alpha}{2}-\frac{\pi}{4}\right)-1 = \frac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}$$

397. $$\cos\alpha = \pm\sqrt{1-\sin^2\alpha} = \pm\sqrt{\frac{1}{2}(1+\cos 2\alpha)} = 2\cos^2\frac{\alpha}{2}-1 = \frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}$$

398. $$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \pm\sqrt{\sec^2\alpha-1} = \frac{\sin 2\alpha}{1+\cos 2\alpha} = \frac{1-\cos 2\alpha}{\sin 2\alpha}$$

$$= \pm\sqrt{\frac{1-\cos 2\alpha}{1+\cos 2\alpha}} = \frac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}$$

399. $$\cot\alpha = \frac{\cos\alpha}{\sin\alpha} = \pm\sqrt{\csc^2\alpha - 1} = \frac{1+\cos 2\alpha}{\sin 2\alpha} = \frac{\sin 2\alpha}{1-\cos 2\alpha}$$
$$= \pm\sqrt{\frac{1+\cos 2\alpha}{1-\cos 2\alpha}} = \frac{1-\tan^2\frac{\alpha}{2}}{2\tan\frac{\alpha}{2}}$$

400. $$\sec\alpha = \frac{1}{\cos\alpha} = \pm\sqrt{1+\tan^2\alpha} = \frac{1+\tan^2\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}}$$

401. $$\csc\alpha = \frac{1}{\sin\alpha} = \pm\sqrt{1+\cot^2\alpha} = \frac{1+\tan^2\frac{\alpha}{2}}{2\tan\frac{\alpha}{2}}$$

4.9 Addition and Subtraction Formulas

402. $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$

403. $\sin(\alpha-y) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$

404. $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

405. $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

406. $\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

407. $\tan(\alpha-\beta)=\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$

408. $\cot(\alpha+\beta)=\dfrac{1-\tan\alpha\tan\beta}{\tan\alpha+\tan\beta}$

409. $\cot(\alpha-\beta)=\dfrac{1+\tan\alpha\tan\beta}{\tan\alpha-\tan\beta}$

4.10 Double Angle Formulas

410. $\sin 2\alpha = 2\sin\alpha\cdot\cos\alpha$

411. $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 1-2\sin^2\alpha = 2\cos^2\alpha - 1$

412. $\tan 2\alpha = \dfrac{2\tan\alpha}{1-\tan^2\alpha} = \dfrac{2}{\cot\alpha - \tan\alpha}$

413. $\cot 2\alpha = \dfrac{\cot^2\alpha - 1}{2\cot\alpha} = \dfrac{\cot\alpha - \tan\alpha}{2}$

4.11 Multiple Angle Formulas

414. $\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha = 3\cos^2\alpha \cdot \sin\alpha - \sin^3\alpha$

415. $\sin 4\alpha = 4\sin\alpha \cdot \cos\alpha - 8\sin^3\alpha \cdot \cos\alpha$

416. $\sin 5\alpha = 5\sin\alpha - 20\sin^3\alpha + 16\sin^5\alpha$

417. $\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha = \cos^3\alpha - 3\cos\alpha \cdot \sin^2\alpha$

418. $\cos 4\alpha = 8\cos^4\alpha - 8\cos^2\alpha + 1$

419. $\cos 5\alpha = 16\cos^5\alpha - 20\cos^3\alpha + 5\cos\alpha$

420. $\tan 3\alpha = \dfrac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha}$

421. $\tan 4\alpha = \dfrac{4\tan\alpha - 4\tan^3\alpha}{1 - 6\tan^2\alpha + \tan^4\alpha}$

422. $\tan 5\alpha = \dfrac{\tan^5\alpha - 10\tan^3\alpha + 5\tan\alpha}{1 - 10\tan^2\alpha + 5\tan^4\alpha}$

423. $\cot 3\alpha = \dfrac{\cot^3\alpha - 3\cot\alpha}{3\cot^2\alpha - 1}$

424. $\cot 4\alpha = \dfrac{1 - 6\tan^2\alpha + \tan^4\alpha}{4\tan\alpha - 4\tan^3\alpha}$

425. $$\cot 5\alpha = \frac{1 - 10\tan^2\alpha + 5\tan^4\alpha}{\tan^5\alpha - 10\tan^3\alpha + 5\tan\alpha}$$

4.12 Half Angle Formulas

426. $$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$

427. $$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

428. $$\tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} = \frac{\sin\alpha}{1+\cos\alpha} = \frac{1-\cos\alpha}{\sin\alpha} = \csc\alpha - \cot\alpha$$

429. $$\cot\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}} = \frac{\sin\alpha}{1-\cos\alpha} = \frac{1+\cos\alpha}{\sin\alpha} = \csc\alpha + \cot\alpha$$

4.13 Half Angle Tangent Identities

430. $$\sin\alpha = \frac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}$$

431. $$\cos\alpha = \frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}$$

432. $$\tan\alpha = \frac{2\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}}$$

433. $$\cot\alpha = \frac{1-\tan^2\frac{\alpha}{2}}{2\tan\frac{\alpha}{2}}$$

4.14 Transforming of Trigonometric Expressions to Product

434. $$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

435. $$\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

436. $$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

437. $$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

438. $\tan\alpha + \tan\beta = \dfrac{\sin(\alpha+\beta)}{\cos\alpha \cdot \cos\beta}$

439. $\tan\alpha - \tan\beta = \dfrac{\sin(\alpha-\beta)}{\cos\alpha \cdot \cos\beta}$

440. $\cot\alpha + \cot\beta = \dfrac{\sin(\beta+\alpha)}{\sin\alpha \cdot \sin\beta}$

441. $\cot\alpha - \cot\beta = \dfrac{\sin(\beta-\alpha)}{\sin\alpha \cdot \sin\beta}$

442. $\cos\alpha + \sin\alpha = \sqrt{2}\cos\left(\dfrac{\pi}{4} - \alpha\right) = \sqrt{2}\sin\left(\dfrac{\pi}{4} + \alpha\right)$

443. $\cos\alpha - \sin\alpha = \sqrt{2}\sin\left(\dfrac{\pi}{4} - \alpha\right) = \sqrt{2}\cos\left(\dfrac{\pi}{4} + \alpha\right)$

444. $\tan\alpha + \cot\beta = \dfrac{\cos(\alpha-\beta)}{\cos\alpha \cdot \sin\beta}$

445. $\tan\alpha - \cot\beta = -\dfrac{\cos(\alpha+\beta)}{\cos\alpha \cdot \sin\beta}$

446. $1 + \cos\alpha = 2\cos^2\dfrac{\alpha}{2}$

447. $1 - \cos\alpha = 2\sin^2\dfrac{\alpha}{2}$

448. $1+\sin\alpha = 2\cos^2\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)$

449. $1-\sin\alpha = 2\sin^2\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)$

4.15 Transforming of Trigonometric Expressions to Sum

450. $\sin\alpha\cdot\sin\beta = \frac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{2}$

451. $\cos\alpha\cdot\cos\beta = \frac{\cos(\alpha-\beta)+\cos(\alpha+\beta)}{2}$

452. $\sin\alpha\cdot\cos\beta = \frac{\sin(\alpha-\beta)+\sin(\alpha+\beta)}{2}$

453. $\tan\alpha\cdot\tan\beta = \frac{\tan\alpha+\tan\beta}{\cot\alpha+\cot\beta}$

454. $\cot\alpha\cdot\cot\beta = \frac{\cot\alpha+\cot\beta}{\tan\alpha+\tan\beta}$

455. $\tan\alpha\cdot\cot\beta = \frac{\tan\alpha+\cot\beta}{\cot\alpha+\tan\beta}$

4.16 Powers of Trigonometric Functions

456. $\sin^2\alpha = \dfrac{1-\cos 2\alpha}{2}$

457. $\sin^3\alpha = \dfrac{3\sin\alpha - \sin 3\alpha}{4}$

458. $\sin^4\alpha = \dfrac{\cos 4\alpha - 4\cos 2\alpha + 3}{8}$

459. $\sin^5\alpha = \dfrac{10\sin\alpha - 5\sin 3\alpha + \sin 5\alpha}{16}$

460. $\sin^6\alpha = \dfrac{10 - 15\cos 2\alpha + 6\cos 4\alpha - \cos 6\alpha}{32}$

461. $\cos^2\alpha = \dfrac{1+\cos 2\alpha}{2}$

462. $\cos^3\alpha = \dfrac{3\cos\alpha + \cos 3\alpha}{4}$

463. $\cos^4\alpha = \dfrac{\cos 4\alpha + 4\cos 2\alpha + 3}{8}$

464. $\cos^5\alpha = \dfrac{10\cos\alpha + 5\sin 3\alpha + \cos 5\alpha}{16}$

465. $\cos^6\alpha = \dfrac{10 + 15\cos 2\alpha + 6\cos 4\alpha + \cos 6\alpha}{32}$

4.17 Graphs of Inverse Trigonometric Functions

466. **Inverse Sine Function**

$$y = \arcsin x, \; -1 \le x \le 1, \; -\frac{\pi}{2} \le \arcsin x \le \frac{\pi}{2}.$$

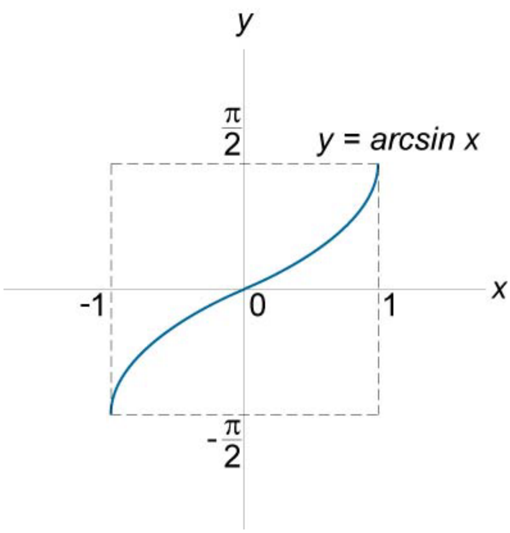


Figure 66.

467. **Inverse Cosine Function**

$$y = \arccos x, \; -1 \le x \le 1, \; 0 \le \arccos x \le \pi.$$

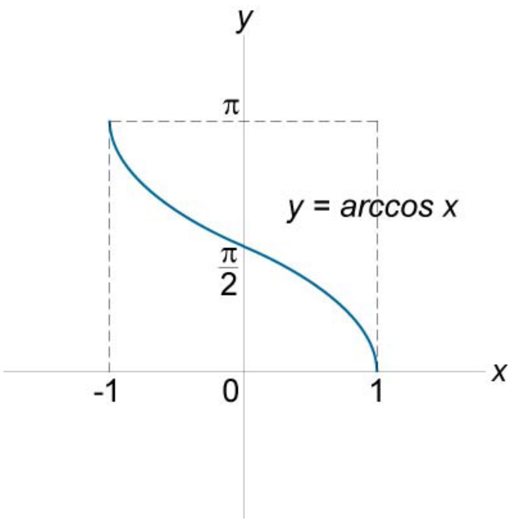


Figure 67.

468. Inverse Tangent Function

$$y = \arctan x\,,\ -\infty \le x \le \infty\,,\ -\frac{\pi}{2} < \arctan x < \frac{\pi}{2}.$$

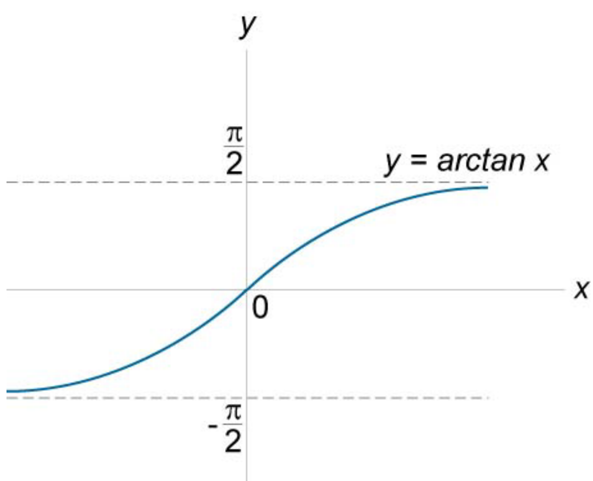


Figure 68.

469. Inverse Cotangent Function

$y = \operatorname{arccot} x, \; -\infty \le x \le \infty, \; 0 < \operatorname{arccot} x < \pi.$

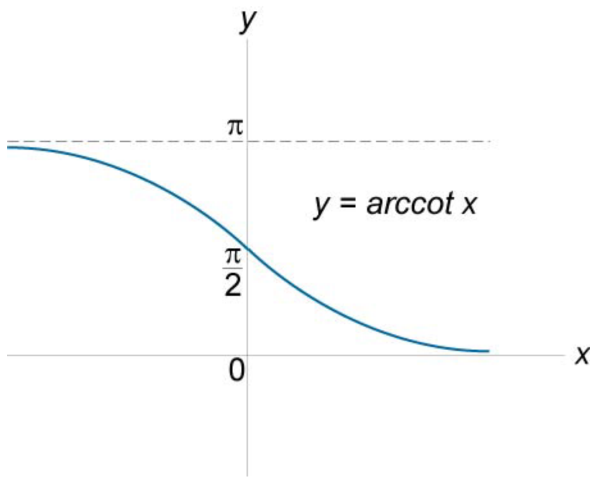


Figure 69.

470. Inverse Secant Function

$y = \operatorname{arcsec} x, \;\; x \in (-\infty, -1] \cup [1, \infty), \;\; \operatorname{arcsec} x \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right].$

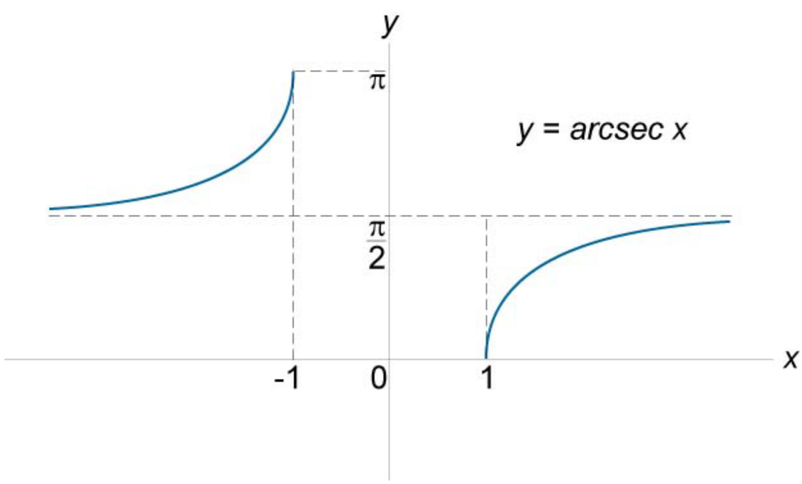


Figure 70.

471. Inverse Cosecant Function

$$y = \operatorname{arccsc} x,\ \ x \in (-\infty, -1] \cup [1, \infty),\ \ \operatorname{arccsc} x \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right].$$

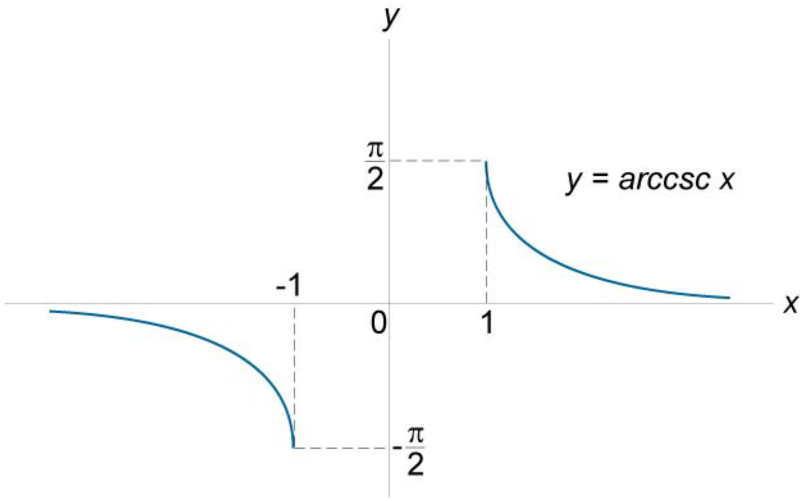


Figure 71.

4.18 Principal Values of Inverse Trigonometric Functions

472.

x	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\arcsin x$	$0°$	$30°$	$45°$	$60°$	$90°$
$\arccos x$	$90°$	$60°$	$45°$	$30°$	$0°$
x	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	
$\arcsin x$	$-30°$	$-45°$	$-60°$	$-90°$	
$\arccos x$	$120°$	$135°$	$150°$	$180°$	

473.

x	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$-\frac{\sqrt{3}}{3}$	-1	$-\sqrt{3}$
$\arctan x$	$0°$	$30°$	$45°$	$60°$	$-30°$	$-45°$	$-60°$
$\operatorname{arccot} x$	$90°$	$60°$	$45°$	$30°$	$120°$	$135°$	$150°$

4.19 Relations between Inverse Trigonometric Functions

474. $\arcsin(-x) = -\arcsin x$

475. $\arcsin x = \frac{\pi}{2} - \arccos x$

476. $\arcsin x = \arccos\sqrt{1-x^2}$, $0 \le x \le 1$.

477. $\arcsin x = -\arccos\sqrt{1-x^2}$, $-1 \le x \le 0$.

478. $\arcsin x = \arctan\frac{x}{\sqrt{1-x^2}}$, $x^2 < 1$.

479. $\arcsin x = \operatorname{arccot}\frac{\sqrt{1-x^2}}{x}$, $0 < x \le 1$.

480. $\arcsin x = \operatorname{arccot}\frac{\sqrt{1-x^2}}{x} - \pi$, $-1 \le x < 0$.

481. $\arccos(-x) = \pi - \arccos x$

482. $\arccos x = \frac{\pi}{2} - \arcsin x$

483. $\arccos x = \arcsin \sqrt{1-x^2}$, $0 \le x \le 1$.

484. $\arccos x = \pi - \arcsin \sqrt{1-x^2}$, $-1 \le x \le 0$.

485. $\arccos x = \arctan \frac{\sqrt{1-x^2}}{x}$, $0 < x \le 1$.

486. $\arccos x = \pi + \arctan \frac{\sqrt{1-x^2}}{x}$, $-1 \le x < 0$.

487. $\arccos x = \operatorname{arccot} \frac{x}{\sqrt{1-x^2}}$, $-1 \le x \le 1$.

488. $\arctan(-x) = -\arctan x$

489. $\arctan x = \frac{\pi}{2} - \operatorname{arccot} x$

490. $\arctan x = \arcsin \frac{x}{\sqrt{1+x^2}}$

491. $\arctan x = \arccos \frac{1}{\sqrt{1+x^2}}$, $x \ge 0$.

492. $\arctan x = -\arccos \frac{1}{\sqrt{1+x^2}}$, $x \le 0$.

493. $\arctan x = \frac{\pi}{2} - \arctan\frac{1}{x},\ x > 0.$

494. $\arctan x = -\frac{\pi}{2} - \arctan\frac{1}{x},\ x < 0.$

495. $\arctan x = \operatorname{arccot}\frac{1}{x},\ x > 0.$

496. $\arctan x = \operatorname{arccot}\frac{1}{x} - \pi,\ x < 0.$

497. $\operatorname{arccot}(-x) = \pi - \operatorname{arccot} x$

498. $\operatorname{arccot} x = \frac{\pi}{2} - \arctan x$

499. $\operatorname{arccot} x = \arcsin\frac{1}{\sqrt{1+x^2}},\ x > 0.$

500. $\operatorname{arccot} x = \pi - \arcsin\frac{1}{\sqrt{1+x^2}},\ x < 0.$

501. $\operatorname{arccot} x = \arccos\frac{x}{\sqrt{1+x^2}}$

502. $\operatorname{arccot} x = \arctan\frac{1}{x},\ x > 0.$

503. $\operatorname{arccot} x = \pi + \arctan\frac{1}{x},\ x < 0.$

4.20 Trigonometric Equations

Whole number: n

504. $\sin x = a, \quad x = (-1)^n \arcsin a + \pi n$

505. $\cos x = a, \quad x = \pm \arccos a + 2\pi n$

506. $\tan x = a, \quad x = \arctan a + \pi n$

507. $\cot x = a, \quad x = \operatorname{arccot} a + \pi n$

4.21 Relations to Hyperbolic Functions

Imaginary unit: i

508. $\sin(ix) = i \sinh x$

509. $\tan(ix) = i \tanh x$

510. $\cot(ix) = -i \coth x$

511. $\sec(ix) = \operatorname{sech} x$

512. $\csc(ix) = -i \operatorname{csch} x$

Chapter 5

Matrices and Determinants

Matrices: A, B, C
Elements of a matrix: a_i, b_i, a_{ij}, b_{ij}, c_{ij}
Determinant of a matrix: $\det A$
Minor of an element a_{ij} : M_{ij}
Cofactor of an element a_{ij} : C_{ij}
Transpose of a matrix: A^T, $\tilde{A}$
Adjoint of a matrix: $\operatorname{adj} A$
Trace of a matrix: $\operatorname{tr} A$
Inverse of a matrix: A^{-1}
Real number: k
Real variables: x_i
Natural numbers: m, n

5.1 Determinants

513. Second Order Determinant

$$\det A = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

514. Third Order Determinant

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} -$$

$$- a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

515. Sarrus Rule (Arrow Rule)

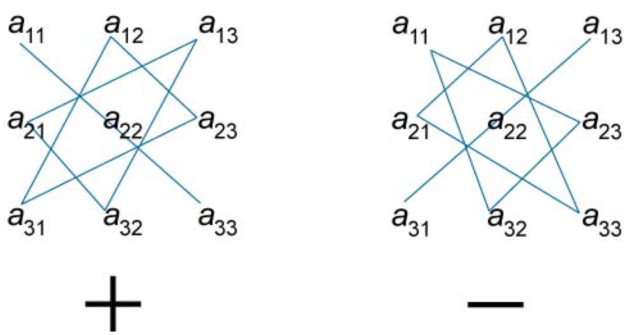


Figure 72.

516. N-th Order Determinant

$$\det A = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nj} & \dots & a_{nn} \end{vmatrix}$$

517. Minor

The minor M_{ij} associated with the element a_{ij} of n-th order matrix A is the $(n-1)$-th order determinant derived from the matrix A by deletion of its i-th row and j-th column.

518. Cofactor

$$C_{ij} = (-1)^{i+j} M_{ij}$$

519. Laplace Expansion of n-th Order Determinant

Laplace expansion by elements of the i-th row

$$\det A = \sum_{j=1}^{n} a_{ij} C_{ij}\ ,\ i = 1, 2, \ldots, n.$$

Laplace expansion by elements of the j-th column

$$\det A = \sum_{i=1}^{n} a_{ij} C_{ij}\ ,\ j = 1, 2, \ldots, n.$$

5.2 Properties of Determinants

520. The value of a determinant remains unchanged if rows are changed to columns and columns to rows.

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

521. If two rows (or two columns) are interchanged, the sign of the determinant is changed.

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = -\begin{vmatrix} a_2 & b_2 \\ a_1 & b_1 \end{vmatrix}$$

522. If two rows (or two columns) are identical, the value of the determinant is zero.

$$\begin{vmatrix} a_1 & a_1 \\ a_2 & a_2 \end{vmatrix} = 0$$

523. If the elements of any row (or column) are multiplied by a common factor, the determinant is multiplied by that factor.

$$\begin{vmatrix} ka_1 & kb_1 \\ a_2 & b_2 \end{vmatrix} = k\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

524. If the elements of any row (or column) are increased (or decreased)by equal multiples of the corresponding elements of any other row (or column), the value of the determinant is unchanged.

$$\begin{vmatrix} a_1 + kb_1 & b_1 \\ a_2 + kb_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

5.3 Matrices

525. Definition

An $m \times n$ matrix A is a rectangular array of elements (numbers or functions) with m rows and n columns.

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

526. Square matrix is a matrix of order $n \times n$.

527. A square matrix $[a_{ij}]$ is symmetric if $a_{ij} = a_{ji}$, i.e. it is symmetric about the leading diagonal.

528. A square matrix $[a_{ij}]$ is skew-symmetric if $a_{ij} = -a_{ji}$.

529. Diagonal matrix is a square matrix with all elements zero except those on the leading diagonal.

530. Unit matrix is a diagonal matrix in which the elements on the leading diagonal are all unity. The unit matrix is denoted by I.

531. A null matrix is one whose elements are all zero.

5.4 Operations with Matrices

532. Two matrices A and B are equal if, and only if, they are both of the same shape $m \times n$ and corresponding elements are equal.

533. Two matrices A and B can be added (or subtracted) of, and only if, they have the same shape $m \times n$. If

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

$$B = [b_{ij}] = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix},$$

then

$$A+B=\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \dots & a_{2n}+b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \dots & a_{mn}+b_{mn} \end{bmatrix}.$$

534. If k is a scalar, and $A=\left[a_{ij}\right]$ is a matrix, then

$$kA=\left[ka_{ij}\right]=\begin{bmatrix} ka_{11} & ka_{12} & \dots & ka_{1n} \\ ka_{21} & ka_{22} & \dots & ka_{2n} \\ \vdots & \vdots & & \vdots \\ ka_{m1} & ka_{m2} & \dots & ka_{mn} \end{bmatrix}.$$

535. Multiplication of Two Matrices

Two matrices can be multiplied together only when the number of columns in the first is equal to the number of rows in the second.

If

$$A=\left[a_{ij}\right]=\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix},$$

$$B=\left[b_{ij}\right]=\begin{bmatrix} b_{11} & b_{12} & \dots & b_{1k} \\ b_{21} & b_{22} & \dots & b_{2k} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nk} \end{bmatrix},$$

then

$$AB = C = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1k} \\ c_{21} & c_{22} & \dots & c_{2k} \\ \vdots & \vdots & & \vdots \\ b_{m1} & c_{m2} & \dots & c_{mk} \end{bmatrix},$$

where

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj} = \sum_{\lambda=1}^{n} a_{i\lambda}b_{\lambda j}$$

$(i = 1, 2, \dots, m; j = 1, 2, \dots, k)$.

Thus if

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}, \; B = [b_i] = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix},$$

then

$$AB = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_{11}b_1 & a_{12}b_2 & a_{13}b_3 \\ a_{21}b_1 & a_{22}b_2 & a_{23}b_3 \end{bmatrix}.$$

536. Transpose of a Matrix
If the rows and columns of a matrix are interchanged, then the new matrix is called the transpose of the original matrix. If A is the original matrix, its transpose is denoted A^T or $\tilde{A}$.

537. The matrix A is orthogonal if $AA^T = I$.

538. If the matrix product AB is defined, then
$(AB)^T = B^T A^T$.

539. Adjoint of Matrix
If A is a square $n \times n$ matrix, its adjoint, denoted by $\operatorname{adj} A$, is the transpose of the matrix of cofactors C_{ij} of A:
$\operatorname{adj} A = \left[C_{ij}\right]^T$.

540. Trace of a Matrix
If A is a square $n \times n$ matrix, its trace, denoted by $\operatorname{tr} A$, is defined to be the sum of the terms on the leading diagonal:
$\operatorname{tr} A = a_{11} + a_{22} + \ldots + a_{nn}$.

541. Inverse of a Matrix
If A is a square $n \times n$ matrix with a nonsingular determinant $\det A$, then its inverse A^{-1} is given by
$A^{-1} = \dfrac{\operatorname{adj} A}{\det A}$.

542. If the matrix product AB is defined, then
$(AB)^{-1} = B^{-1}A^{-1}$.

543. If A is a square $n \times n$ matrix, the eigenvectors X satisfy the equation
$AX = \lambda X$,
while the eigenvalues λ satisfy the characteristic equation
$|A - \lambda I| = 0$.

5.5 Systems of Linear Equations

Variables: x, y, z, x_1, $x_2, \ldots$
Real numbers: $a_1, a_2, a_3, b_1, a_{11}, a_{12}, \ldots$

Determinants: D, D_x, D_y, D_z

Matrices: A, B, X

544. $\begin{cases} a_1x + b_1y = d_1 \\ a_2x + b_2y = d_2 \end{cases}$,

$x = \frac{D_x}{D}$, $y = \frac{D_y}{D}$ (Cramer's rule),

where

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1,$$

$$D_x = \begin{vmatrix} d_1 & b_1 \\ d_2 & b_2 \end{vmatrix} = d_1b_2 - d_2b_1,$$

$$D_y = \begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix} = a_1d_2 - a_2d_1.$$

545. If $D \neq 0$, then the system has a single solution:

$x = \frac{D_x}{D}$, $y = \frac{D_y}{D}$.

If $D = 0$ and $D_x \neq 0$ (or $D_y \neq 0$), then the system has no solution.

If $D = D_x = D_y = 0$, then the system has infinitely many solutions.

546. $\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$,

$x = \frac{D_x}{D}$, $y = \frac{D_y}{D}$, $z = \frac{D_z}{D}$ (Cramer's rule), where

$$D=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},\ D_x=\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix},$$

$$D_y=\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix},\ D_z=\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}.$$

547. If $D \neq 0$, then the system has a single solution:

$$x=\frac{D_x}{D},\ y=\frac{D_y}{D},\ z=\frac{D_z}{D}.$$

If $D=0$ and $D_x \neq 0$ (or $D_y \neq 0$ or $D_z \neq 0$), then the system has no solution.

If $D=D_x=D_y=D_z=0$, then the system has infinitely many solutions.

548. Matrix Form of a System of n Linear Equations in n Unknowns

The set of linear equations

$$\begin{cases} a_{11}x_1+a_{12}x_2+\ldots+a_{1n}x_n=b_1 \\ a_{21}x_1+a_{22}x_2+\ldots+a_{2n}x_n=b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}x_1+a_{n2}x_2+\ldots+a_{nn}x_n=b_n \end{cases}$$

can be written in matrix form

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix},$$

i.e.

$$A \cdot X = B,$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix},\ X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},\ B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

549. Solution of a Set of Linear Equations $n \times n$

$X = A^{-1} \cdot B$,

where A^{-1} is the inverse of A.

Chapter 6

Vectors

Vectors: $\vec{u}$, $\vec{v}$, $\vec{w}$, $\vec{r}$, $\overrightarrow{AB}$, ...
Vector length: $|\vec{u}|$, $|\vec{v}|$, ...
Unit vectors: $\vec{i}$, $\vec{j}$, $\vec{k}$
Null vector: $\vec{0}$
Coordinates of vector $\vec{u}$: X_1, Y_1, Z_1
Coordinates of vector $\vec{v}$: X_2, Y_2, Z_2
Scalars: λ, μ
Direction cosines: $\cos\alpha$, $\cos\beta$, $\cos\gamma$
Angle between two vectors: θ

6.1 Vector Coordinates

550. Unit Vectors

$$\vec{i} = (1, 0, 0),$$
$$\vec{j} = (0, 1, 0),$$
$$\vec{k} = (0, 0, 1),$$
$$|\vec{i}| = |\vec{j}| = |\vec{k}| = 1.$$

551. $\vec{r} = \overrightarrow{AB} = (x_1 - x_0)\vec{i} + (y_1 - y_0)\vec{j} + (z_1 - z_0)\vec{k}$

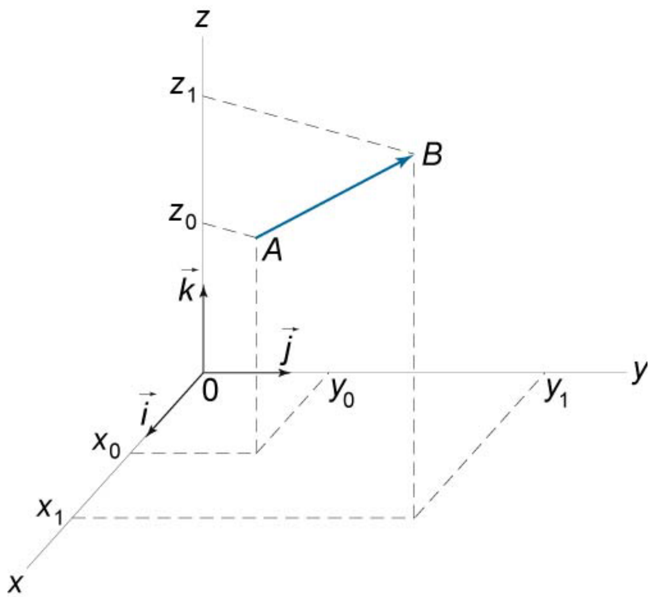


Figure 73.

552. $|\vec{r}| = \left|\overrightarrow{AB}\right| = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2}$

553. If $\overrightarrow{AB} = \vec{r}$, then $\overrightarrow{BA} = -\vec{r}$.

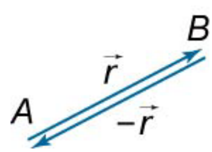


Figure 74.

554. $X = |\vec{r}|\cos\alpha$,

$Y = |\vec{r}|\cos\beta$,

$Z = |\vec{r}|\cos\gamma$.

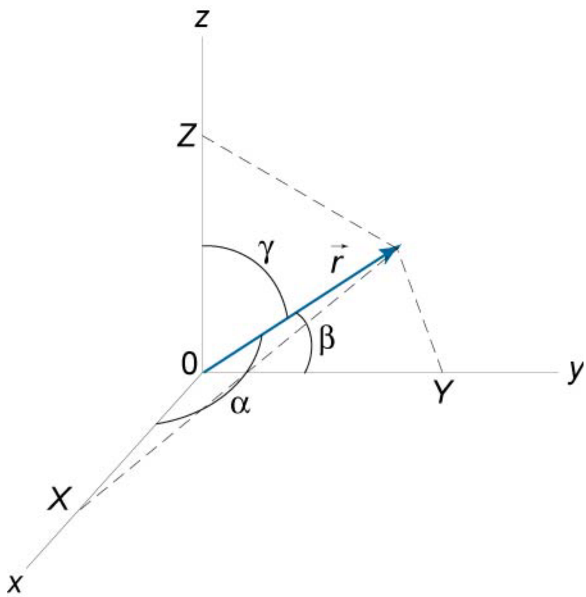


Figure 75.

555. If $\vec{r}(X, Y, Z) = \vec{r}_1(X_1, Y_1, Z_1)$, then
$X = X_1$, $Y = Y_1$, $Z = Z_1$.

6.2 Vector Addition

556. $\vec{w} = \vec{u} + \vec{v}$

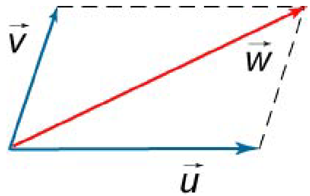


Figure 76.

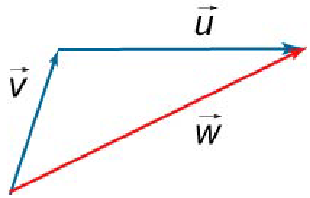


Figure 77.

557. $\vec{w} = \vec{u}_1 + \vec{u}_2 + \vec{u}_3 + \ldots + \vec{u}_n$

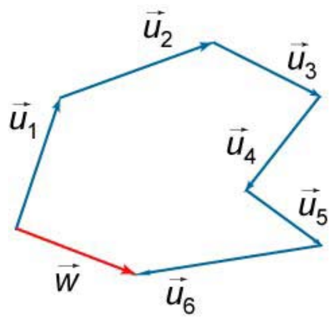


Figure 78.

558. Commutative Law

$\vec{u} + \vec{v} = \vec{v} + \vec{u}$

559. Associative Law

$(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

560. $\vec{u} + \vec{v} = (X_1 + X_2, Y_1 + Y_2, Z_1 + Z_2)$

6.3 Vector Subtraction

561. $\vec{w} = \vec{u} - \vec{v}$ if $\vec{v} + \vec{w} = \vec{u}$.

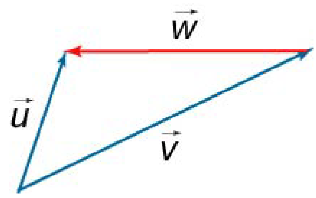


Figure 79.

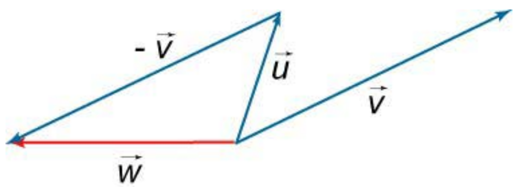


Figure 80.

562. $\vec{u} - \vec{v} = \vec{u} + (-\vec{v})$

563. $\vec{u} - \vec{u} = \vec{0} = (0, 0, 0)$

564. $|\vec{0}| = 0$

565. $\vec{u} - \vec{v} = (X_1 - X_2, Y_1 - Y_2, Z_1 - Z_2)$,

6.4 Scaling Vectors

566. $\vec{w} = \lambda\vec{u}$

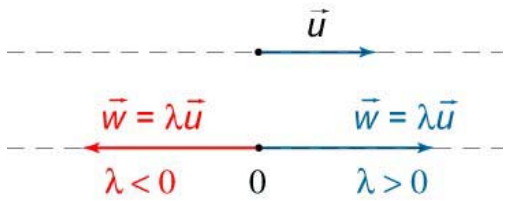


Figure 81.

567. $|\vec{w}| = |\lambda| \cdot |\vec{u}|$

568. $\lambda\vec{u} = (\lambda X, \lambda Y, \lambda Z)$

569. $\lambda\vec{u} = \vec{u}\lambda$

570. $(\lambda + \mu)\vec{u} = \lambda\vec{u} + \mu\vec{u}$

571. $\lambda(\mu\vec{u}) = \mu(\lambda\vec{u}) = (\lambda\mu)\vec{u}$

572. $\lambda(\vec{u} + \vec{v}) = \lambda\vec{u} + \lambda\vec{v}$

6.5 Scalar Product

573. Scalar Product of Vectors $\vec{u}$ and $\vec{v}$
$\vec{u} \cdot \vec{v} = |\vec{u}| \cdot |\vec{v}| \cdot \cos\theta$,
where θ is the angle between vectors $\vec{u}$ and $\vec{v}$.

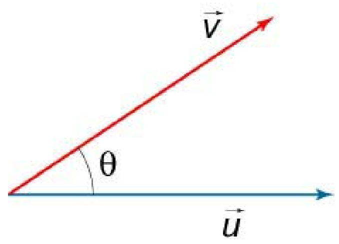


Figure 82.

574. Scalar Product in Coordinate Form

If $\vec{u} = (X_1, Y_1, Z_1)$, $\vec{v} = (X_2, Y_2, Z_2)$, then

$\vec{u} \cdot \vec{v} = X_1 X_2 + Y_1 Y_2 + Z_1 Z_2$.

575. Angle Between Two Vectors

If $\vec{u} = (X_1, Y_1, Z_1)$, $\vec{v} = (X_2, Y_2, Z_2)$, then

$$\cos\theta = \frac{X_1 X_2 + Y_1 Y_2 + Z_1 Z_2}{\sqrt{X_1^2 + Y_1^2 + Z_1^2}\sqrt{X_2^2 + Y_2^2 + Z_2^2}}.$$

576. Commutative Property

$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

577. Associative Property

$(\lambda \vec{u}) \cdot (\mu \vec{v}) = \lambda \mu \vec{u} \cdot \vec{v}$

578. Distributive Property

$\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$

579. $\vec{u} \cdot \vec{v} = 0$ if $\vec{u}, \vec{v}$ are orthogonal ($\theta = \frac{\pi}{2}$).

580. $\vec{u} \cdot \vec{v} > 0$ if $0 < \theta < \frac{\pi}{2}$.

581. $\vec{u}\cdot\vec{v}<0$ if $\frac{\pi}{2}<\theta<\pi$.

582. $\vec{u}\cdot\vec{v}\le|\vec{u}|\cdot|\vec{v}|$

583. $\vec{u}\cdot\vec{v}=|\vec{u}|\cdot|\vec{v}|$ if $\vec{u},\vec{v}$ are parallel ($\theta=0$).

584. If $\vec{u}=(X_1, Y_1, Z_1)$, then
$\vec{u}\cdot\vec{u}=\vec{u}^2=|\vec{u}|^2=X_1^{\ 2}+Y_1^{\ 2}+Z_1^{\ 2}$.

585. $\vec{i}\cdot\vec{i}=\vec{j}\cdot\vec{j}=\vec{k}\cdot\vec{k}=1$

586. $\vec{i}\cdot\vec{j}=\vec{j}\cdot\vec{k}=\vec{k}\cdot\vec{i}=0$

6.6 Vector Product

587. Vector Product of Vectors $\vec{u}$ and $\vec{v}$
$\vec{u}\times\vec{v}=\vec{w}$, where

- $|\vec{w}|=|\vec{u}|\cdot|\vec{v}|\cdot\sin\theta$, where $0\le\theta\le\frac{\pi}{2}$;
- $\vec{w}\perp\vec{u}$ and $\vec{w}\perp\vec{v}$;
- Vectors $\vec{u}$, $\vec{v}$, $\vec{w}$ form a right-handed screw.

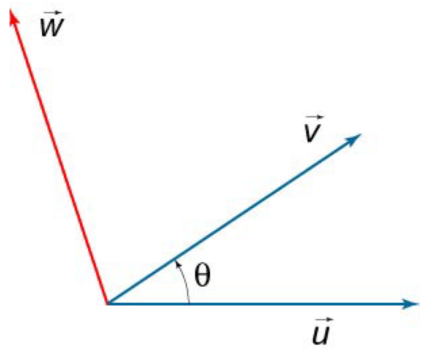


Figure 83.

588. $\vec{w} = \vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix}$

589. $\vec{w} = \vec{u} \times \vec{v} = \left(\begin{vmatrix} Y_1 & Z_1 \\ Y_2 & Z_2 \end{vmatrix}, -\begin{vmatrix} X_1 & Z_1 \\ X_2 & Z_2 \end{vmatrix}, \begin{vmatrix} X_1 & Y_1 \\ X_2 & Y_2 \end{vmatrix} \right)$

590. $S = |\vec{u} \times \vec{v}| = |\vec{u}| \cdot |\vec{v}| \cdot \sin\theta$ (Fig.83)

591. Angle Between Two Vectors (Fig.83)

$$\sin\theta = \frac{\vec{u} \times \vec{v}}{|\vec{u}| \cdot |\vec{v}|}$$

592. Noncommutative Property

$\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$

593. Associative Property

$(\lambda\vec{u}) \times (\mu\vec{v}) = \lambda\mu\vec{u} \times \vec{v}$

594. Distributive Property
$\vec{u}\times(\vec{v}+\vec{w})=\vec{u}\times\vec{v}+\vec{u}\times\vec{w}$

595. $\vec{u}\times\vec{v}=\vec{0}$ if $\vec{u}$ and $\vec{v}$ are parallel ($\theta=0$).

596. $\vec{i}\times\vec{i}=\vec{j}\times\vec{j}=\vec{k}\times\vec{k}=\vec{0}$

597. $\vec{i}\times\vec{j}=\vec{k}$, $\vec{j}\times\vec{k}=\vec{i}$, $\vec{k}\times\vec{i}=\vec{j}$

6.7 Triple Product

598. Scalar Triple Product
$[\vec{u}\vec{v}\vec{w}]=\vec{u}\cdot(\vec{v}\times\vec{w})=\vec{v}\cdot(\vec{w}\times\vec{u})=\vec{w}\cdot(\vec{u}\times\vec{v})$

599. $[\vec{u}\vec{v}\vec{w}]=[\vec{w}\vec{u}\vec{v}]=[\vec{v}\vec{w}\vec{u}]=-[\vec{v}\vec{u}\vec{w}]=-[\vec{w}\vec{v}\vec{u}]=-[\vec{u}\vec{w}\vec{v}]$

600. $k\vec{u}\cdot(\vec{v}\times\vec{w})=k[\vec{u}\vec{v}\vec{w}]$

601. Scalar Triple Product in Coordinate Form

$$\vec{u}\cdot(\vec{v}\times\vec{w})=\begin{vmatrix} X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \\ X_3 & Y_3 & Z_3 \end{vmatrix},$$

where
$\vec{u}=(X_1,Y_1,Z_1)$, $\vec{v}=(X_2,Y_2,Z_2)$, $\vec{w}=(X_3,Y_3,Z_3)$.

602. Volume of Parallelepiped
$V=|\vec{u}\cdot(\vec{v}\times\vec{w})|$

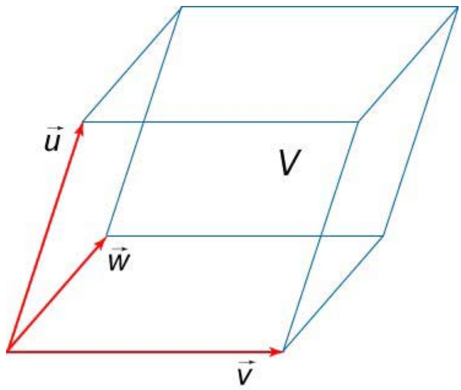


Figure 84.

603. Volume of Pyramid

$$V = \frac{1}{6}\left|\vec{u}\cdot\left(\vec{v}\times\vec{w}\right)\right|$$

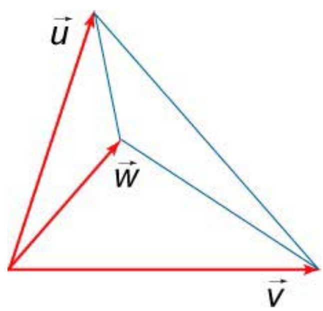


Figure 85.

604. If $\vec{u}\cdot\left(\vec{v}\times\vec{w}\right)=0$, then the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ are linearly dependent, so $\vec{w}=\lambda\vec{u}+\mu\vec{v}$ for some scalars λ and μ.

605. If $\vec{u}\cdot\left(\vec{v}\times\vec{w}\right)\neq 0$, then the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ are linearly independent.

606. Vector Triple Product

$$\vec{u}\times(\vec{v}\times\vec{w})=(\vec{u}\cdot\vec{w})\vec{v}-(\vec{u}\cdot\vec{v})\vec{w}$$

Chapter 7

Analytic Geometry

7.1 One-Dimensional Coordinate System

Point coordinates: x_0, x_1, x_2, y_0, y_1, y_2
Real number: λ
Distance between two points: d

607. Distance Between Two Points
$d = AB = |x_2 - x_1| = |x_1 - x_2|$

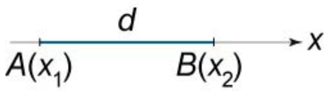


Figure 86.

608. Dividing a Line Segment in the Ratio λ
$x_0 = \dfrac{x_1 + \lambda x_2}{1+\lambda}$, $\lambda = \dfrac{AC}{CB}$, $\lambda \neq -1$.

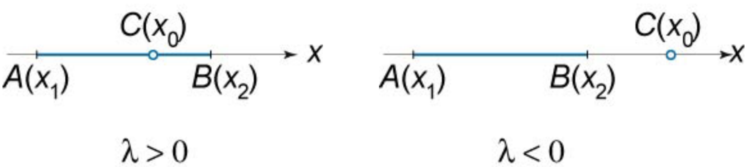


Figure 87.

609. Midpoint of a Line Segment

$$x_0 = \frac{x_1 + x_2}{2}, \ \lambda = 1.$$

7.2 Two-Dimensional Coordinate System

Point coordinates: $x_0, x_1, x_2, y_0, y_1, y_2$
Polar coordinates: r, φ
Real number: λ
Positive real numbers: a, b, c,
Distance between two points: d
Area: S

610. Distance Between Two Points

$$d = AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

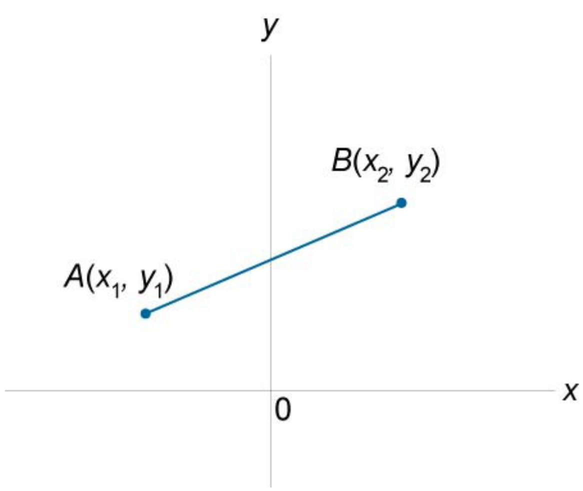


Figure 88.

611. **Dividing a Line Segment in the Ratio λ**

$$x_0 = \frac{x_1 + \lambda x_2}{1+\lambda},\ y_0 = \frac{y_1 + \lambda y_2}{1+\lambda},$$

$$\lambda = \frac{AC}{CB},\ \lambda \neq -1.$$

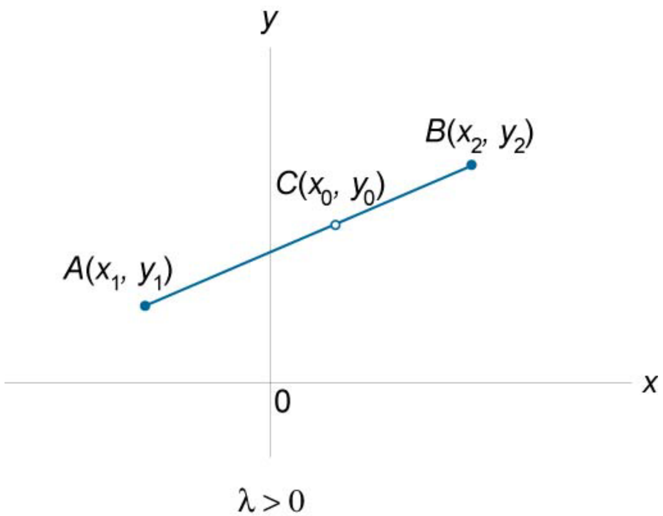
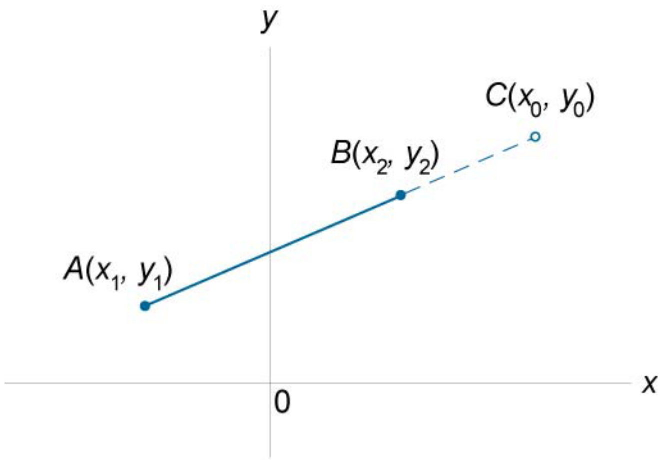


Figure 89.



$\lambda < 0$

Figure 90.

612. Midpoint of a Line Segment

$$x_0 = \frac{x_1 + x_2}{2},\ y_0 = \frac{y_1 + y_2}{2},\ \lambda = 1.$$

613. Centroid (Intersection of Medians) of a Triangle

$$x_0 = \frac{x_1 + x_2 + x_3}{3},\ y_0 = \frac{y_1 + y_2 + y_3}{3},$$

where $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$ are vertices of the triangle ABC.

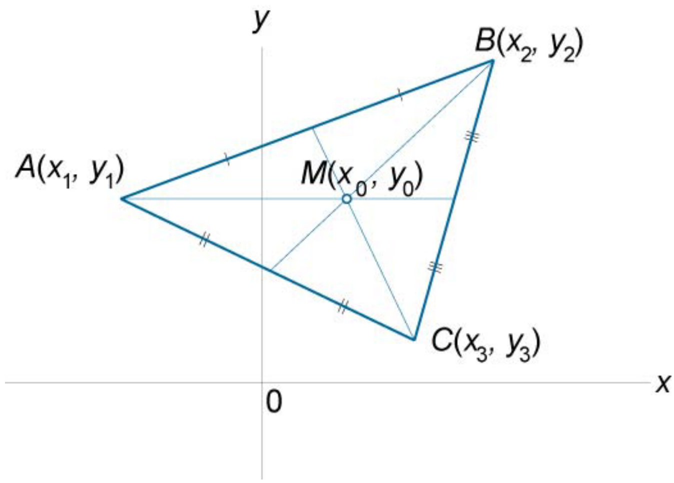


Figure 91.

614. Incenter (Intersection of Angle Bisectors) of a Triangle

$$x_0 = \frac{ax_1 + bx_2 + cx_3}{a + b + c}, \quad y_0 = \frac{ay_1 + by_2 + cy_3}{a + b + c},$$

where $a = BC$, $b = CA$, $c = AB$.

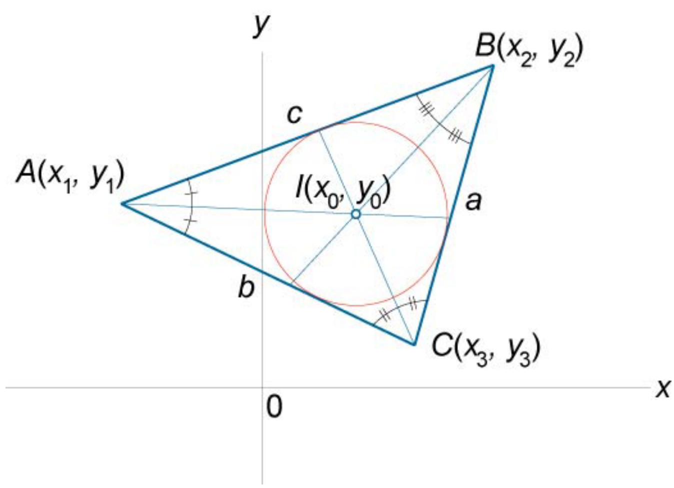


Figure 92.

615. **Circumcenter (Intersection of the Side Perpendicular Bisectors) of a Triangle**

$$x_0 = \frac{\begin{vmatrix} x_1^2+y_1^2 & y_1 & 1 \\ x_2^2+y_2^2 & y_2 & 1 \\ x_3^2+y_3^2 & y_3 & 1 \end{vmatrix}}{2\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}}, \quad y_0 = \frac{\begin{vmatrix} x_1 & x_1^2+y_1^2 & 1 \\ x_2 & x_2^2+y_2^2 & 1 \\ x_3 & x_3^2+y_3^2 & 1 \end{vmatrix}}{2\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}}$$

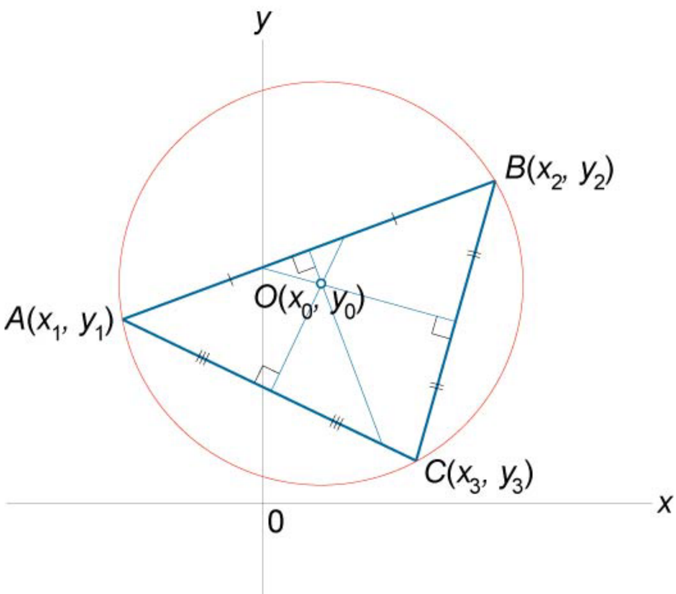


Figure 93.

616. Orthocenter (Intersection of Altitudes) of a Triangle

$$x_0 = \frac{\begin{vmatrix} y_1 & x_2x_3 + y_1^2 & 1 \\ y_2 & x_3x_1 + y_2^2 & 1 \\ y_3 & x_1x_2 + y_3^2 & 1 \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}}, \quad y_0 = \frac{\begin{vmatrix} x_1^2 + y_2y_3 & x_1 & 1 \\ x_2^2 + y_3y_1 & x_2 & 1 \\ x_3^2 + y_1y_2 & x_3 & 1 \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}}$$

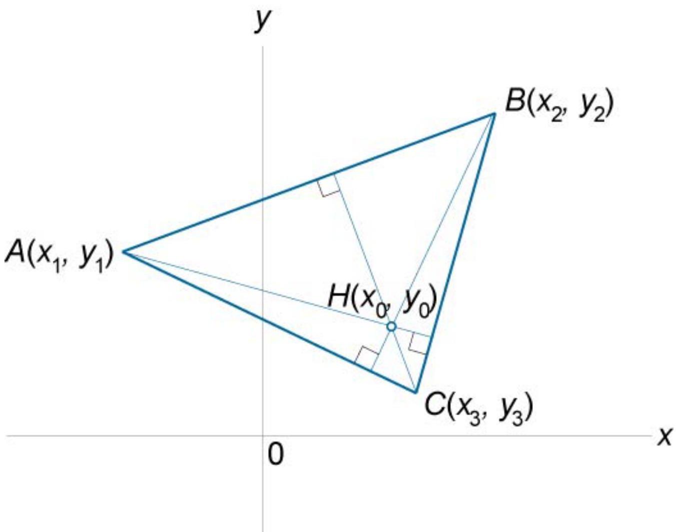


Figure 94.

617. Area of a Triangle

$$S = (\pm)\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = (\pm)\frac{1}{2}\begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix}$$

618. Area of a Quadrilateral

$$S=(\pm)\frac{1}{2}[(x_1-x_2)(y_1+y_2)+(x_2-x_3)(y_2+y_3)+$$
$$+(x_3-x_4)(y_3+y_4)+(x_4-x_1)(y_4+y_1)]$$

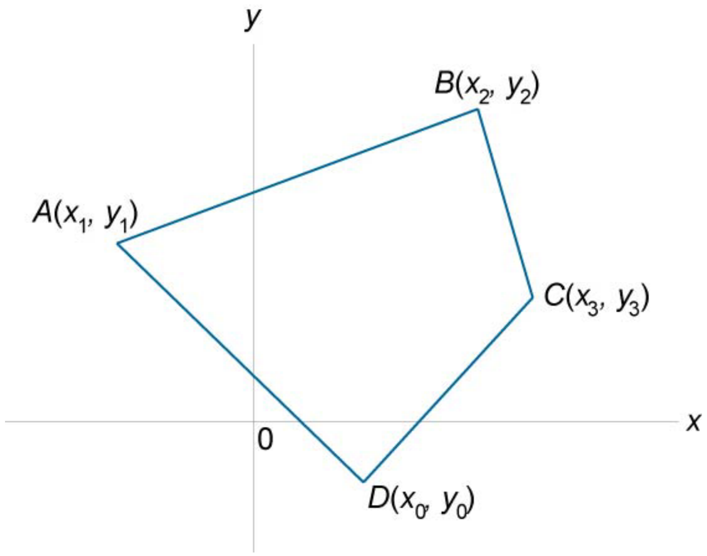


Figure 95.

Note: In formulas 617, 618 we choose the sign (+) or (–) so that to get a positive answer for area.

619. Distance Between Two Points in Polar Coordinates

$$d=AB=\sqrt{r_1^2+r_2^2-2r_1r_2\cos(\varphi_2-\varphi_1)}$$

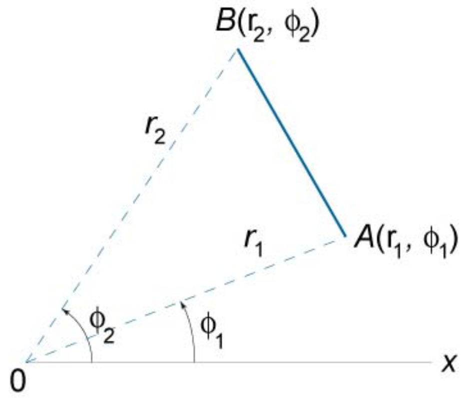


Figure 96.

620. Converting Rectangular Coordinates to Polar Coordinates

$x = r\cos\varphi$, $y = r\sin\varphi$.

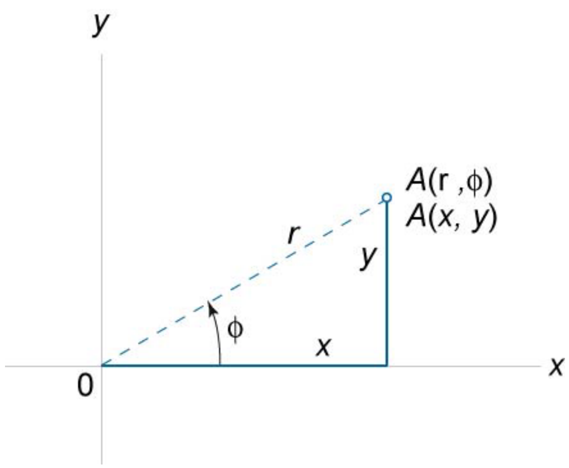


Figure 97.

621. Converting Polar Coordinates to Rectangular Coordinates

$r = \sqrt{x^2 + y^2}$, $\tan\varphi = \dfrac{y}{x}$.

7.3 Straight Line in Plane

Point coordinates: X, Y, x, x_0, x_1, y_0, y_1, a_1, a_2, ...
Real numbers: k, a, b, p, t, A, B, C, A_1, A_2, ...
Angles: α, β
Angle between two lines: φ
Normal vector: $\vec{n}$
Position vectors: $\vec{r}$, $\vec{a}$, $\vec{b}$

622. General Equation of a Straight Line
$Ax + By + C = 0$

623. Normal Vector to a Straight Line
The vector $\vec{n}(A, B)$ is normal to the line $Ax + By + C = 0$.

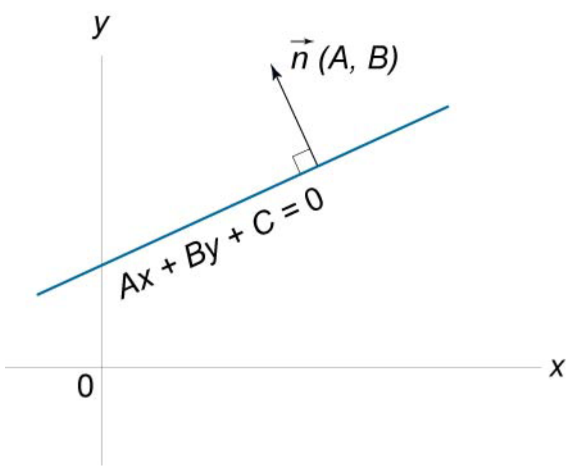


Figure 98.

624. Explicit Equation of a Straight Line (Slope-Intercept Form)
$y = kx + b$.

The gradient of the line is $k = \tan\alpha$.

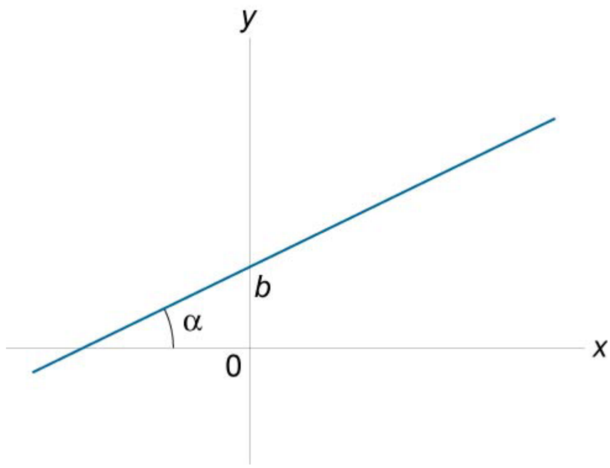


Figure 99.

625. Gradient of a Line

$$k = \tan\alpha = \frac{y_2 - y_1}{x_2 - x_1}$$

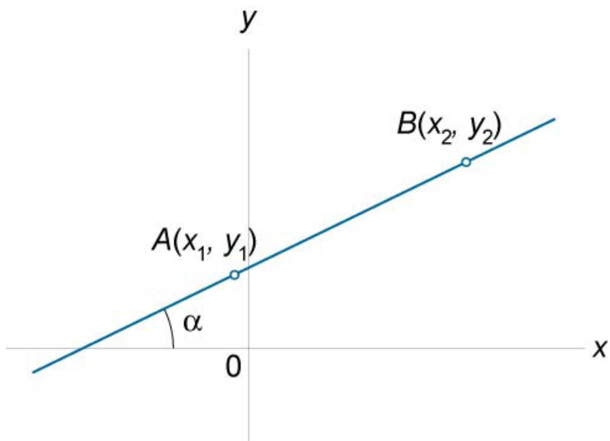


Figure 100.

626. Equation of a Line Given a Point and the Gradient

$$y = y_0 + k(x - x_0),$$

where k is the gradient, $P(x_0, y_0)$ is a point on the line.

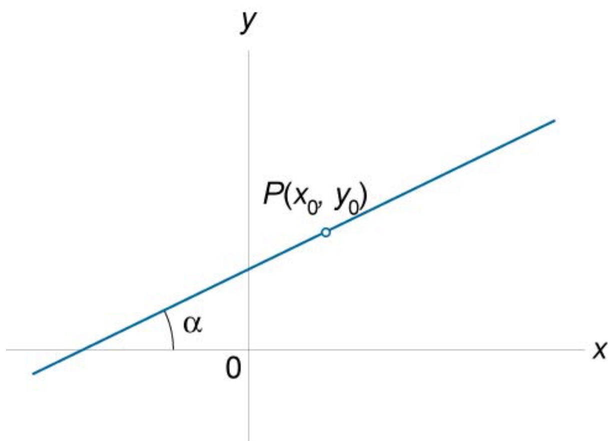


Figure 101.

627. Equation of a Line That Passes Through Two Points

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

or

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$$

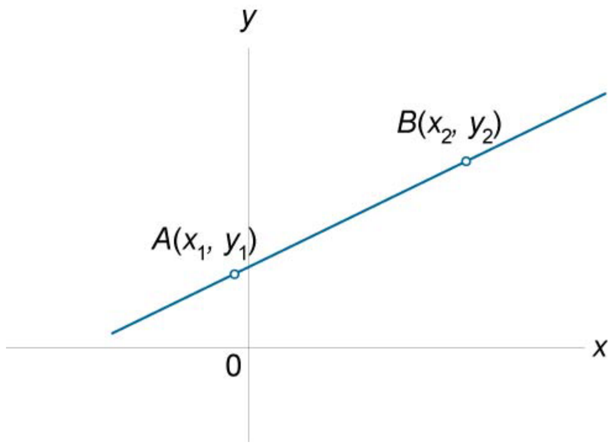


Figure 102.

628. Intercept Form

$$\frac{x}{a}+\frac{y}{b}=1$$

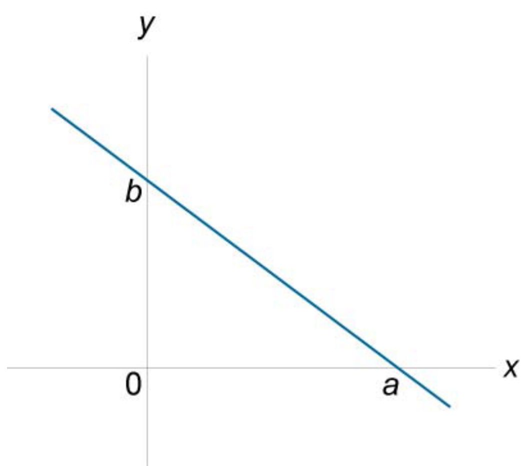


Figure 103.

629. Normal Form

$$x \cos\beta + y \sin\beta - p = 0$$

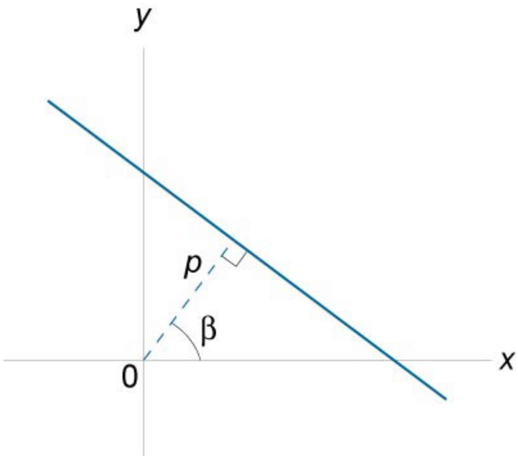


Figure 104.

630. Point Direction Form

$$\frac{x - x_1}{X} = \frac{y - y_1}{Y},$$

where (X, Y) is the direction of the line and $P_1(x_1, y_1)$ lies on the line.

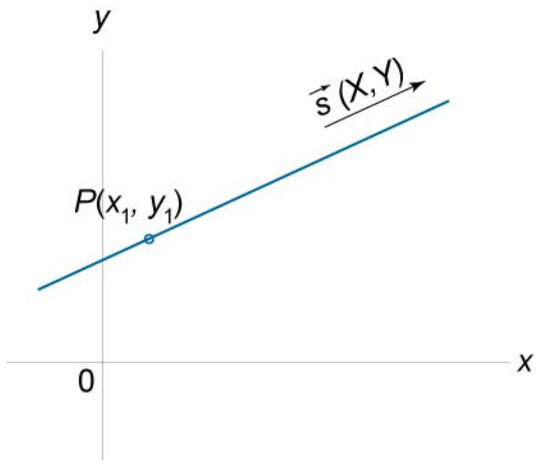


Figure 105.

631. Vertical Line
$x = a$

632. Horizontal Line
$y = b$

633. Vector Equation of a Straight Line
$\vec{r} = \vec{a} + t\vec{b}$,
where
O is the origin of the coordinates,
X is any variable point on the line,
$\vec{a}$ is the position vector of a known point A on the line ,
$\vec{b}$ is a known vector of direction, parallel to the line,
t is a parameter,
$\vec{r} = \overrightarrow{OX}$ is the position vector of any point X on the line.

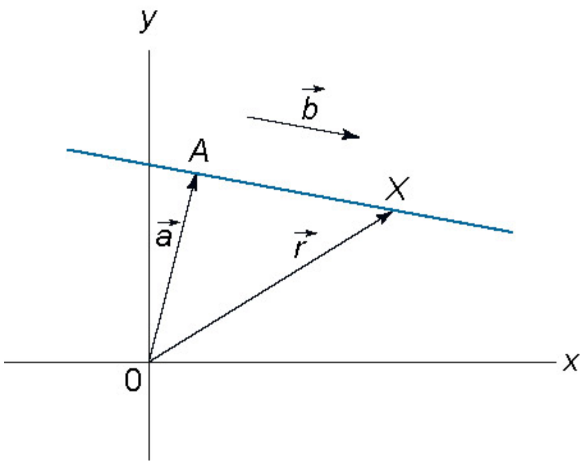


Figure 106.

634. Straight Line in Parametric Form

$$\begin{cases} x = a_1 + tb_1 \\ y = a_2 + tb_2 \end{cases},$$

where

(x, y) are the coordinates of any unknown point on the line,

(a_1, a_2) are the coordinates of a known point on the line,

(b_1, b_2) are the coordinates of a vector parallel to the line,

t is a parameter.

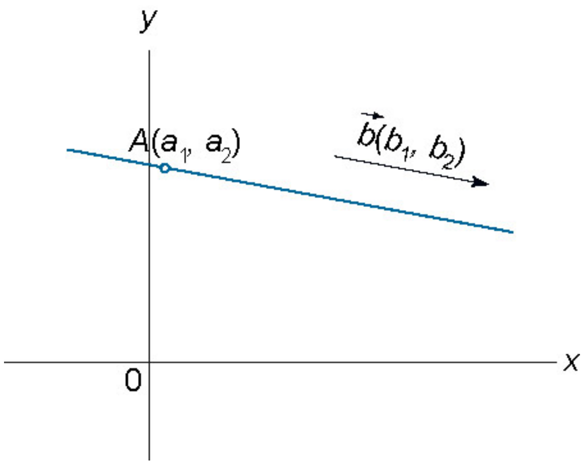


Figure 107.

635. Distance From a Point To a Line
The distance from the point $P(a, b)$ to the line $Ax + By + C = 0$ is

$$d = \frac{|Aa + Bb + C|}{\sqrt{A^2 + B^2}}.$$

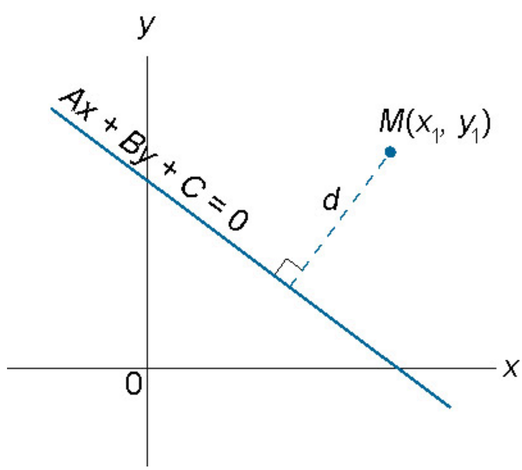


Figure 108.

636. Parallel Lines
Two lines $y = k_1 x + b_1$ and $y = k_2 x + b_2$ are parallel if
$k_1 = k_2$.
Two lines $A_1 x + B_1 y + C_1 = 0$ and $A_2 x + B_2 y + C_2 = 0$ are parallel if
$\frac{A_1}{A_2} = \frac{B_1}{B_2}$.

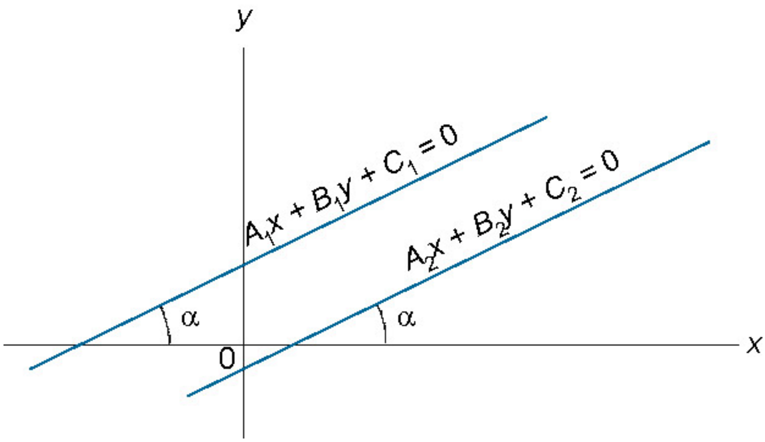


Figure 109.

637. Perpendicular Lines
Two lines $y = k_1 x + b_1$ and $y = k_2 x + b_2$ are perpendicular if
$k_2 = -\frac{1}{k_1}$ or, equivalently, $k_1 k_2 = -1$.
Two lines $A_1 x + B_1 y + C_1 = 0$ and $A_2 x + B_2 y + C_2 = 0$ are perpendicular if
$A_1 A_2 + B_1 B_2 = 0$.

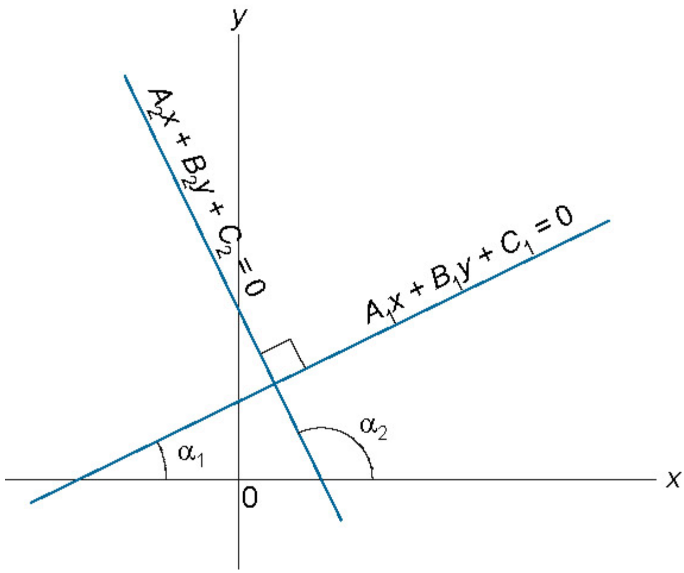


Figure 110.

638. **Angle Between Two Lines**

$$\tan\varphi = \frac{k_2 - k_1}{1 + k_1 k_2},$$

$$\cos\varphi = \frac{A_1A_2 + B_1B_2}{\sqrt{A_1^2 + B_1^2} \cdot \sqrt{A_2^2 + B_2^2}}.$$

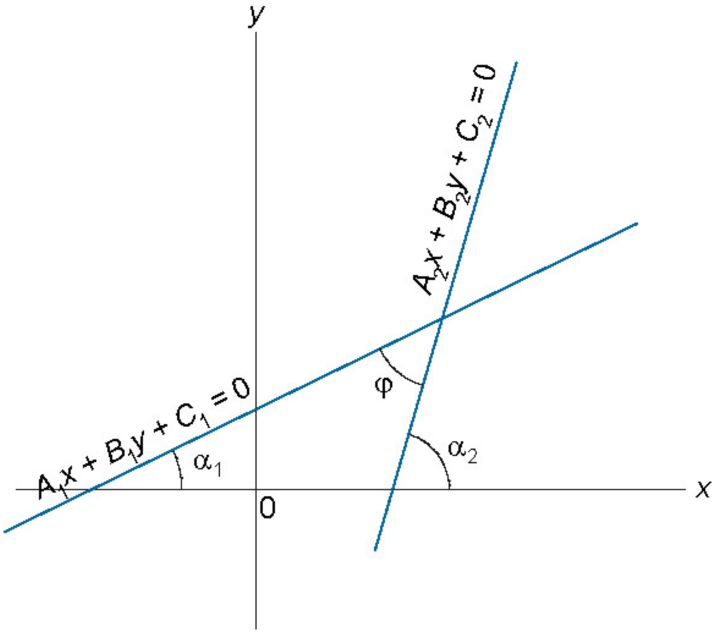


Figure 111.

639. Intersection of Two Lines

If two lines $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$ intersect, the intersection point has coordinates

$$x_0 = \frac{-C_1B_2 + C_2B_1}{A_1B_2 - A_2B_1}, \quad y_0 = \frac{-A_1C_2 + A_2C_1}{A_1B_2 - A_2B_1}.$$

7.4 Circle

Radius: R

Center of circle: (a, b)

Point coordinates: x, y, x_1, y_1, …

Real numbers: A, B, C, D, E, F, t

640. Equation of a Circle Centered at the Origin (Standard Form)

$x^2 + y^2 = R^2$

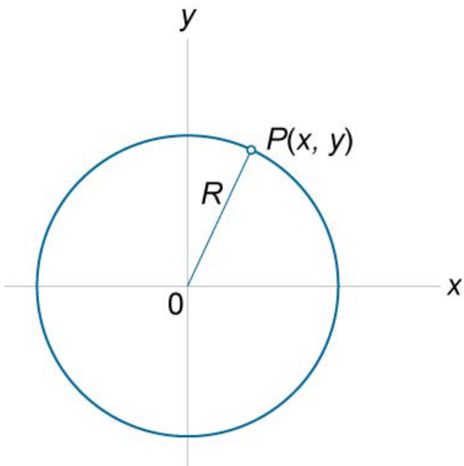


Figure 112.

641. Equation of a Circle Centered at Any Point (a, b)

$(x - a)^2 + (y - b)^2 = R^2$

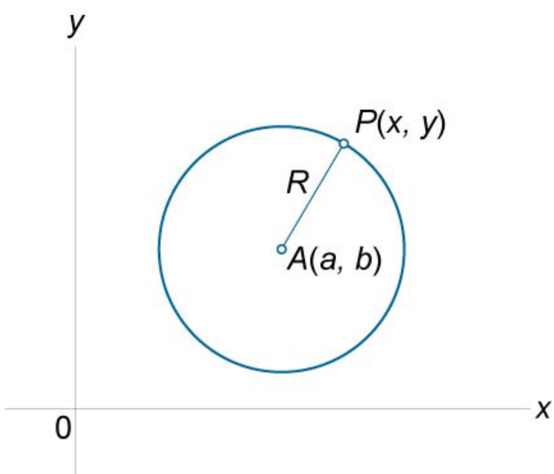


Figure 113.

642. Three Point Form

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

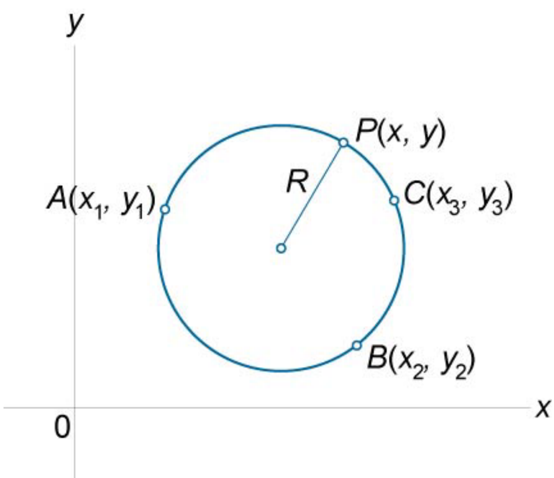


Figure 114.

643. Parametric Form

$$\begin{cases} x = R\cos t \\ y = R\sin t \end{cases}, \quad 0 \le t \le 2\pi .$$

644. General Form

$Ax^2 + Ay^2 + Dx + Ey + F = 0$ (A nonzero, $D^2 + E^2 > 4AF$).

The center of the circle has coordinates (a, b), where

$$a = -\frac{D}{2A}, \quad b = -\frac{E}{2A}.$$

The radius of the circle is

$$R = \sqrt{\frac{D^2 + E^2 - 4AF}{2|A|}}.$$

7.5 Ellipse

Semimajor axis: a
Semiminor axis: b
Foci: $F_1(-c, 0)$, $F_2(c, 0)$
Distance between the foci: 2c
Eccentricity: e
Real numbers: A, B, C, D, E, F, t
Perimeter: L
Area: S

645. Equation of an Ellipse (Standard Form)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

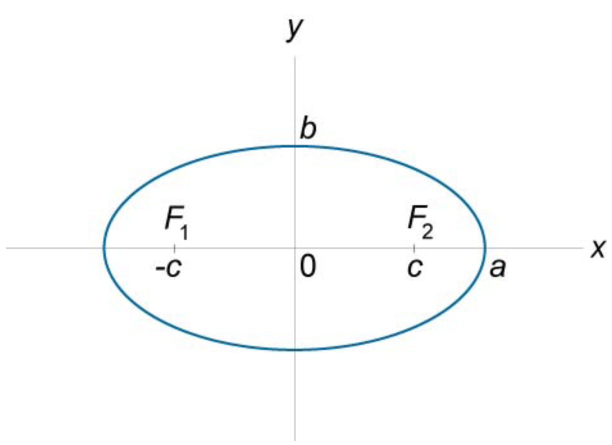


Figure 115.

646. $r_1 + r_2 = 2a$,
where r_1, r_2 are distances from any point $P(x, y)$ on the ellipse to the two foci.

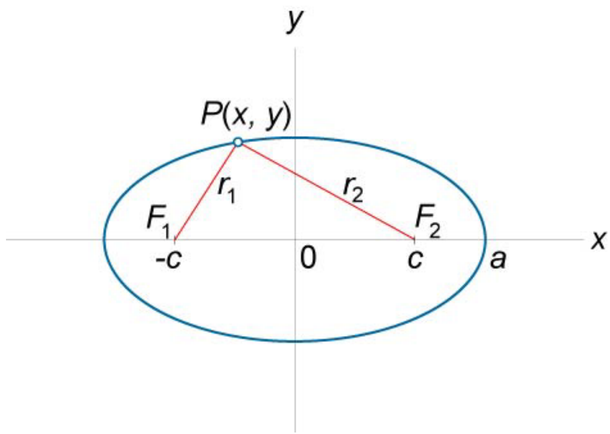


Figure 116.

647. $a^2 = b^2 + c^2$

648. Eccentricity

$$e = \frac{c}{a} < 1$$

649. Equations of Directrices

$$x = \pm\frac{a}{e} = \pm\frac{a^2}{c}$$

650. Parametric Form

$$\begin{cases} x = a\cos t \\ y = b\sin t \end{cases}, \; 0 \le t \le 2\pi.$$

651. General Form
$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$,
where $B^2 - 4AC < 0$.

652. General Form with Axes Parallel to the Coordinate Axes
$Ax^2 + Cy^2 + Dx + Ey + F = 0$,
where $AC > 0$.

653. Circumference
$L = 4aE(e)$,
where the function E is the complete elliptic integral of the second kind.

654. Approximate Formulas of the Circumference
$L = \pi\left(1.5(a+b) - \sqrt{ab}\right)$,
$L = \pi\sqrt{2(a^2 + b^2)}$.

655. $S = \pi ab$

7.6 Hyperbola

Transverse axis: a
Conjugate axis: b
Foci: $F_1(-c, 0)$, $F_2(c, 0)$
Distance between the foci: 2c
Eccentricity: e
Asymptotes: s, t
Real numbers: A, B, C, D, E, F, t, k

656. Equation of a Hyperbola (Standard Form)

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

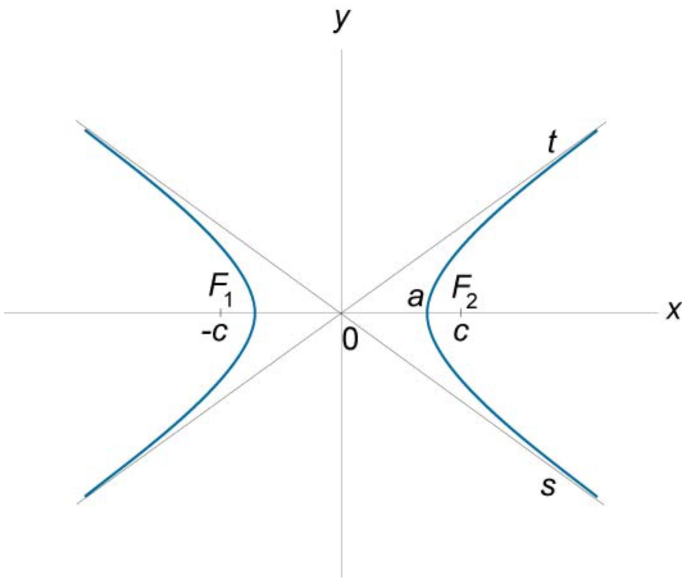


Figure 117.

657. $|r_1 - r_2| = 2a$,

where r_1, r_2 are distances from any point $P(x, y)$ on the hyperbola to the two foci.

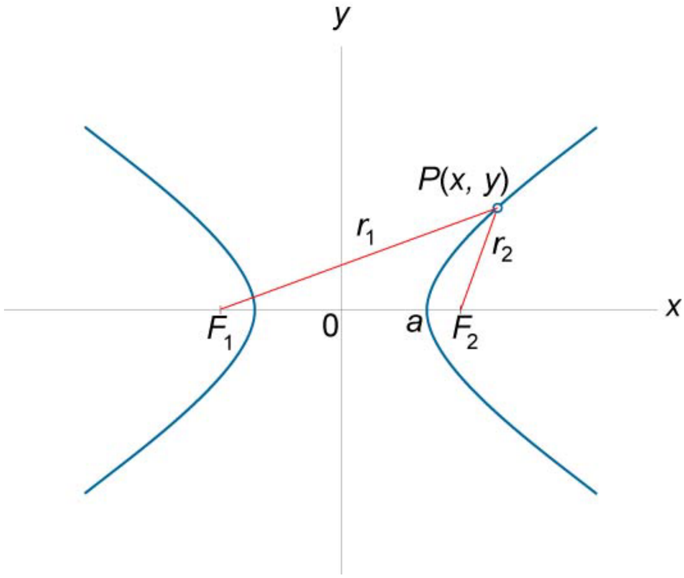


Figure 118.

658. Equations of Asymptotes

$$y = \pm \frac{b}{a} x$$

659. $c^2 = a^2 + b^2$

660. Eccentricity

$$e = \frac{c}{a} > 1$$

661. Equations of Directrices

$$x = \pm \frac{a}{e} = \pm \frac{a^2}{c}$$

662. Parametric Equations of the Right Branch of a Hyperbola

$$\begin{cases} x = a\cosh t \\ y = b\sinh t \end{cases}, \quad 0 \le t \le 2\pi .$$

663. General Form

$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$,

where $B^2 - 4AC > 0$.

664. General Form with Axes Parallel to the Coordinate Axes

$Ax^2 + Cy^2 + Dx + Ey + F = 0$,

where $AC < 0$.

665. Asymptotic Form

$$xy = \frac{e^2}{4},$$

or

$$y = \frac{k}{x}, \text{ where } k = \frac{e^2}{4}.$$

In this case , the asymptotes have equations $x = 0$ and $y = 0$.

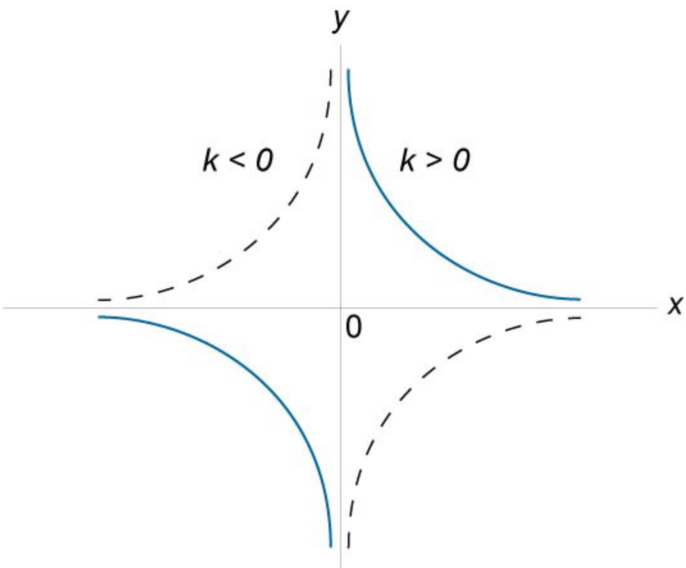


Figure 119.

7.7 Parabola

Focal parameter: p
Focus: F
Vertex: $M(x_0, y_0)$
Real numbers: A, B, C, D, E, F, p, a, b, c

666. Equation of a Parabola (Standard Form)
$y^2 = 2px$

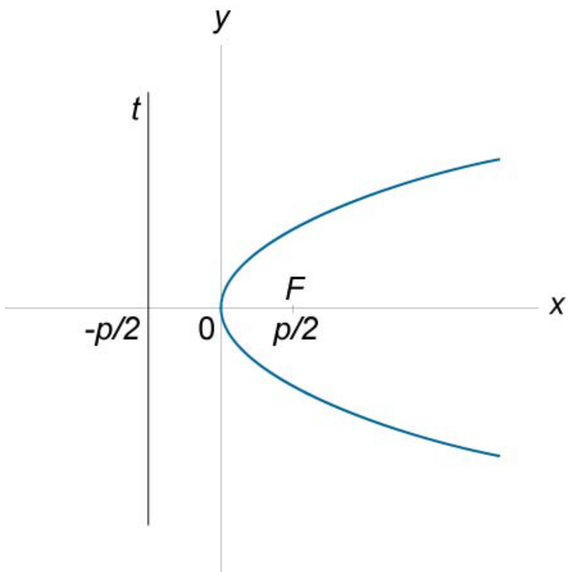


Figure 120.

Equation of the directrix

$x = -\frac{p}{2}$,

Coordinates of the focus

$F\left(\frac{p}{2}, 0\right)$,

Coordinates of the vertex

$M(0, 0)$.

667. General Form

$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$,

where $B^2 - 4AC = 0$.

668. $y = ax^2$, $p = \frac{1}{2a}$.

Equation of the directrix

$y = -\frac{p}{2}$,

Coordinates of the focus

$F\left(0, \frac{p}{2}\right)$,

Coordinates of the vertex

$M(0, 0)$.

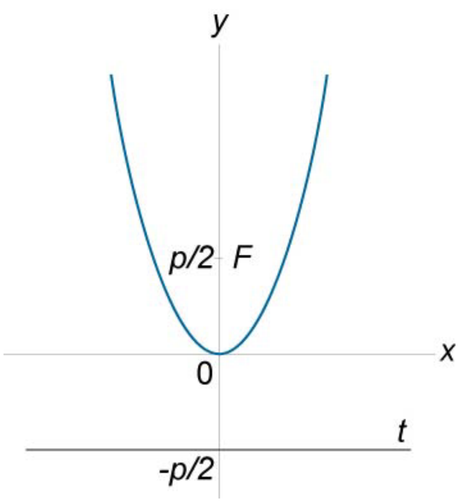


Figure 121.

669. General Form, Axis Parallel to the y-axis

$Ax^2 + Dx + Ey + F = 0$ (A, E nonzero),

$y = ax^2 + bx + c$, $p = \frac{1}{2a}$.

Equation of the directrix

$y = y_0 - \frac{p}{2}$,

Coordinates of the focus