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2005 – ST

Test Paper Code : ST

Time : 3 Hours **Max. Marks : 300**

INSTRUCTIONS

1. The question-cum-answer book has **40** pages and has **29** questions. Please ensure that the copy of the question-cum-answer book you have received contains all the questions.
2. Write your **Roll Number, Name and the name of the Test Centre** in the appropriate space provided on the right side.
3. Write the answers to the objective questions against each Question No. in the **Answer Table for Objective Questions**, provided on page No. **9**. Do not write anything else on this page.
4. Each objective question has **4 choices** for its answer: (A), (B), (C) and (D). Only **ONE** of them is the correct answer. There will be **negative marking** for wrong answers to objective questions. The following marking scheme for objective questions shall be used :
   (a) For each objective question, you will be awarded **6 (six)** marks if you have written only the correct answer.
   (b) In case you have not written any answer for a question you will be awarded **0 (zero)** mark for that question.
   (c) In all other cases, you will be awarded **–2 (minus two)** marks for the question.
5. Answer the subjective question only in the space provided after each question.
6. Do not write more than one answer for the same question. In case you attempt a subjective question more than once, please cancel the answer(s) you consider wrong. Otherwise, the answer appearing later only will be evaluated.
7. All answers must be written in blue/black/blue-black ink only. Sketch pen, pencil or ink of any other colour should not be used.
8. All rough work should be done in the space provided and scored out finally.
9. No supplementary sheets will be provided to the candidates.
10. **Logarithmic Tables / Calculator of any kind / cellular phone / pager / electronic gadgets are not allowed.**
11. The question-cum-answer book must be returned in its entirety to the Invigilator before leaving the examination hall. Do not remove any page from this book.
12. Refer to special instruction/useful data on the reverse.

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READ THE INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

ROLL NUMBER					

Name :

Test Centre :

Do not write your Roll Number or Name anywhere else in this question-cum-answer book.

I have read all the instructions and shall abide by them.

....

Signature of the Candidate

I have verified the information filled by the Candidate above.

....

Signature of the Invigilator

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> **NOTE : Attempt ALL the 29 questions. Questions 1 – 15 (objective questions) carry <u>six</u> marks each and questions 16 – 29 (subjective questions) carry <u>fifteen</u> marks each.**
>
> **Write the answers to the objective questions ONLY in the *Answer Table for Objective Questions* provided on page 9.**

1. Let

$$P = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 1 \\ 2 & 3 & 4 & 8 & 6 & 3 \\ 2 & 4 & 6 & 7 & 10 & 3 \\ 4 & 7 & 10 & 14 & 16 & 7 \end{bmatrix}.$$

Then the rank of the matrix P is

(A) 1

(B) 2

(C) 3

(D) 4

2. Consider the following system of linear equations:

$$x + y + z = 3,$$
$$x + az = b,$$
$$y + 2z = 3.$$

This system has infinite number of solutions if

(A) $a = -1, \quad b = 0$

(B) $a = 1, \quad b = 2$

(C) $a = 0, \quad b = 1$

(D) $a = -1, \quad b = 1$

3. Six identical fair dice are thrown independently. Let S denote the number of dice showing even numbers on their upper faces. Then the variance of the random variable S is

(A) $\frac{1}{2}$

(B) 1

(C) $\frac{3}{2}$

(D) 3

4. Let $X_1, X_2, \ldots, X_{21}$ be a random sample from a distribution having the variance 5. Let $\overline{X} = \frac{1}{21}\sum_{i=1}^{21} X_i$ and $S = \sum_{i=1}^{21}(X_i - \overline{X})^2$. Then the value of $E(S)$ is

(A) 5

(B) 100

(C) 0.25

(D) 105

5. Let X and Y be independent standard normal random variables. Then the distribution of $U = \left(\frac{X-Y}{X+Y}\right)^2$ is

(A) chi-square with 2 degrees of freedom

(B) chi-square with 1 degree of freedom

(C) F with (2,2) degrees of freedom

(D) F with (1,1) degrees of freedom

6. In three independent throws of a fair dice, let X denote the number of upper faces showing six. Then the value of $E(3-X)^2$ is

(A) $\frac{20}{3}$

(B) $\frac{2}{3}$

(C) $\frac{5}{2}$

(D) $\frac{5}{12}$

7. Let

$$P = \begin{bmatrix} 1 & 0 & 1+x & 1+x \\ 0 & 1 & 1 & 1 \\ 1 & 1+x & 0 & 1+x \\ 1 & 1+x & 1+x & 0 \end{bmatrix}.$$

Then the determinant of the matrix P is

(A) $3(x+1)^3$

(B) $3(x+1)^2$

(C) $3(x+1)$

(D) $(x+1)(2x+3)$

8. The area of the region $\left\{(x,y): 0 \le x,\ y \le 1,\ \frac{3}{4} \le x+y \le \frac{3}{2}\right\}$ is

(A) $\frac{9}{16}$

(B) $\frac{7}{16}$

(C) $\frac{13}{32}$

(D) $\frac{19}{32}$

9. Let E, F and G be three events such that the events E and F are mutually exclusive, $P(E \cup F) = 1$, $P(E \cap G) = \frac{1}{4}$ and $P(G) = \frac{7}{12}$. Then $P(F \cap G)$ equals

(A) $\frac{1}{12}$

(B) $\frac{1}{4}$

(C) $\frac{5}{12}$

(D) $\frac{1}{3}$

10. Let X and Y have the joint probability mass function

$$P(X = x, Y = y) = \frac{1}{3x}, \quad y = 1,\ 2,\ \ldots,\ x; \quad x = 1, 2, 3.$$

Then the value of the conditional expectation $E(Y \mid X = 3)$ is

(A) 1

(B) 2

(C) 1.5

(D) 2.5

11. Let X_1 and X_2 be independent random variables with respective moment generating functions

$$M_1(t) = \left(\frac{3}{4} + \frac{1}{4}e^t\right)^3 \text{ and } M_2(t) = e^{2(e^t - 1)}, \quad -\infty < t < \infty$$

Then the value of $P(X_1 + X_2 = 1)$ is

(A) $\frac{81}{64}e^{-2}$

(B) $\frac{27}{64}e^{-2}$

(C) $\frac{11}{64}e^{-2}$

(D) $\frac{27}{32}e^{-2}$

12. $\lim_{n\to\infty}\left[\frac{1}{2^{\frac{n}{2}}\,\Gamma\left(\frac{n}{2}\right)}\int_{n+\sqrt{2n}}^{\infty} e^{\frac{-t}{2}}\, t^{\frac{n}{2}-1}\,dt\right]$ equals

(A) 0.5

(B) 0

(C) 0.0228

(D) 0.1587

13. Let X_1 and X_2 be two independent random variables having the same mean θ. Suppose that $E(X_1 - \theta)^2 = 1$ and $E(X_2 - \theta)^2 = 2$. For estimating θ, consider the estimators $T_\alpha(X_1, X_2) = \alpha X_1 + (1-\alpha) X_2$, $\alpha \in [0,1]$. The value of $\alpha \in [0, 1]$, for which the variance of $T_\alpha(X_1, X_2)$ is minimum, equals

(A) $\frac{2}{3}$

(B) $\frac{1}{2}$

(C) $\frac{1}{4}$

(D) $\frac{3}{4}$

14. Let $x_1 = 3$, $x_2 = 4$, $x_3 = 3.5$, $x_4 = 2.5$ be the observed values of a random sample from the probability density function

$$f(x \mid \theta) = \frac{1}{3}\left[\frac{1}{\theta} e^{\frac{-x}{\theta}} + \frac{1}{\theta^2} e^{\frac{-x}{\theta^2}} + e^{-x}\right], \qquad x > 0, \theta > 0$$

Then the method of moments estimate (MME) of θ is

(A) 3.5

(B) 4

(C) 2.5

(D) 1

15. Let $x_1 = -2$, $x_2 = 1$, $x_3 = 3$, $x_4 = -4$ be the observed values of a random sample from the distribution having probability density function

$$f(x \mid \theta) = \frac{e^{-x}}{e^{\theta} - e^{-\theta}}, \qquad -\theta \le x \le \theta, \qquad \theta > 0.$$

Then the maximum likelihood estimate of θ is

(A) 3

(B) 0.5

(C) 4

(D) Any value between 1 and 2

A

16. Let X and Y be independent and identically distributed uniform random variables over the interval $(0, 1)$ and let $S = X + Y$. Find the probability that the quadratic equation $9x^2 + 9Sx + 1 = 0$ has no real root.

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17. Find the number of real roots of the polynomial $f(x) = x^5 + x^3 - 2x + 1$.

A

18. Consider the $n \times n$ matrix

$$P = \begin{bmatrix} \frac{2}{n+1} & \frac{1}{n+1} & \frac{1}{n+1} & \cdots\cdots & \frac{1}{n+1} \\ \frac{1}{n+1} & \frac{2}{n+1} & \frac{1}{n+1} & \cdots\cdots & \frac{1}{n+1} \\ \frac{1}{n+1} & \frac{1}{n+1} & \frac{2}{n+1} & \cdots\cdots & \frac{1}{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n+1} & \frac{1}{n+1} & \frac{1}{n+1} & \cdots\cdots & \frac{2}{n+1} \end{bmatrix}.$$

Let I be the $n \times n$ identity matrix and let J be the $n \times n$ matrix whose all entries are 1. Express the matrix P as $aI + bJ$, for suitable constants a and b. It is known that the inverse of P is of the form $cI + dJ$, for some constants c and d. Find the constants c and d in terms of n.

A

19. Let

$$f(x) = (x+1)\,|\,x^2 - 1\,|, \quad -\infty < x < \infty.$$

Verify the differentiability of the function $f(\cdot)$ at the points $x = -1$ and $x = 1$.

20. There are four urns labeled U_1, U_2, U_3 and U_4, each containing 3 blue and 5 red balls. The fifth urn, labeled U_5, contains 4 blue and 4 red balls. An urn is selected at random from these five urns and a ball is drawn at random from it. Given that the selected ball is red, find the probability that it came from the urn U_5.

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21. Let

$$F(x) = \begin{cases} 0, & \text{if} \quad x < 0 \\ \dfrac{x^2}{10}, & \text{if} \quad 0 \le x < 1 \\ \dfrac{x+2}{8}, & \text{if} \quad 1 \le x < 2 \\ \dfrac{c(6x - x^2 - 1)}{2}, & \text{if} \quad 2 \le x \le 3 \\ 1, & \text{if} \quad x > 3. \end{cases}$$

Find the value of c for which $F(\cdot)$ is a cumulative distribution function of a random variable X. Also evaluate $P(1 \le X < 2)$.

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22. Let A, B and C be pair-wise independent events such that

$$P(A \cap B) = 0 \cdot 1 \quad \text{and} \quad P(B \cap C) = 0 \cdot 2.$$

Show that $P(A^c \cup C) \geq \dfrac{7}{8}$.

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23. Let the random variables X and Y have the joint probability density function

$$f(x,y) = \begin{cases} ce^{-(x+y)}, & y > x > 0 \\ 0, & \text{otherwise} \end{cases}$$

Evaluate c and $E(Y \mid X = 2)$.

A

24. Find the area of the region enclosed between the curves $y = (x - 2)^2$ and $y = 4 - x^2$.

A

25. Let $\overline{X}$ be the sample mean of a random sample of size 10 from a distribution having the probability density function

$$f(x \mid \theta) = \frac{1}{\theta} e^{\frac{-x}{\theta}}, \quad x > 0, \quad \theta > 0.$$

Using Chebyshev's inequality, show that $P(0 < \overline{X} \leq 2\theta) \geq 0 \cdot 9$.

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26. Let X be a continuous random variable having the probability density function

$$f(x) = \begin{cases} \dfrac{2}{25}(x+2), & -2 \le x \le 3 \\ 0, & \text{otherwise} \end{cases}$$

Find the cumulative distribution function of $Y = X^2$. Hence derive the expression for probability density function of Y.

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27. Let X be a single observation from the probability density function

$$f(x \mid \theta) = (\theta+1)x^{\theta}, \quad 0 < x < 1,$$

where $\theta \in \{1, 2\}$ is unknown. Find the most powerful test of level $\alpha = \frac{13}{49}$ for testing $H_0 : \theta = 1$ against $H_1 : \theta = 2$. What is the power of the test?

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28. Let $X_1, X_2, \cdots, X_n$ be a random sample from a population having the probability density function

$$f(x|\theta,\sigma)=\begin{cases}\frac{1}{\sigma}e^{\frac{-(x-\theta)}{\sigma}}, & x>\theta, \\ 0, & \text{otherwise,}\end{cases} \qquad -\infty<\theta<\infty, \quad \sigma>0.$$

Find the method of moments and maximum likelihood estimators of θ and σ.

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29. Let

$$f(x) = 3(x-2)^{\frac{2}{3}} - (x-2), \qquad 0 \le x \le 20.$$

Let x_0 and y_0 be the points of the global minima and the global maxima, respectively, of $f(\cdot)$ in the interval [0, 20]. Evaluate $f(x_0) + f(y_0)$.

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