

TEST CODE: MM 2012

SYLLABUS

Convergence and divergence of sequence and series; Cauchy sequence and completeness; Bolzano-Weierstrass theorem; continuity, uniform continuity, differentiability, directional derivatives, Jacobians, Taylor Expansion; integral calculus of one variable – existence of Riemann integral, Fundamental theorem of calculus, change of variable; elementary topological notions for metric space – open, closed and compact sets, connectedness; elements of ordinary differential equations.

Equivalence relations and partitions; vector spaces, subspaces, basis, dimension, direct sum; matrices, systems of linear equations, determinants; diagonalization, triangular forms; linear transformations and their representation as matrices; groups, subgroups, quotients, homomorphisms, products, Lagrange's theorem, Sylow's theorems; rings, ideals, maximal ideals, prime ideals, quotients, integral domains, unique factorization domains, polynomial rings; fields, algebraic extensions, separable and normal extensions, finite fields.

SAMPLE QUESTIONS

1. Let k be a field and $k[x, y]$ denote the polynomial ring in the two variables x and y with coefficients from k . Prove that for any $a, b \in k$ the ideal generated by the linear polynomials $x - a$ and $y - b$ is a maximal ideal of $k[x, y]$.
2. Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a linear transformation. Show that there is a line L such that $T(L) = L$.
3. Let $A \subseteq \mathbb{R}^n$ and $f : A \rightarrow \mathbb{R}^m$ be a uniformly continuous function. If $\{x_n\}_{n \geq 1} \subseteq A$ is a Cauchy sequence then show that $\lim_{n \rightarrow \infty} f(x_n)$ exists.
4. Let $N > 0$ and let $f : [0, 1] \rightarrow [0, 1]$ be denoted by $f(x) = 1$ if $x = 1/i$ for some integer $i \leq N$ and $f(x) = 0$ for all other values of x . Show that f is Riemann integrable.
5. Let $F : \mathbb{R}^n \rightarrow \mathbb{R}$ be defined by

$$F(x_1, x_2, \dots, x_n) = \max\{|x_1|, |x_2|, \dots, |x_n|\}.$$

Show that F is a uniformly continuous function.

6. Show that every isometry of a compact metric space into itself is onto.
7. Let $\mathbf{T} = \{z \in \mathbf{C} : |z| = 1\}$ and $f : [0, 1] \rightarrow \mathbf{C}$ be continuous with $f(0) = 0, f(1) = 2$. Show that there exists at least one t_0 in $[0, 1]$ such that $f(t_0)$ is in \mathbf{T} .

8. Let f be a continuous function on $[0, 1]$. Evaluate

$$\lim_{n \rightarrow \infty} \int_0^1 x^n f(x) dx.$$

9. Find the most general curve whose normal at each point passes through $(0, 0)$. Find the particular curve through $(2, 3)$.
10. Suppose f is a continuous function on \mathbf{R} which is periodic with period 1, that is, $f(x+1) = f(x)$ for all x . Show that
- (i) the function f is bounded above and below,
 - (ii) it achieves both its maximum and minimum and
 - (iii) it is uniformly continuous.
11. Let $A = (a_{ij})$ be an $n \times n$ matrix such that $a_{ij} = 0$ whenever $i \geq j$. Prove that A^n is the zero matrix.
12. Determine the integers n for which \mathbf{Z}_n , the set of integers modulo n , contains elements x, y so that $x + y = 2$, $2x - 3y = 3$.
13. Let a_1, b_1 be arbitrary positive real numbers. Define

$$a_{n+1} = \frac{a_n + b_n}{2}, b_{n+1} = \sqrt{a_n b_n}$$

for all $n \geq 1$. Show that a_n and b_n converge to a common limit.

14. Show that the only field automorphism of \mathbf{Q} is the identity. Using this prove that the only field automorphism of \mathbf{R} is the identity.
15. Consider a circle which is tangent to the y -axis at 0. Show that the slope at any point (x, y) satisfies $\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$.
16. Consider an $n \times n$ matrix $A = (a_{ij})$ with $a_{12} = 1, a_{ij} = 0 \forall (i, j) \neq (1, 2)$. Prove that there is no invertible matrix P such that PAP^{-1} is a diagonal matrix.
17. Let G be a nonabelian group of order 39. How many subgroups of order 3 does it have?
18. Let $n \in \mathbf{N}$, let p be a prime number and let \mathbf{Z}_{p^n} denote the ring of integers modulo p^n under addition and multiplication modulo p^n . Let $f(x)$ and $g(x)$ be polynomials with coefficients from the ring \mathbf{Z}_{p^n} such that $f(x) \cdot g(x) = 0$. Prove that $a_i b_j = 0 \forall i, j$ where a_i and b_j are the coefficients of f and g respectively.
19. Show that the fields $\mathbf{Q}(\sqrt{2})$ and $\mathbf{Q}(\sqrt{3})$ are isomorphic as \mathbf{Q} -vector spaces but not as fields.
20. Suppose $a_n \geq 0$ and $\sum a_n$ is convergent. Show that $\sum 1/(n^2 a_n)$ is divergent.