Entrance Examination : M.Sc. Mathematics, 2009

Hall Ticket Number

Time : 2 hours
Max. Marks. 75

Part A : 25 marks
Part B : 50 marks

Instructions

1. Write your Booklet Code and Hill Ticket Number in the OMR Answer Sheet given to you. Also write the Hall Ticket Number in the space provided above.
2. There is negative marking. Do not gamble.
3. Answers are to be marked on the OMR answer sheet following the instructions provided there upon.
4. Hand over the question paper booklet and the OMR answer sheet at the end of the examination.
5. No additional sheets will be provided. Rough work can be done in the question paper itself/space provided at the end of the booklet.
6. Calculators are not allowed.
7. There are a total of 50 questions in Part A and Part B together.
8. The appropriate answer should be coloured in either a blue or black ball point or sketch pen. DO NOT USE A PENCIL.

PART A

Each question carries 1 mark. 0.33 marks will be deducted for each wrong answer. There will be no penalty if the question is left unanswered.

The set of real numbers is denoted by $\mathbb{R}$, the set of complex numbers by $\mathbb{C}$, the set of rational numbers by $\mathbb{Q}$ and the set of integers by $\mathbb{Z}$.

1. Let P, Q, R be three sets such that $P \bigcup Q = P \bigcup R$ and $P \bigcap Q = P \bigcap R$. Then
   (A) $P = Q$ (B) $Q = R$
   (C) $P = R$ (D) P, Q, R are distinct sets.

2. The remainder of 3^{1989} when divided by 7 is
   (A) 1. (B) 2. (C) 5. (D) 6.

3. Let ABC be a triangle such that the altitude, angular bisector, and the median from A divides the angle $\angle BAC$ into four equal angles. Then the triangle ABC is
   (A) an equilateral triangle. (B) an isosceles triangle.
   (C) a right angled triangle. (D) none of these.

4. Let R be a rectangle having length and breadth 8 and 6 units respectively. If a circle passes through the four vertices of R, then the perimeter of the circle is
   (A) 6π. (B) 10π. (C) 14π. (D) 18π.

5. If the circles $x^2 + y^2 + 2ax + 2by = 0$ and $x^2 + y^2 + 2cx + 2dy = 0$ touch each other, then
   (A) $ad = bc$. (B) $ac = bd$. (C) $ab = cd$. (D) $a^2 + b^2 = c^2 + d^2$.

6. For two positive real numbers x and y, which of the combinations is impossible?

   (A) $x < x^y$ and $y < y^x$.
   (B) $x^y < x$ and $y^x < y$.
   (C) $x < x^y$ and $y^x < y$.
   (D) $x = x^y$ and $y = y^x$.

7. The following non-homogeneous system of linear equations

$$3x - y + 2z = a$$
$$2x - y + z = b$$
$$x + z = c$$

where a, b, c are real numbers, has
(A) solutions for all values of a, b, c.
(B) solutions for those a, b, c satisfying $a = b = c$.
(C) solutions for those a, b, c satisfying $a + b + c = 0$.
(D) solutions for those a, b, c satisfying $a - b - c = 0$.

8. A polynomial equation that the matrix $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ satisfies is
(A) $x^2 - 5x - 2 = 0$. (B) $x^2 - 5x + 2 = 0$.
(C) $x^2 + 5x - 2 = 0$. (D) $x^2 + 5x + 2 = 0$.

9. If α, β , γ are the roots of $x^3 + ax^2 + b = 0$ then the value of the determinant $\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix}$ is
(A) $a^3 - 3b$. (B) $a^2 - 3b$. (C) a^3. (D) $-a^3$.

10. Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers such that $\lim\limits_{n\to\infty} a_n = 1$ and $\lim\limits_{n\to\infty} b_n = -1$. Then the sequence $\{c_n\}$ where $c_n = a_{2n} + b_{2n+1}$, $n \in \mathbb{N}$,
(A) converges to -1. (B) converges to 0.
(C) converges to 1. (D) does not converge.

11. The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n+1} + (n+1)\sqrt{n}}$
(A) is divergent. (B) converges to 2.
(C) converges to 1. (D) converges to 1/2.

12. If f is a function defined on the set $S = \{0, 1, 2, \ldots, n, \ldots\}$ into itself such that $f(0) = 1$, and $f(n+1) = f(n) + n$ for all $n \in S$, then the value of $f(25)$ is
(A) 326. (B) 301. (C) 277. (D) 254.

13. The domain of the real valued function f defined by $f(x) = \dfrac{1}{\sqrt{|x| - x}}$ is

(A) $(-\infty, 0)$. (B) $(-\infty, 0]$. (C) $(0, \infty)$. (D) $\mathbb{R} \setminus \{0\}$.

14. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = \dfrac{1}{1 + \exp(x)}$. Then the range $f(\mathbb{R})$ of f is

(A) $\mathbb{R}$. (B) $(0, \infty)$. (C) $(0, 1)$. (D) $(1, \infty)$.

15. Let $f, g : \mathbb{R} \to \mathbb{R}$ be two functions defined by $f(x) = 1 + x - [x]$ and $g(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0 \end{cases}$ where $[x]$ is the greatest integer less than or equal to x. Then $(fog)(x)$ is equal to

(A) x. (B) $f(x)$. (C) $g(x)$. (D) 1.

16. The value of $\lim\limits_{x \to 0^+} \dfrac{\sqrt{1 + 2x} - \sqrt{1 + 3x}}{x + 2x^2}$ is

(A) $1/2$. (B) $-1/2$. (C) 0. (D) 1.

17. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and $g : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x) = \sin(x) + g(x)$, $x \in \mathbb{R}$. Then
(A) g is continuous but may not be differentiable.
(B) g is differentiable.
(C) g is differentiable and unbounded.
(D) g is not continuous.

18. Let $x = 1 + t^3$ and $y = 2t - 5$. Then $\dfrac{d^2y}{dx^2}$ is

(A) $\dfrac{4}{9t^3}$. (B) $\dfrac{-4}{9t^3}$. (C) $\dfrac{-4}{9t^5}$. (D) $\dfrac{4}{9t^5}$.

19. The value of $\displaystyle\int_0^{\pi/2} \frac{dx}{\sqrt{2} + \cos(x)}$ is

(A) π. (B) $2 \arctan(3 - 2\sqrt{2})$.
(C) $2 \arctan(2 - 3\sqrt{2})$. (D) none of the above.

20. The general solution of $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = 0$ is

(A) $c_1 + c_2 x$. (B) $c_1 x + c_2 \log(x)$.
(C) $c_1 \log(x) + c_2$. (D) $c_1 x + c_2 x^2$.

21. If a finite group G has an element of order 2 then

(A) $O(G)$ is divisible by 2.
(B) $O(G) = 2$.
(C) G is abelian and $O(G)$ is an even integer.
(D) G is cyclic and $O(G)$ is an even integer.

22. The function $f(z) = z\bar{z}$, where z is a complex variable is
(A) differentiable at each z. (B) differentiable only at 0.
(C) analytic at 0. (D) none of the above.

23. In a bag there are 10 balls numbered 1, 2, ..., 10. Two balls are taken out without replacement. The probability that the sum of numbers on them is at most 11 is
(A) less than 1/4. (B) equals to 1/4.
(C) equal to 1/2. (D) more than 1/2.

24. There are 10 girls out of 100 students in College 1, and 20 girls out of 200 students in College 2. From each college, 2 students are selected randomly to represent their respective colleges in a meeting. If p_i denotes the probability that from College i $(i = 1, 2)$ there is one girl and one boy representative, then
(A) $p_1 = p_2/2$. (B) p_1 is almost equal to p_2.
(C) $p_1 < p_2/2$. (D) $p_2/2 < p_1 \leq 3p_2/4$.

25. If all the arrangements of 4 exactly alike red balls and 6 exactly alike blue balls are equally likely then the probability that an arrangement will have 3 red balls in a row is
(A) $\binom{8}{6} / \binom{10}{6}$ (B) 8!/10! (C) 3! 6!/10 ! (D) 2! 6! /10!

PART B

Each question carries 2 marks. 0.66 marks will be deducted for a wrong answer. There will be no penalty if a question is unanswered.

26. Let X, Y be two subsets of $\mathbb{R}$ and $X + Y = \{x + y : x \in X, y \in Y\}$. If $X = X + Y$, then
(A) $Y \subseteq X$.
(B) $Y = X$.
(C) $Y = \{0\}$, the singleton set containing 0.
(D) none of the above.

27. Let $X = \{x \in \mathbb{R} : x^2 \text{is rational}\}$. Then
(A) $X = \mathbb{R}$.
(B) $X = \mathbb{Q}$.
(C) X is a countable subset of $\mathbb{R}$ and $X \neq \mathbb{Q}$.
(D) X is an uncountable subset of $\mathbb{R}$ and $X \neq \mathbb{R}$.

28. Suppose $X_1, X_2, \ldots, X_6$ are six sets each containing 5 elements and $Y_1, Y_2, \ldots, Y_k$ are k sets each containing 3 elements. Further, suppose $S = \cup_{i=1}^{6} X_i = \cup_{j=1}^{k} Y_j$. If each element of S belongs exactly to 5 X_i's and exactly to 3 Y_j's, then the value of k is
(A) 5. (B) 6. (C) 7. (D) 8

29. Let a, b, c be rational numbers such that $a + b\, 2^{1/3} + c\, 4^{1/3} = 0$. Then
(A) $a = b = c = 0$. (B) $a = 0,\ b \neq 0,\ c \neq 0$.
(C) $a \neq 0, b = 0, c \neq 0$. (D) $a \neq 0,\ b \neq 0,\ c = 0$.

30. Consider the sequence $\{x_n\}$, where $x_n = n \sin^2(\frac{1}{2} n\pi)$, $n \in \mathbb{N}$. Then
(A) $\liminf x_n = -\infty$, $\limsup x_n = +\infty$.
(B) $\liminf x_n = 0$, $\limsup x_n = +\infty$.
(C) $\liminf x_n = -\infty$, $\limsup x_n = 0$.
(D) $\liminf x_n = \limsup x_n = +\infty$.

31. If $Y = \{y \in \mathbb{R} : y \text{ is irrational}\}$ and $x \in \mathbb{R}$ such that $\inf\{|x - y| : y \in Y\} = 0$, then
(A) $x = 0$.
(B) $x = \sqrt{2}$.
(C) x is a rational number.
(D) none of the above.

32. The sequence $\{x_n\}$, where $x_n = \sin(\frac{n\pi}{3}), n \in \mathbb{N}$
(A) is convergent.
(B) diverges to $+\infty$.
(C) has a convergent subsequence.
(D) is unbounded.

33. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $\lim_{x\to 0} f(x) = \ell$. Suppose $g : \mathbb{R} \to \mathbb{R}$ is defined by $g(x) = f(2x)$. Then $\lim_{x\to 0} g(x)$
(A) exists and is equal to 2ℓ.
(B) exists and is equal to ℓ.
(C) exists and is equal to $\frac{1}{2}\ell$.
(D) does not exist.

34. If $f : \mathbb{R} \to \mathbb{R}$ is a function such that $|f(x) - f(y)| \leq M(x - y)^2$ for all $x, y \in \mathbb{R}$, $M > 0$ is a fixed constant, then
(A) f is a constant function.
(B) $f(x) = 0$ for all $x \in \mathbb{R}$.
(C) $f(x) = x$ for all $x \in \mathbb{R}$.
(D) none of the above.

35. If f is a real valued continuous function defined on $[0, 1]$ such that $f(0) = f(1)$, then which of the following statements is not correct?
(A) $f(\frac{1}{2}) - f(0)$ and $f(1) - f(\frac{1}{2})$ have opposite signs.
(B) There is a $c \in [0, \frac{1}{2}]$ such that $f(C) = f(c + \frac{1}{2})$.
(C) There are c_1, c_2 such that $0 < c_1 < c_2 < 1$ and $f(c_1) = f(c_2)$.
(D) f is a constant function.

36. Consider the polynomial $a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$, where the coefficients $a_0, a_1, \cdots, a_n$ are all integers. If a rational number $\frac{p}{q}$, where p, q have no common factor except 1, is a root of this polynomial, then

(A) p divides a_n and q divides a_0.
(B) p divides a_n but q does not divide a_0.
(C) p does not divide a_n but q divides a_0.
(D) p does not divide a_n and q does not divide a_0.

37. The interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n3^n} x^{n-1}$ is
(A) $[-3, 3]$. (B) $(-3, 3]$. (C) $[-3, 3)$. (D) $(-\infty, \infty)$.

38. In the group $\mathbb{Z}_{35}$ of all integers modulo 35
(A) only 1 is of order 35.
(B) only 1 and 2 are of order 35.
(C) there are 34 elements of order 35.
(D) there are 24 elements of order 35.

39. An example of an infinite group in which every element has a finite order is
(A) $SL(2, \mathbb{Z})$, the group of 2×2 matrices with integer entries and of determinant 1.
(B) the quotient group $\mathbb{Q}/\mathbb{Z}$.
(C) the quotient group $\mathbb{R}/\mathbb{Z}$.
(D) none of the above.

40. Consider the following two statements.

I. In $\mathbb{R}^3$ the vector $(1, 2, 3)$ belongs to the subspace spanned by $(2, 1, -1)$ and $(-1, 2, 1)$.
II. If M is an $m \times n$ matrix and P is a $k \times m$ matrix, then the row space of $N = PM$ is a subspace of the row space of M.

Then

(A) both I and II are true. (B) both I and II are false.
(C) I is false but II is true. (D) I is true but II is false.

41. Among the sets listed below, which one does not form a basis for $\mathbb{R}^3$?
(A) $\{(1, 0, 1), (2, 5, 1), (0, -4, 3)\}$.
(B) $\{(-1, 3, 1), (2, -4, -3), (-3, 8, 2)\}$.
(C) $\{(1, 2, -1), (1, 0, 2), (2, 1, 1)\}$.
(D) $\{(2, -4, 1), (0, 3, -1), (6, 0, -1)\}$.

42. The rank of the matrix $\begin{pmatrix} 1 & 2 & 0 & 1 & 1 \\ 2 & 4 & 1 & 3 & 0 \\ 3 & 6 & 2 & 5 & 1 \\ -4 & -8 & 1 & -3 & 1 \end{pmatrix}$ is

(A) 4. (B) 3. (C) 2. (D) 1.

43. Which of the following functions of a complex variable z is not bounded?

(A) $\exp(-|\mathrm{Re}\ z|)$. (B) $\exp(i\mathrm{Re}\ z)$. (C) $\exp(z)$. (D) $\exp(-|z|)$.

44. The orthogonal trajectories of the family of parabolas $y = cx^2$ is given by

(A) $\frac{x^2}{4} + \frac{y^2}{2} = \text{constant}$. (B) $\frac{x^2}{2} + \frac{y^2}{4} = \text{constant}$.

(C) $\frac{x^2}{4} - \frac{y^2}{2} = \text{constant}$. (D) $\frac{x^2}{2} - \frac{y^2}{4} = \text{constant}$.

45. General solution of the differential equation is $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = 0$ is

(A) $(c_1 + c_2 x)\, x^2$. (B) $(c_1 + c_2 \log(x))\, x^2$.
(C) $(c_1 + c_2 \exp(x))x^2$. (D) $c_1 x^2 + c_2 x^{-2}$.

46. Consider the following statements

I. In a group, if the order of an element x is 2 and the order of another element y is 3, then the order of xy is finite.

II. In an abelian group, if the order of an element x is 2 and the order of another element y is 3, then order of xy is 6.

Then

(A) both I and II are true. (B) both I and II are false.
(C) I is false but II is true. (D) I is true but II is false.

47. Two fair dice are thrown simultaneously. The probability that the product of the numbers that show up is an even number is
(A) 1/4. (B) 1/2. (C) 3/4. (D) 1.

48. In Bag 1, there are 4 slips numbered 1, 2, 3, 4. In Bag 2 also there are 4 slips numbered 1, 2, 3, 4. Two slips without replacement are taken out from each of the bags. The probability that the sum of the

numbers on the slips taken out from the bags is equal is
(A) 1/9. (B) 2/9. (C) 1/3. (D) 4/9.

49. If $\cos\alpha, \cos\beta, \cos\gamma$ are the direction cosines of a line in space, then the value of $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$ is
(A) 0. (B) 1. (C) 2. (D) none of these.

50. The equation of the plane through $(2, -3, 1)$ normal to the line joining $(3, 4, -1), (2, -1, 5)$ (in the rectangular axes) is
(A) $x + 5y + 6z - 19 = 0$.
(B) $x + 5y - 6z + 19 = 0$.
(C) $x - 5y + 6z + 19 = 0$.
(D) $x - 5y + 6z - 19 = 0$.