ADVANCED GMAT QUANT

Math Strategy Guide

This supplemental guide provides in-depth and comprehensive explanations of the advanced math skills necessary for the highest-level performance on the GMAT.

Advanced GMAT Quant Strategy Guide

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May 31st, 2011

Dear Student,

Thank you for picking up a copy of *Advanced GMAT Math*. This book is aptly titled—it's designed to provide you with the most advanced knowledge and skills you'll need to perform at the highest level on the GMAT quantitative section. Be sure that you have mastered the content found in our core strategy guides before tackling these tougher math concepts. We hope it ends up being the most challenging and rewarding GMAT quant book you'll put your hands on.

As with most accomplishments, there were many people involved in the creation of the book you're holding. First and foremost is Zeke Vanderhoek, the founder of MG Prep and Manhattan GMAT. Zeke was a lone tutor in New York when he started the Company in 2000. Now, eleven years later, the Company has Instructors and offices nationwide and contributes to the studies and successes of thousands of students each year.

Our Manhattan GMAT Strategy Guides are based on the continuing experiences of our Instructors and students. For this *Advanced GMAT Math Guide*, we are particularly indebted to Tate Shafer and Emily Sledge who drove the development of this book over a long period of time and deserve kudos for the countless hours spent creating this book. Many other instructors, including Josh Braslow, Matt Cressy, Steven Jupiter, Jad Lee, and Dave Mahler, made valuable contributions. Dan McNaney and Cathy Huang provided their design expertise to make the books as user-friendly as possible, and Liz Krisher made sure all the moving pieces came together at just the right time. Finally, many thanks to Chris Ryan. Beyond providing content additions and edits for this book (and more than a few sleepless nights), Chris continues to be the driving force behind all of our curriculum and instruction efforts. His leadership is invaluable.

At Manhattan GMAT, we continually aspire to provide the best Instructors and resources possible. We hope that you'll find our commitment manifest in this book. If you have any questions or comments, please email me at dgonzalez@manhattangmat.com. I'll look forward to reading your comments, and I'll be sure to pass them along to our curriculum team.

Thanks again, and best of luck preparing for the GMAT!

Sincerely,

Dan Gonzalez
President
Manhattan GMAT

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Your purchase includes ONLINE ACCESS to the following:

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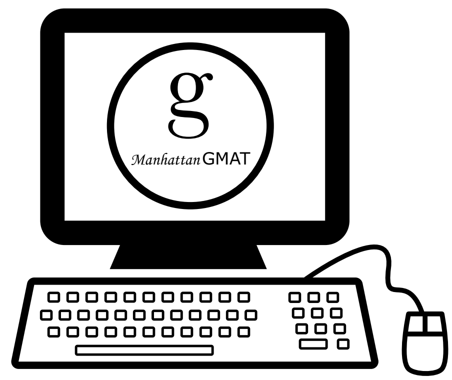
The Bonus Online Drill Sets for ADVANCED GMAT QUANT consist of extra practice questions (with detailed explanations) that test the variety of Advanced Math concepts and skills covered in this book. These questions provide you with extra practice beyond the problem sets contained in this book. You may use our online timer to practice your pacing by setting time limits for each question in the banks.



Online Updates to the Contents in this Book

The content presented in this book is updated periodically to ensure that it reflects the GMAT's most current trends. You may view all updates, including any known errors or changes, upon registering for online access.

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A Qualified Welcome

Welcome to Advanced GMAT Quant! In this venue, we decided to be a little nerdy and call the introduction "Chapter 0." After all, the point (0, 0) in the coordinate plane is called the *origin*, right? (That's the first and last math joke in this book.)

Unfortunately, we have to qualify our welcome right away, because **this book isn't for everyone**. At least, it's not for everyone *right away*.

Who Should Use This Book

You should use this book if you meet the following conditions:

- You have achieved at least 70^{th} percentile math scores on GMAT practice exams.
- You have worked through the 5 math-focused Manhattan GMAT Strategy Guides, which are organized around broad topics:
    - Number Properties
    - Fractions, Decimals, & Percents
    - Equations, Inequalities, & VICs (Algebra)
    - Word Translations
    - Geometry
- Or you have worked through similar material from another company.
- You are already comfortable with the core principles in these topics.
- You want to raise your performance to the 90^{th} percentile or higher.
- You want to become a significantly smarter test-taker.

If you match this description, then please turn the page!

If you don't match this description, then please recognize that you will probably find this book too difficult at this stage of your preparation.

For now, you are better off working on topic-focused material, such as our Strategy Guides, and ensuring that you have mastered that material *before* you return to this book.

Try Them

Take a look at the following three problems, which are very difficult. They are at least as hard as any real GMAT problem—probably even harder.

Go ahead and give these problems a try. You should not expect to solve any of them in 2 minutes. In fact, you might find yourself completely stuck. If that's the case, switch gears. Do your best to eliminate some wrong choices and take an educated guess.

Try-It #0-1

A jar is filled with red, white, and blue tokens that are equivalent except for their color. The chance of randomly selecting a red token, replacing it, then randomly selecting a white token is the same as the chance of randomly selecting a blue token. If the number of tokens of every color is a multiple of 3, what is the smallest possible total number of tokens in the jar?

(A) 9 (B) 12 (C) 15 (D) 18 (E) 21

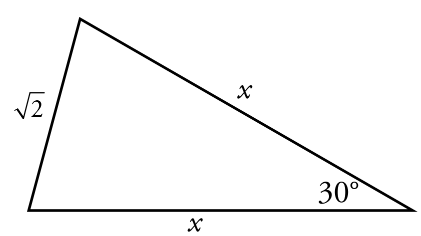
Try-It #0-2

Arrow $\overleftarrow{AB}$, which is a line segment exactly 5 units long with an arrowhead at *A*, is to be constructed in the *xy*-plane. The *x*- and *y*-coordinates of *A* and *B* are to be integers that satisfy the inequalities $0 \le x \le 9$ and $0 \le y \le 9$. How many different arrows with these properties can be constructed?

(A) 50 (B) 168 (C) 200 (D) 368 (E) 536

Try-It #0-3

In the diagram to the right, what is the value of *x*?



(A) $1+\sqrt{2}$ (B) $1+\sqrt{3}$ (C) $2\sqrt{2}$ (D) $\sqrt{2}+\sqrt{3}$ (E) $2\sqrt{3}$

(Note: this problem does not require any non-GMAT math, such as trigonometry.)

The Purpose of This Book

This book is designed to prepare you for the *most difficult* math problems on the GMAT.

So… what *is* a difficult math problem, from the point of view of the GMAT?

A difficult math problem is one that *most* GMAT test takers get wrong under exam conditions. In fact, this is essentially how the GMAT measures difficulty: by the percent of test takers who get the problem wrong.

So, what kinds of math questions do most test takers get wrong? What characterizes these problems? There are two kinds of features:

1) **Topical nuances or obscure principles**

    - Connected to a particular topic
    - Inherently hard to grasp, or simply unfamiliar
    - Easy to mix up

    These topical nuances are largely covered in the Advanced sections of the Manhattan GMAT Strategy Guides. The book you are holding includes many problems that involve topical nuances. However, the complete theory of Advanced Divisibility & Primes, for instance, is not repeated here.

2) **Complex structures**

    - Based only on simple principles but have non-obvious solution paths
    - May require multiple steps
    - May make you consider many cases
    - May combine more than one topic
    - May need a flash of real insight to complete
    - May make you change direction or switch strategies along the way

    Complex structures are essentially *disguises* for simpler content. These disguises may be difficult to pierce. The path to the answer is twisted or clouded somehow.

    To solve problems that have simple content but complex structures, **we need approaches that are both *more general* and *more creative*. This book concentrates on such approaches.**

    The three problems on the previous page have complex structures. We will return to them shortly. In the meantime, let's look at another problem.

An Illustration

Give this problem a whirl. Don't go on until you have spent a few minutes on it—or until you have figured it out!

Try-It #0-4

What should the next number in this sequence be?

1 2 9 64 ___

Note: this problem is *not* exactly GMAT-like, because there is no mathematically definite rule. However, you'll know when you've solved the problem. The answer will be elegant.

This problem has very simple content but a complex structure. Researchers in cognitive science have used sequence-completion problems such as this one to develop realistic models of human thought. Here is one such model, simplified but practical.

Top-Down Brain and Bottom-Up Brain

To solve the sequence-completion problem above, we need two kinds of thinking:

Top-Down ↓ and ↑ Bottom-Up

We can even say that we need two types of brain.

The Top-Down brain is your conscious self. If you imagine the contents of your head as a big corporation, then your Top-Down brain is the CEO, responding to input, making decisions and issuing orders. In cognitive science, the Top-Down brain is called the "executive function." Top-Down thinking and planning is indispensible to any problem-solving process.

But the corporation in your head is a big place. For one thing, how does information get to the CEO? And how pre-processed is that information?

The Bottom-Up brain is your PRE-conscious processor. After raw sensory input arrives, your Bottom-Up brain processes that input extensively *before* it reaches your Top-Down brain.

For instance, to your optic nerve, every word on this page is simply a lot of black squiggles. Your Bottom-Up brain immediately turns these squiggles into letters, joins the letters into words, summons relevant images and concepts, and finally serves these images and concepts to your Top-Down brain. This all happens automatically and swiftly. In fact, it takes effort to *interrupt* this process. Also, unlike your Top-Down brain, which does things one at a time, your Bottom-Up brain can easily do many things at once.

How does all this relate to solving the sequence problem above?

Each of your brains needs the other one to solve difficult problems.

Your Top-Down brain needs your Bottom-Up brain to notice patterns, sniff out valuable leads, and make quick, intuitive leaps and connections.

But your Bottom-Up brain is inarticulate and distractible. Only your Top-Down brain can build plans, pose explicit questions, follow procedures, and state findings.

An analogy may clarify the ideal relationship between your Top-Down and your Bottom-Up brains. Imagine that you are trying to solve a tough murder case. To find all the clues in the woods, you need both a savvy detective and a sharp-nosed bloodhound.

Your Top-Down brain is the detective. **Your Bottom-Up brain is the bloodhound.**	Be **organized**, **fast**, and **flexible** to crack the case.

To solve difficult GMAT problems, try to harmonize the activity of your two brains by following an organized, fast, and flexible problem-solving process.

Organized

You need a general step-by-step approach to guide you. One such approach, inspired by the expert mathematician George Polya, is ***Understand, Plan, Solve***:

1) ***Understand*** the problem first.

2) ***Plan*** your attack by adapting known techniques in new ways.

3) ***Solve*** by executing your plan.

You may never have thought you needed steps 1 and 2 before. It may have been easy or even automatic for you to *Understand* easier problems and to *Plan* your approach to them. As a result, you may tend to dive right into the *Solve* stage. This is a bad strategy. Mathematicians know that the real math on hard problems is not *Solve*; the real math is *Understand* and *Plan*.

Fast

Speed is important for its own sake on the GMAT, of course. What you may not have thought as much about is that being fast can also lower your stress level and promote good process. If you know you can *Solve* quickly, then you can take more time to comprehend the question, consider the given information, and select a strategy. To this end, make sure that you can rapidly complete calculations and manipulate algebraic expressions.

At the same time, avoid focusing too much on speed, especially in the early *Understand* and *Plan* stages of your problem-solving process. A little extra time invested upfront can pay off handsomely later.

Flexible

To succeed against difficult problems, you sometimes have to "unstick" yourself. Expect to run into brick walls and encounter dead ends. Returning to first principles and to the general process (e.g., making sure that you fully *Understand* the problem) can help you back up out of the mud.

Let's return to the sequence problem and play out a sample interaction between the two brains. The path is not linear; there are several dead ends, as you would expect. This dialogue will lead to the answer, so don't start reading until you've given the problem a final shot (if you haven't already solved it). The Top-Down brain is labeled TD; the Bottom-Up brain is labeled BU.

1 2 9 64 ___	TD: "Okay, let's *Understand* this thing. They've given us some numbers, and they want us to find the number that 'should' go in the blank, whatever 'should' means. What's our *Plan*? Hmm. No idea. Stare at the numbers they gave us?" *BU notices that $9 = 3^2$ and $64 = 8^2$. Likes the two squares.*
1 2 3^2 8^2 ___	TD: "Let's write in the two squares."
1 2 3^2 8^2 ___ sq no sq sq ___	*BU notices that 1 is a square too.* TD: "Are they all perfect squares? No, since 2 isn't." *BU doesn't like this break in the pattern.*
1 2 3^2 2^6 ___	TD: "Let's back up. We could try primes, factoring all the way. 8 = 2^3, and so $8^2 = (2^3)^2 = 2^6$." *BU notices $6 = 2 \times 3$, but so what.*
1 2 3^2 $(2^3)^2$ ___	TD: "Let's write 2^6 as $(2^3)^2$. Anything there?" *BU notices lots of 2's and 3's, but so what.*
1 2 3^2 $(2^3)^2$ ___	TD: "Let's keep looking at this. Are the 2's and 3's stacked somehow?" *BU notices no real pattern. There's 2–3–2 twice as you go across, but so what. And the 1 is weird by itself.*
1 2 9 64 ___ 1 7 55	TD: "No good leads there. Hmm... Let's go back to the original and try taking differences." *BU notices no pattern. The numbers look even uglier.*
1 2 9 64 ___	TD: "Hmm. No good. Go back to original numbers again. What's going on there?" *BU notices that the numbers are growing quickly, like squares or exponentials.*
1^2 2 3^2 8^2 ___	TD: "Must have something to do with those squares. Let's look at those again." *BU notices a gap on the left, among the powers.*

$1^?$	2^1	3^2	8^2	___	TD: "Let's look at 2. Write it with exponents... $2 = 2^1$. Actually, 1 doesn't have to be 1^2. 1 can be to any power and still be 1. The power is a question mark." *BU notices 2^1 then 3^2. Likes the counting numbers.* *BU really wants 1, 2, 3, 4 somehow.*
$1^?$	2^1	3^2	$4^{??}$	___	TD: "Let's try 4 in that last position. Could the last term be 4 somehow?" *BU likes the look of this. 8 and 4 are related.*
$1^?$	2^1	3^2	4^3	___	TD: "64 is 4 to the what... $4^2 = 16$, times another 4 equals 64, so it's 4 to the third power. That fits." *BU is thrilled. 1, 2, 3, 4 below and 1, 2, 3 up top.*
1^0	2^1	3^2	4^3	___	TD: "Extend left. It's 1^0. Confirmed. The bases are 1, 2, 3, 4, etc. and the powers are 0, 1, 2, 3, etc." *BU is content.*
1^0	2^1	3^2	4^3	$\underline{5^4}$	TD: "So the answer is 5^4, which is 25^2, or 625."

Your own process was almost certainly different in the details. Also, your internal dialogue was very rapid—parts of it probably only took fractions of a second to transpire. After all, you think at the speed of thought.

The important thing is to recognize how the Bottom-Up bloodhound and the Top-Down detective worked together in the case above. The TD detective set the overall agenda and then pointed the BU bloodhound at the clues. The bloodhound did practically all the "noticing," which in some sense is where all the magic happened. But sometimes the bloodhound got stuck, so the detective had to intervene, consciously trying a new path. For instance, 64 reads so strongly as 8^2 that the detective had to actively give up on that reading.

There are so many possible meaningful sequences that it wouldn't have made sense to apply a strict *recipe* from the outset: "Try X first, then Y, then Z…" Such an algorithm would require hundreds of possibilities. Should we always look for 1, 2, 3, 4? Should we *never* find differences or prime factors, because they weren't that useful here? Of course not! A computer can rapidly and easily apply a complicated algorithm with hundreds of steps, but humans can't. (If you are an engineer or programmer, maybe you *wish* you could program your own brain, but so far, that's not possible!)

What we *are* good at, though, is noticing patterns. Our Bottom-Up brain is extremely powerful—far more powerful than any computer yet built.

As we gather problem-solving tools, the task becomes **knowing when to apply which tool.** This task becomes harder as problem structures become more complex. But if we deploy our Bottom-Up bloodhound according to a general problem-solving process such as *Understand*, *Plan*, *Solve*, then we can count on the bloodhound to notice the relevant aspects of the problem—the aspects that tell us which tool to use.

Train your Top-Down brain to be:

- **Organized** in an overall approach to difficult problems,
- **Fast** at executing mechanical steps, and
- **Flexible** enough to abandon unpromising leads and try new paths.

This way, your Bottom-Up brain can do its best work. You will be able to solve problems that you might have given up on before. Speaking of which…

Giving Up Without Giving Up

This book is intended to make you smarter.

It is also intended to make you *scrappier*.

You might have failed to solve any of these Try-It problems, even with unlimited time.

The real question is this: what do you do when you run into a brick wall?

The problems in this book are designed to push you to the limit—and past. If you have traditionally been good at paper-based standardized tests, then you may be used to being able to solve practically every problem the *"textbook"* way. Problems that forced you to get down & dirty—to work backwards from the choices, to estimate and eliminate—may have annoyed you.

Well, you need to shift your thinking. As you know, the GMAT is an adaptive test. This means that if you keep getting questions right, the test will keep getting harder… and harder… and harder…

At some point, there will appear a monster problem, one that announces "I must break you." In your battle with this problem, you could lose the bigger war—whether or not you ultimately conquer this particular problem. Maybe it takes you 8 minutes, or it beats you up so badly that your head starts pounding. This will take its toll on your score.

A major purpose of this book is to help you learn to give up on the stereotypical "textbook" approach when necessary, without completely giving up on the problem. We will sometimes present a scrappy approach as a secondary method or even as the primary line of attack.

After all, the right way to deal with a monster problem can be to look for a scrappy approach right away. Or it can be to switch gears after you've looked for a "textbook" solution for a little while. Unfortunately, advanced test-takers are sometimes very stubborn. Sometimes they feel they *should* solve a problem according to some theoretical approach. Or they fail to move to Plan B or C rapidly enough, so they don't have enough gas in the tank to execute that plan. In the end, they might wind up guessing purely at random—and that's a shame.

GMAT problems often have ***backdoors***—ways to solve them that don't involve crazy amounts of computation or genius-level insights. Remember that in theory, GMAT problems can all be solved in 2 minutes. Simply by searching for the backdoor, you might avoid all the bear traps that the problem writer set out by the front door!

Plan of This Book

The rest of this book has three parts:

Part I: Question Formats	Chapter 1 – Problem Solving: Advanced Principles Chapter 2 – Problem Solving: Advanced Strategies & Guessing Tactics Chapter 3 – Data Sufficiency: Advanced Principles Chapter 4 – Data Sufficiency: Advanced Strategies & Guessing Tactics
Part II: Cross-Topic Content	Chapter 5 – Pattern Recognition Chapter 6 – Common Terms & Quadratic Templates Chapter 7 – Visual Solutions Chapter 8 – Hybrid Problems
Part III: Workouts	Workouts 1–15—15 sets of 10 difficult problems

The four chapters in Part I focus on principles, strategies, and tactics related to the two types of GMAT math problems: Problem Solving and Data Sufficiency. The next four chapters, in Part II, focus on techniques that apply across several topics but are more specific than the approaches in Part I.

Each of the 8 chapters in Part I and Part II contains

Try-It Problems, embedded throughout the text, and
In-Action Problems at the end of the chapter.

Many of these problems will be GMAT-like in format, but many will not.

Part III contains sets of GMAT-like **Workout Problems**, designed to exercise your skills as if you were taking the GMAT and seeing its hardest problems. Several of these sets contain clusters of problems relating to the chapters in Parts I and II, although the problems within each set do not all resemble each other in obvious ways. Other Workout Problem sets are mixed by both approach and topic.

Note that these problems are *not* arranged in order of difficulty! Also,x you should know that some of these problems draw on advanced content covered in the 5 Manhattan GMAT Strategy Guides devoted to math.

Solutions to Try-It Problems

If you haven't tried to solve the first three *Try-It* problems on page 14, then go back and try them now. Think about how to get your Top-Down brain and your Bottom-Up brain to work together like a detective and a bloodhound. Come back when you've tackled the problems, or at least you've tried to. Get scrappy if necessary. Be sure to take a stab at the answer and write it down.

In these solutions, we'll outline sample dialogues between the Top-Down detective and the Bottom-Up bloodhound.

Try-It #0-1

An jar is filled with red, white, and blue tokens that are equivalent except for their color. The chance of randomly selecting a red token, replacing it, then randomly selecting a white token is the same as the chance of randomly selecting a blue token. If the number of tokens of every color is a multiple of 3, what is the smallest possible total number of tokens in the jar?

(A) 9 (B) 12 (C) 15 (D) 18 (E) 21

Solution to Try-It #0-1

…**jar** is filled with **red, white, and blue tokens**… **chance** of **randomly** selecting…	TD: "Let's *Understand* this problem first. There's a jar, and it's got red, white, and blue 'tokens' in it." *BU notices "chance" and "randomly." That's probability.* TD: "All right, this is a probability problem. Now, what's the situation?" *BU notices that there are two situations.*
…**chance of** randomly selecting **a red token**, replacing it, **then** randomly selecting **a white token** is the **same** as the **chance of** randomly selecting **a blue token**.	TD: "Let's rephrase. In simpler words, if I pick a red, then a white, that's the same chance as if I pick a blue. Okay, what else?" *BU doesn't want to deal with this "multiple of 3" thing yet.*
…number of tokens of every color is a multiple of 3…	TD: "Okay, what are they asking us?"
…**smallest possible total number** of tokens in the jar?	*BU notices "smallest possible total number." Glances at answer choices. They're small, but not tiny. Hmm.*
$\frac{R}{R+W+B}\times\frac{W}{R+W+B}=\frac{B}{R+W+B}$ $\frac{RW}{(R+W+B)^2}=\frac{B}{R+W+B}$ $RW=B(R+W+B)$ $RW=BR+BW+B^2$	TD: "Let's *Plan* for a moment. How can we approach this? How about algebra—if we name the number of each color, then we can represent each fact and also what we're looking for. Okay, let's use R, W, and B. Make probability fractions. Multiply red and white fractions. Simplify algebraically." *BU is now unsure. No obvious path forward.*
…The chance of randomly selecting a red token, replacing it, then randomly selecting a white token is the same as the chance of randomly selecting a blue token…	TD: "Let's start over conceptually. Reread the problem. Can we learn anything interesting?" *BU notices that blues are different.* TD: "How are blues different? Hmm. Picking a red, then a white is as likely as picking a blue. What does that mean?" *BU notices that it's unlikely to pick a blue. So there aren't many blues, compared to reds or whites.*

Fewer blues than reds or whites. $B < R$ and $B < W$ In the very first equation above, each fraction on the left is less than 1, so their product is even smaller. The denominators of the three fractions are all the same. So the numerator of the product (B) must be smaller than either of the other numerators (R and W).	TD: "Are there fewer blues? Yes. Let's justify this. Focus on the algebraic setup." *BU notices fractions less than 1. All positive.* TD: "Two positive fractions less than 1, multiplied together, give you an even smaller number." TD: "Yes, there are fewer blues." *BU is quiet.*
… If the number of tokens of every color is a multiple of 3, what is the smallest possible total number of tokens in the jar? $B = 3$ $RW = 3R + 3W + 9$ $1 = \frac{3}{W} + \frac{3}{R} + \frac{9}{RW}$	TD: "Let's go back and reread the rest of the problem." *BU again notices "multiple of 3," also in answer choices. Small multiples.* TD: "Change of *Plan*: algebra by itself isn't getting us there. Let's plug a number in. Try the most constrained variable: *B*. Since it's the smallest quantity, but still positive, let's pretend *B* is 3. Execute this algebraically. Divide by *RW*." *BU likes just having 2 variables.*
Neither R nor W can equal 3 (since B is smaller than either). Let $R = W = 6$. (A) and (B) are out now. The smallest possible total is now 15. Is $1 = \frac{3}{6} + \frac{3}{6} + \frac{9}{36}$? No	TD: "Let's test other numbers. Apply constraints we know—*B* is the smallest number. Rule out answer choices as we go." TD: "6 and 6 don't work, because the right side adds up to larger than 1. (C) is out too. Try the next possibility."
Let $R = 6$ and $W = 9$. Is $1 = \frac{3}{6} + \frac{3}{9} + \frac{9}{54}$? Is $1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}$? Yes	*BU doesn't like breaking the symmetry between R and W. They seem to be alike.* TD: "Does it matter whether we do $R = 6$ and $W = 9$ or the other way around? No, it doesn't. One is 6, the other is 9. Plug in and go." TD: "This works. The answer is 3 + 6 + 9 = 18."

The correct answer is D.

Let's look at a scrappier pathway—one that moves more quickly to the backdoors.

Alternative Solution to Try-It #0-1

… chance of randomly selecting…	*BU notices "chance." BU doesn't like probability.* TD: "Oh man, probability. Okay, let's make sense of this and see whether there are any backdoors. That's the *Plan.*"
… the number of tokens of every color is a multiple of 3…	*BU notices that there are only limited possibilities for each number.* TD: "Okay, every quantity is a multiple of 3. That simplifies things. We have 3, 6, 9, etc. of each color."
A jar is filled with red, white, and blue tokens…	*BU is alert—what about 0?* TD: "What about 0? Hmm… The wording at the beginning assumes that there actually are tokens of each color. So we can't have 0 tokens of any kind."
(A) 9 (B) 12 (C) 15 (D) 18 (E) 21	TD: "Now let's look at the answer choices." *BU notices that they're small.*
(A) 9	TD: "Well, we can try plugging in the choices. Let's start at the easy end—in this case, the smallest number." *BU notices 9 = 3 + 3 + 3.*
Select a red: 3/9 = 1/3 Select a white: 3/9 = 1/3 1/3 × 1/3 = 1/9, which is not "select a blue" (3/9)	TD: "The only possible way to have 9 total tokens is to have 3 reds, 3 whites, and 3 blues. So… does that work? Plug into probability formula."
~~(A) 9~~ (B) 12 (C) 15 (D) 18 (E) 21 (B) 12	TD: "No, that doesn't work. This is good. Knock out A. Let's keep going. Try B." *BU notices 12 = 3 + 3 + 6.*
Select a red: 3/12 = 1/4 Select a white: 6/12 = 1/2 1/4 × 1/2 = 1/8, which is not "select a blue" (3/12)	TD: "Only way to have 12 total is 3, 3, and 6. Which one's which… we probably want blue to be one of the 3's, to make it small. Let's say red is 3 and white is 6. Let's try it."
~~(A) 9 (B) 12~~ (C) 15 (D) 18 (E) 21 (C) 15	TD: "That doesn't work either. Knock out B. Keep going." *BU notices 15 has a few options.*

Select a red: 6/15 = 2/5 Select a white: 6/15 = 2/5 2/5 × 2/5 = 4/25, which is not "select a blue" (3/15)	TD: "We can either make 15 by 3, 6, and 6 or by 3, 3, and 9. Let's try 3–6–6; make blue the 3."
3/15 × 9/15 = 3/25, which is not "select a blue" (3/15)	TD: "Nope. Let's try 3–3–9."
~~(A) 9~~ ~~(B) 12~~ ~~(C) 15~~ (D) 18 (E) 21 (D) 18	TD: "Knock out C. Try D." TD: "Let's try 3-6-9 first. Make blue the 3."
6/18 × 9/18 = 1/3 × 1/2 = 1/6 = 3/18, which IS "select a blue"	TD: "Thank goodness! Answer's D."

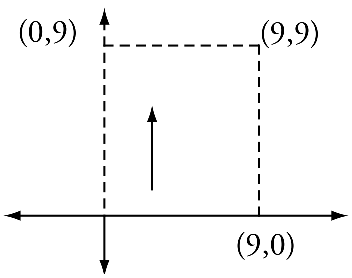
In hindsight, this second approach turned out to be less stressful and more efficient than the textbook approach. That's because in the end, there is no way to find the right answer by pure algebra. Ultimately, you have to test suitable numbers.

Try-It #0-2

Arrow $\overleftarrow{AB}$, which is a line segment exactly 5 units long with an arrowhead at *A*, is to be constructed in the *xy*-plane. The *x*- and *y*-coordinates of *A* and *B* are to be integers that satisfy the inequalities $0 \le x \le 9$ and $0 \le y \le 9$. How many different arrows with these properties can be constructed?

(A) 50 (B) 168 (C) 200 (D) 368 (E) 536

Solution to Try-It #0-2



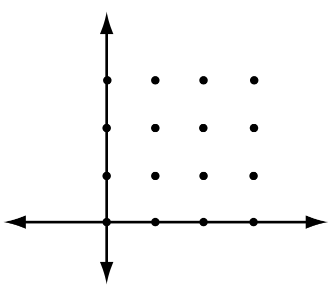
BU notices "xy-plane."
TD: "Let's *Understand* first. This is a coordinate-plane problem. Draw a quick general picture. Put in boundaries as necessary."

BU wonders where we're going.

How many different arrows with these properties can be constructed?

TD: "What are we asked for again? Reread the question."
BU wonders which properties.
TD: "What are the properties of the arrows supposed to be again? Each arrow is 5 units long."

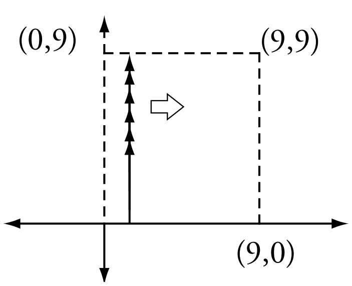
…exactly 5 units long with an arrowhead at *A*... the *x*- and *y*-coordinates of *A* and *B* are to be integers that satisfy the inequalities $0 \le x \le 9$. and $0 \le x \le 9$.



BU notices "integers" and "coordinates" and pictures a pegboard.

TD: "Both the tip and the end of the arrow have to touch holes in the pegboard exactly. Okay. The *Plan* is to start counting."

BU imagines many possible arrows. Brute force can't be the right way forward. The arrows can point in all sorts of different ways.



TD: "Let's simplify the *Plan*. Let's *focus* on just one orientation of arrows—pointing straight up. Draw this situation. How many places can the arrow be?"

BU wants to go up & down, then right & left.

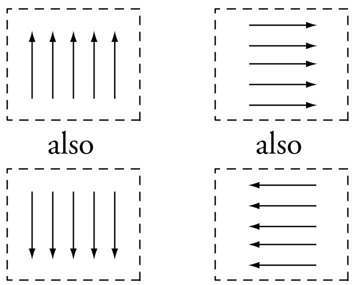
In one column, there are 5 positions for the arrowhead: $y = 5, 6, 7, 8$, or 9. That's the same as $9 - 5 + 1$, by the way.

There are 10 identical columns: $x = 0$ through $x = 9$. $5 \times 10 = 50$ possible positions for the arrow pointing straight up.

TD: "Keep simplifying the *Plan*. Count the positions in *one* column, then multiply by the number of columns. Be careful to count endpoints."

TD: "Great. We've *Solved* one part. Other possibilities?"

$50 \times 2 = 100$ possible positions for the arrow if it points straight up or to the right.



BU notices the square is the same vertically as horizontally. Go right.
TD: "We get the same result for arrows pointing right. 50 more positions. Is that it? Are we done?"

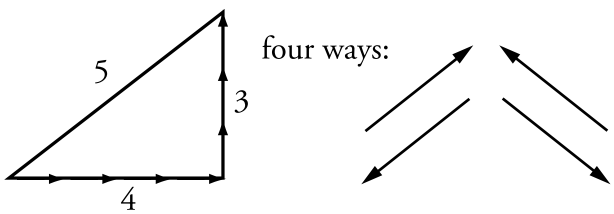
BU feels "up" and "right" are incomplete. What about "down" and "left"?
TD: "These arrows can point straight *down* or straight *left*, too. We'll get the same results. So we have 50 positions in each of the four directions. Calculate at this point and evaluate answers. Eliminate A and B."

$50 \times 4 = 200$ possible positions

Answer seems to be C.

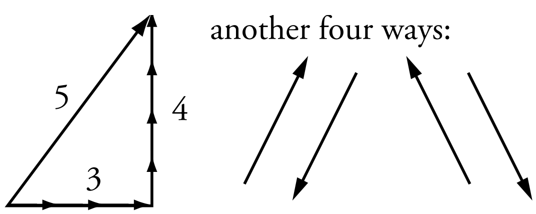
BU is suspicious: somehow too easy.
TD: "Tentative answer is C, but we're not done."
BU wonders about still other ways for the arrows to point.
TD: "Could the arrows be at an angle?"
BU notices that the arrow is 5 units long, associated with 3–4–5 triangles.

Three up, four across:



TD: "3–4–5 triangles. Yes. We can put the arrow as the hypotenuse of a 3–4–5 triangle. How can this be done? Try to place the arrow. Remember the reversal. Looks like there are four ways if we go 3 up and 4 across: up right, up left, down right, down left."

Four up, three across:



BU is happy. This is the trick.

TD: "Likewise, there must be four ways if we go 4 up and 3 across: again, up right, up left, down right, and down left. By the way, the answer must be D or E."

Three up, four across, pointing up to the right: There are 7 positions vertically for the arrowhead $(9 - 3 + 1)$ and 6 positions horizontally $(9 - 4 + 1)$, for a total of $7 \times 6 = 42$ positions.

TD: "Now count just one of these ways. Same ideas as before. Be sure to include endpoints."
BU notices the symmetry. The 3 up, 4 across is the same as the 4 up, 3 across, if you turn the square.

$8 \times 42 = 336$ possible positions at an angle.

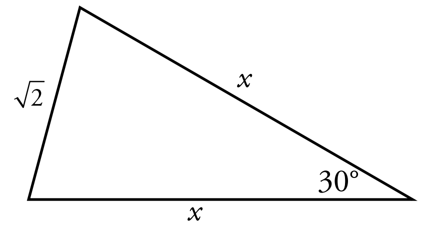
In total, we have $200 + 336 = 536$ positions.

TD: "Each of these angled ways will be the same. There are 8 ways to point the arrow at an angle. Finish the calculation and confirm the answer."

The correct answer is E. There isn't much of an alternative to the approach above. With counting problems, it can often be very difficult to estimate the answer or work backwards from the answer choices.

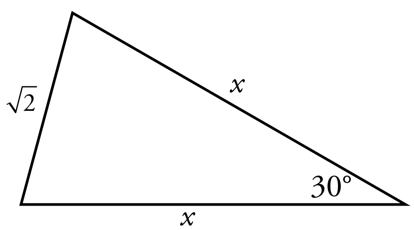
Try-It #0-3.

In the diagram to the right, what is the value of x?



(A) $1+\sqrt{2}$ (B) $1+\sqrt{3}$ (C) $2\sqrt{2}$ (D) $\sqrt{2}+\sqrt{3}$ (E) $2\sqrt{3}$

Solution to Try-It #0-3

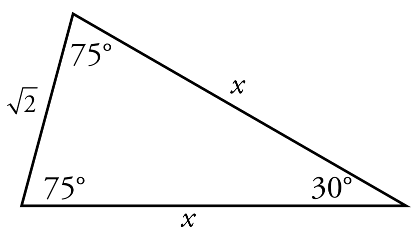


TD: "Okay, let's *Understand* this. Redraw the figure. We're looking for x. Now… how about a *Plan*?"

BU notices this is an isosceles triangle, because there are two sides labeled x. How about the two equal angles?

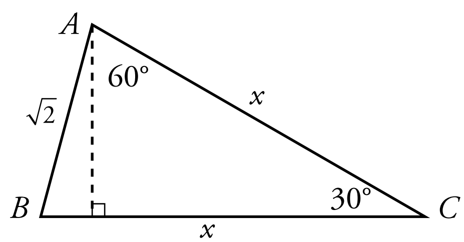
TD: "Figure out the two missing angles. Use the 180° rule."

180° – 30° = 150°.
Divide 150° equally across the two missing angles. So each angle is 75°.



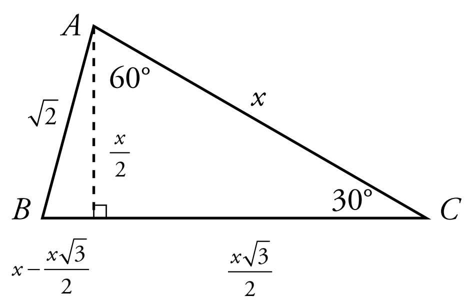
BU doesn't recognize this triangle.

TD: "Hmm... Here's a *Plan*: add a perpendicular line to make right triangles. Drop the line from the top point. Let's label corners while we're at it. Now fill in angles."



BU notices 30–60–90 and is happy.

TD: "Use the 30–60–90 to write expressions for its sides. Then side $\overline{BC}$ can be split up into 2 pieces, and we can set up the Pythagorean Theorem."



BU feels that this process is kind of ugly.

TD: "Let's push through. Write the Pythagorean Theorem for the small triangle on the left, using the $\sqrt{2}$ as the hypotenuse."

$\left(\frac{x}{2}\right)^2+\left(x-\frac{x\sqrt{3}}{2}\right)^2=\left(\sqrt{2}\right)^2$ $\frac{x^2}{4}+x^2-x^2\sqrt{3}+\frac{3x^2}{4}=2$ $\left(2-\sqrt{3}\right)x^2=2$	*BU thinks this equation is really ugly.* TD: “Push through. Execute the algebra quickly. Expand the quadratic & simplify.”
$x^2=\frac{2}{2-\sqrt{3}}\left(\frac{2+\sqrt{3}}{2+\sqrt{3}}\right)$ $=\frac{4+2\sqrt{3}}{4-3}=4+2\sqrt{3}$	TD: “Rationalize the denominator to get x^2 equal to a simpler expression.” *BU has no idea how to take the square root of this expression.*
Skip choice A, because it doesn’t contain $\sqrt{3}$.	TD: “Almost there. Change of *Plan*. Let’s just test answer choices by squaring them. Focus only on the ones involving $\sqrt{3}$.”
Square choice B. $\left(1+\sqrt{3}\right)\left(1+\sqrt{3}\right)=1+2\sqrt{3}+3=4+2\sqrt{3}$	TD: “This is it. Done.”

The answer is B.

The method we just saw is algebraically intensive, and so our Bottom-Up bloodhound might have kicked up a fuss along the way. Sometimes, your Top-Down brain needs to ignore the Bottom-Up brain. Remember, when you’re actually taking the GMAT, you have to solve problems quickly—and you don’t need to publish your solutions in a mathematics journal. What you want is to get the right answer as quickly and as easily as possible. In this regard, the solution above works perfectly well.

As an alternative method, we can estimate lengths. Draw the triangle *carefully* and start with the same perpendicular line as before. This line is a little shorter than the side of length $\sqrt{2}$ (which is about 1.4). So we can estimate the length of the perpendicular to be 1.2 or 1.3. Since this length is the “30” side of the 30-60-90 triangle, it’s half of the hypotenuse. Thus, we can estimate x to be 2.4 to 2.6.

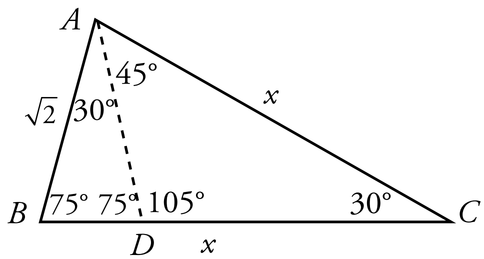
Now we can examine the answer choices. Approximate them using 1.4 for $\sqrt{2}$ and 1.7 for $\sqrt{3}$.

A) 2.4 B) 2.7 C) 2.8 D) 3.1 E) 3.4

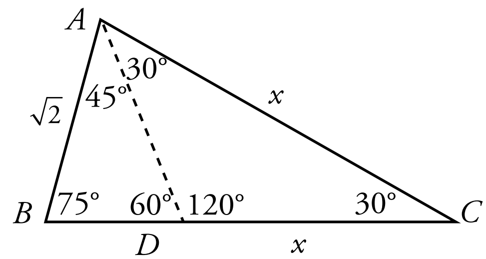
They’re all close, but we can pretty confidently eliminate D and E, and probably C for that matter. Now we’re guessing between A and B. Unfortunately, we might guess wrong at this point! But the odds are much better than they were at the outset.

A third method involves drawing different interior lines. It’s a good instinct to drop a perpendicular from the top point, but are there other possibilities?

Once we find that the two equal angles are 75°, we could try to split up one of the 75° angles in other ways. 75 = 30 + 45, both "friendly" angles (they show up in triangles we know). So let's split 75° two ways: 30° and 45°, and 45° and 30°.

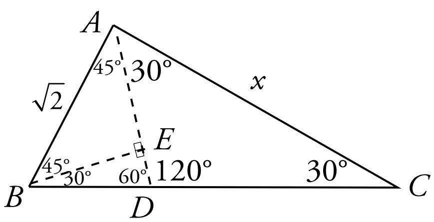


$\triangle ABD$ is similar to $\triangle CAB$, but otherwise, we don't have a whole lot to go on.



$\triangle ADC$ is isosceles (and it's made up of two 30–60–90 triangles). $\triangle ABD$ now has a 60° angle.

The case on the right seems more promising. Maybe if we add one more line? We may also be inspired by the $\sqrt{2}$ together with the 45°. This might remind us of a 45–45–90 triangle. So let's try to make one.



Now $\triangle ABE$ is a 45–45–90 triangle, $\triangle BED$ is a 30–60–90 triangle, and $\triangle ADC$ is still an isosceles triangle that, in the worst case, we can cut in half to make two 30–60–90 triangles.

We now have the problem cracked. From here, we just need to fill in side lengths.

The two short legs of $\triangle ABE$ each have length 1. Side $\overline{BE}$ is also part of the 30–60–90 triangle, so we can find the other two sides of that triangle. Now we find the length of side $\overline{AD}$, which also gives us side $\overline{DC}$. Finally, we get side $\overline{BC}$, which is *x*. Again, the answer is B, $1+\sqrt{3}$.

This third pathway is *extremely* difficult! It requires significant experimentation and at least a couple of flashes of insight. Thus, it is unlikely that the GMAT would *require* you to take such a path, even at the highest levels of difficulty.

However, it is still worthwhile to look for these sorts of solutions as you practice. Your Top-Down brain will become faster, more organized, and more flexible, enabling your Bottom-Up brain to have more flashes of insight.

That was a substantial introduction. Now, on to Chapter 1!